ON A LONG STANDING CONJECTURE:
POSITIVE LIOUVILLE THEOREM
FOR HYPOELLIPTIC ORNSTEIN–UHLENBECK OPERATORS

SU UNA CONGETTURA ANCORA APERTA:
TEOREMA DI LIOUVILLE POSITIVO
PER OPERATORI DI ORNSTEIN–UHLENBECK IPOELLITTICI

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ABSTRACT. Let $\mathcal{L}$ be the hypoelliptic Ornstein–Uhlenbeck operator associated with the pair of matrices (A, B). In 2004, Priola and Zabczyk proved the following Liouville-type theorem: every bounded entire solution of $\mathcal{L}u = 0$ is constant if and only if $(*)$ every eigenvalue of B has real part less than or equal to zero. This remarkable result raised the following problem, which is still not completely solved: if condition $(*)$ holds, is it true that every non-negative entire solution of $\mathcal{L}u = 0$ is constant? In this note, along with a review of the current state of research on this problem, we present some recent new results.

SUNTO. Sia $\mathcal{L}$ l'operatore di Ornstein-Uhlenbeck ipoellittico relativo alla coppia di matrici (A, B). Nel 2004, Priola e Zabczyk dimostrarono il seguente teorema di tipo Liouville: ogni soluzione intera e limitata di $\mathcal{L}u = 0$ è costante se e solo se $(*)$ ogni autovalore di B ha parte reale minore o uguale a zero. Questo notevole risultato pose il seguente problema non ancora completamente risolto: se la condizione $(*)$ è verificata è vero che ogni soluzione intera non-negativa di $\mathcal{L}u = 0$ è costante? In questa nota, insieme con una rassegna sullo "stato dell'arte" delle ricerche su questo problema, presentiamo alcuni recenti nuovi risultati.

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1. Introduction: Past and Recent History

Let A and B be $n \times n$ matrices with real constant entries, and let A be non-negative definite. The linear second order Partial Differential Operator in $\mathbb{R}^n$

$$\mathcal{L} = \mathcal{L}_{A,B} := \operatorname{div}(A\nabla u) + \langle Bx, \nabla u \rangle \tag{1}$$

is usually called the finite dimensional Ornstein–Uhlenbeck operator related to the pair (A, B). In (1) div, ∇, and $\langle \cdot, \cdot \rangle$ denote, respectively, the divergence, the gradient, and the inner product in $\mathbb{R}^n$.

Being $A \geq 0$, the operator $\mathcal{L} = \mathcal{L}_{A,B}$ is possibly degenerate; however it is *hypoelliptic*, i.e., every distributional solution to

$$\mathcal{L}u = f$$

is smooth whenever f is smooth – if and only if the pair (A, B) satisfies the following Kalman condition:

$$\operatorname{rank}\left[\sqrt{A},\ B\sqrt{A},\ B^2\sqrt{A},\ \ldots,\ B^{n-1}\sqrt{A}\right] = n. \tag{2}$$

It is quite well known that this condition is equivalent to the celebrated Hörmander rank condition for $\mathcal{L}_{A,B}$ (see, e.g., [8]).

When $A = I_n$ and $B = -I_n$ (with I_n denoting the $n \times n$ identity matrix), the case $(A, B) = (I_n, -I_n)$ leads to the classical Ornstein–Uhlenbeck operator, given by

$$\mathcal{L}_n := \mathcal{L}_{(I_n, -I_n)} := \Delta - \langle x, \nabla \rangle, \tag{3}$$

where

$$\Delta := \sum_{i=1}^{n} \partial_{x_i}^2$$

is the classical Laplace operator.

Before proceeding, it is appropriate to introduce some definitions.

We say that $\mathcal{L}$ has the *L^∞-Liouville property* if

$$\mathcal{L}u = 0 \text{ in } \mathbb{R}^n, \quad u \in L^\infty(\mathbb{R}^n) \ \Rightarrow \ u = \text{constant}.$$

We say that $\mathcal{L}$ has the *positive-Liouville property* if

$$\mathcal{L}u = 0 \text{ in } \mathbb{R}^n, \quad u \geq 0 \implies u = \text{constant}.$$

The spectrum of B is denoted by

$$\sigma(B) := \{\lambda \in \mathbb{C} \; : \; \lambda \text{ eigenvalue of } B\}.$$

We emphasize, although it is almost obvious, that the positive-Liouville property is stronger than the L^∞-Liouville property.

It is quite well known that the classical Ornstein–Uhlenbeck operator $\mathcal{L}_n$ has the *positive Liouville* property. This result naturally raises the following question.

> (Q) Let $\mathcal{L} = \mathcal{L}_{A,B}$ be hypoelliptic. Then: is it true that $\mathcal{L}$ has the positive Liouville property?

