LIOUVILLE THEOREMS FOR NONLOCAL OPERATORS WITH CONICAL DIFFUSION

TEOREMI DI LIOUVILLE PER OPERATORI NON LOCALI CON DIFFUSIONE CONICA

GIULIO GALISE

Abstract. We consider linear stable operators whose spectral measure is assumed to be positive only on a relatively open subset of the unit sphere, the aim being to present semilinear Liouville-type results for positive supersolutions in a half-space.

Sunto. Vengono presentati risultati di tipo Liouville, nel semispazio, per soprasoluzioni positive di operatori lineari stabili la cui misura spettrale è supposta essere positiva soltanto in un aperto relativo della sfera unitaria.

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1. Introduction

The aim of this note is to review recent results, jointly obtained with I. Birindelli and L. Du [9], about Liouville type theorems for positive supersolutions of semilinear integral equations in the half-space $\mathbb{R}^N_+ := \left\{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, x_N > 0\right\}$. We also include an additional new result related to this framework.

We focus on the existence/nonexistence of classical solutions of the problem

$