In general the answer is *no*, as the following example shows.

Example 1.1. *Consider the Ornstein–Uhlenbeck operator related to the pair (I_n, I_n), i.e.*

$$\tilde{\mathcal{L}}_n = \Delta + \langle x, \nabla \rangle.$$

The function
$$x \longmapsto u(x) = u(x_1, \ldots, x_n) := \int_{-\infty}^{x_1} \exp\left(-\frac{s^2}{2}\right) ds$$
is a strictly positive solution to

$$\tilde{\mathcal{L}}_n u = 0 \quad \text{in } \mathbb{R}^n,$$

and, obviously, u is non-constant.

A partial, though remarkable, answer to question (Q) was given by Priola and Zabczyk in the paper [PZ], in the following theorem.

Theorem 1.1 (Priola–Zabczyk)**.** *Let $\mathcal{L} = \mathcal{L}_{A,B}$ be the Ornstein–Uhlenbeck operator related to the pair (A, B). Assume that $\mathcal{L}$ is hypoelliptic. Then $\mathcal{L}$ has the L^∞–Liouville*

property if and only if

(4) $\mathrm{Re}(\lambda) \leq 0$ *for every* $\lambda \in \sigma(B)$.

In their work, Priola and Zabczyk used probabilistic techniques and formulated the following - quite natural - conjecture:

(C) *If* $\mathcal{L} = \mathcal{L}_{A,B}$ *is hypoelliptic and condition* (4) *is satisfied, then* $\mathcal{L}$ *also has the positive Liouville property.*

So far, (C) has been proven to hold in the following specific cases in which $\mathcal{L} = \mathcal{L}_{A,B}$ is assumed to be hypoelliptic:

(I) $\mathcal{L} = \mathcal{L}_{A,B}$ is homogeneous of degree two with respect to a group of dilations in $\mathbb{R}^n$. See [4]. The techniques used in that paper are purely analytic. We also stress that, in this case, $\sigma(B) = \{0\}$.

(II) $\mathcal{L} = \mathcal{L}_{A,B}$, with $A > 0$, B diagonalizable over the complex field $\mathbb{C}$, and $\sigma(B) \subseteq i\mathbb{R}$. See [5]. Also in this case, the authors used purely analytic techniques.

(III) $\mathcal{L} = \mathcal{L}_{A,B}$ with $A \geq 0$ and B having the following block form:

$$B = \begin{pmatrix} B_0 & 0 \\ 0 & B_1 \end{pmatrix},$$

where either $B_1 = \{0\}$ or

$$B_1 = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}, \qquad \alpha \in \mathbb{R},$$

and B_0 satisfies

$$\sigma(B_0) \subseteq \{\, \lambda \in \mathbb{C} : \mathrm{Re}(\lambda) < 0 \,\}.$$

(See again [5]). In this case the authors proved the *positive Liouville property* for $\mathcal{L}$ using probabilistic techniques.

Very recently, in [6], using *analytic techniques*, we have proved the following theorem, which establishes the positive Liouville property for a much larger class of operators than those in cases (I) and (II).

Theorem 1.2. *The Ornstein–Uhlenbeck operator*

$$\mathcal{L} = \mathcal{L}_{A,B}$$

has the positive Liouville property if the following conditions are satisfied:

(i) $\mathcal{L}$ *is hypoelliptic;*

(ii) $\sigma(B) \subseteq i\mathbb{R}.$

In the following paragraph we shall outline the main ideas underlying our approach to the positive Liouville properties of $\mathcal{L}_{A,B}$ operators.

2. Our methods

2a. One of our starting points in looking for positive Liouville property for the hypoelliptic Ornstein–Uhlenbeck operators is the following proposition.

Proposition 2.1. *Let* $\mathcal{L} = \mathcal{L}_{A,B}$ *be hypoelliptic. Then the following conditions are equivalent:*

(i) $\mathcal{L}$ *has the positive Liouville property;*

(ii) *every non-negative solution of*

$$\mathcal{L}u = 0 \quad in \ \mathbb{R}^n$$

attains a minimum.

Proof. The implication $(i) \Rightarrow (ii)$ is trivial. Let us prove the reverse implication. $(ii) \Rightarrow (i)$: Let u be a non-negative solution of

$$\mathcal{L}u = 0 \quad in \ \mathbb{R}^n.$$

By assumption (ii), there exists a point $x_0 \in \mathbb{R}^n$ such that

$$u(x_0) = \min_{\mathbb{R}^n} u.$$

Since $\mathcal{L} = \mathcal{L}_{A,B}$ is hypoelliptic, the pair (A, B) satisfies the Kalman condition (2), and thus, by the controllability Theorem 5.1.2 in the monograph [2], for every fixed $z \in \mathbb{R}^n$

there exists a control $u \in L^2([0,1], \mathbb{R}^n)$ such that the solution $x : [0,1] \to \mathbb{R}^n$ of

$$(5) \qquad \begin{cases} \dot{x}(t) = Bx(t) + A\,u(t), \\ x(0) = x_0, \end{cases}$$

satisfies $x(1) = z$. Since x_0 is a minimum point of u, by Theorem 11.7 in the monograph [1], the minimum of u propagates along the trajectory $t \mapsto x(t)$ in (5). Hence

$$u(x(t)) = u(x_0) \qquad \text{for every } t \in [0,1].$$

In particular, being $x(1) = z$,

$$u(z) = u(x_0).$$

Since z is an arbitrary point of $\mathbb{R}^n$, we conclude that u is constant on $\mathbb{R}^n$. Therefore, as desired, $\mathcal{L} = \mathcal{L}_{A,B}$ has the *positive Liouville property*.   □

A sufficient condition ensuring that a non-negative solution of $\mathcal{L}u = 0$ in $\mathbb{R}^n$ attains a minimum is provided by the following lemma.

Lemma 2.1. *Let $\mathcal{L} = \mathcal{L}_{A,B}$ be an Ornstein–Uhlenbeck operator. Assume that, for a suitable $R > 0$, there exists a C^2-function*

$$h : \{x \in \mathbb{R}^n : |x| \geq R\} \to \mathbb{R}$$

such that

$$(6) \qquad \mathcal{L}h \leq 0 \qquad \text{for every } x \in \mathbb{R}^n, \ |x| \geq R,$$

and

$$(7) \qquad \lim_{|x| \to \infty} h(x) = \infty.$$

Then every non-negative solution to

$$\mathcal{L}u = 0 \quad \text{in } \mathbb{R}^n$$

attains a minimum in $\mathbb{R}^n$.

Proof. Let u be a non-negative solution to $\mathcal{L}u = 0$ in $\mathbb{R}^n$. Denote by B the Euclidean ball

$$B = \{x \in \mathbb{R}^n : |x| < 2R\}.$$

By Picone's minimum principle (see, e.g., [1]),

$$(8) \qquad u(x) \geq \min_{\partial B} u \qquad \text{for every } x \in B.$$

We want to prove that the following inequality also holds:

$$(9) \qquad u(x) \geq \min_{\partial B} u \qquad \text{for every } x \notin B.$$

To this end, for $\varepsilon > 0$ we let

$$u_\varepsilon = u + \varepsilon h.$$

Since $\mathcal{L}u = 0$ and $\mathcal{L}h \leq 0$, we have

$$\mathcal{L}u_\varepsilon \leq 0 \qquad \text{in } \mathbb{R}^n \setminus B.$$

Moreover, since $u \geq 0$, by (7) we obtain

$$\lim_{|x| \to \infty} u_\varepsilon(x) = \infty.$$

As a consequence, by Picone's minimum principle again,

$$u_\varepsilon(x) \geq \min_{\partial B} u_\varepsilon \qquad \text{for every } x \in \mathbb{R}^n \setminus B,$$

hence,

$$(10) \qquad u(x) + \varepsilon h(x) \geq \min_{\partial B} u - \varepsilon \sup_{\partial B} |h| \qquad \text{for every } x \in \mathbb{R}^n \setminus B.$$

Letting $\varepsilon \to 0$ in (10) we obtain (9). This latter, together with inequality (8), implies

$$u(x) \geq \min_{\partial B} u \qquad \text{for every } x \in \mathbb{R}^n.$$

Therefore, there exists $x_0 \in \partial B$ such that

$$u(x_0) = \min_{\mathbb{R}^n} u.$$

This concludes the proof of the lemma.   □

Having Proposition 2.1 and Lemma 2.1 at our disposal, we are now in a position to prove the following theorem.

Theorem 2.1. *Let $\mathcal{L} = \mathcal{L}_{A,B}$ be a hypoelliptic Ornstein–Uhlenbeck operator. If*

$$\mathrm{Re}(\lambda) < 0 \quad \text{for every } \lambda \in \sigma(B)$$

then $\mathcal{L}$ has the positive Liouville property.

Proof. Since the eigenvalues of B have strictly negative real part, the Lyapunov matrix equation

$$PB + B^T P + I_n = 0,$$

has a solution P which is symmetric and strictly positive definite: see [2], Lemma 3.1.7.

Then

$$\langle PBx, x \rangle + \langle B^T Px, x \rangle = -|x|^2 \qquad \text{for every } x \in \mathbb{R}^n.$$

On the other hand,

$$\langle B^T Px, x \rangle = \langle x, PBx \rangle.$$

Thus

(11) $$2 \langle PBx, x \rangle = -|x|^2.$$

Let us now consider the function

$$h(x) := \langle Px, x \rangle, \qquad x \in \mathbb{R}^n.$$

Since P is strictly positive definite, we have

(12) $$\lim_{|x| \to \infty} h(x) = \infty.$$

Moreover, by (11),

$$\mathcal{L}h(x) = 2 \, \mathrm{tr}(AP) + 2\langle Bx, Px \rangle$$

$$= 2 \, \mathrm{tr}(AP) + 2\langle PBx, x \rangle$$

$$= 2 \operatorname{tr}(AP) - |x|^2.$$

Therefore,

(13) $$\mathcal{L}h(x) \leq 0 \qquad \text{if } |x| \geq R_1 := \sqrt{2 \operatorname{tr}(AP)}.$$

By (12) and (13) we may apply Lemma 2.1 and conclude that every non-negative solution of $\mathcal{L}u = 0$ in $\mathbb{R}^n$ attains a minimum. Thus, by Proposition 2.1, $\mathcal{L}$ has the positive Liouville property. $\qquad \square$

Remark 2.1. *Under the assumptions of case (III) listed in the Introduction, we are able to construct a function h satisfying the assumptions of Lemma 2.1, by adding to a function like the one constructed in the previous proof, a suitable logarithmic function of two real variables. We refer the reader to ([7]) for the explicit construction of the function h. Thus, also in this case, the positive Liouville property for $\mathcal{L} = \mathcal{L}_{A,B}$ can be proved by means of purely analytic techniques.*

2.b. If $\sigma(B) \subset i\mathbb{R}$, the method to prove the positive Liouville property used in the previous sub-section does *not* work or at least, we were not able to get it to work.

To prove Theorem 1.2 we used a completely different technique, based on certain properties of the Kolmogorov operator evolution, which is the counterpart of the classical Ornstein–Uhlenbeck operator. Let

$$\mathcal{L} = \mathcal{L}_{A,B}$$

be the Ornstein–Uhlenbeck operator in $\mathbb{R}^n$ related to the pair (A, B), and consider the Kolmogorov operator on $\mathbb{R}^n \times \mathbb{R}$:

(14) $$\mathcal{K} = \mathcal{K}_{A,B} := \mathcal{L}_{A,B} - \partial_t = \operatorname{div}(AD) + \langle Bx, D \rangle - \partial_t.$$

This operator is hypoelliptic if and only if $\mathcal{L}_{A,B}$ is hypoelliptic.

It is quite well known that the solutions to

$$\mathcal{K}u = 0$$

satisfy a mean value formula on the level set of the fundamental solution of $\mathcal{K}$; see e.g. [3, 8].

With a procedure based on this mean value property, in [6] we proved the following theorem.

Theorem 2.2. *Let $\mathcal{K} = \mathcal{K}_{A,B}$ be the Kolmogorov operator in* (14). *Assume $\mathcal{K}$ hypoelliptic and*

$$\sigma(B) \subset i\mathbb{R}.$$

Then there exists a constant $c > 0$, depending only on $\mathcal{K}$, such that:

for every non–negative solution v to

$$\mathcal{K}v = 0 \qquad in \ \mathbb{R}^n \times \mathbb{R},$$

and for every $z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$, and for every $x \in \mathbb{R}^n$, one has

$$\limsup_{t \to -\infty} v(x, t) \ \leq \ c \, v(x_0, t_0).$$

With this theorem at hand it is very easy to prove the positive Liouville Theorem 1.2.

Proof of Theorem 1.2. Let u be a non-negative solution to

$$\mathcal{L}_{A,B} u = 0 \qquad in \ \mathbb{R}^n.$$

Let

$$m = \inf_{\mathbb{R}^n} u$$

and define

$$v : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}, \qquad v(x, t) = u(x) - m.$$

Then $v \geq 0$, and

$$\mathcal{K}_{A,B} v = 0 \quad in \ \mathbb{R}^n \times \mathbb{R}.$$

Fix $\varepsilon > 0$ and choose a point $z \in \mathbb{R}^n \times \mathbb{R}$ such that

$$v(z) < \varepsilon.$$

Then, by Theorem 2.2,

$$0 \leq u(x) - m = \limsup_{t \to -\infty} v(x, t) \leq c \, v(z) \leq c\varepsilon$$

for every $x \in \mathbb{R}^n$, where $c > 0$ is independent of u and ε.

Therefore,

$$0 \leq u(x) - m \leq c\varepsilon \qquad \text{for every } x \in \mathbb{R}^n,$$

and for every $\varepsilon > 0$. This clearly implies

$$u(x) = m \qquad \forall x \in \mathbb{R}^n.$$

This completes the proof of Theorem 1.2. $\qquad\qquad\qquad\qquad\qquad\qquad \square$

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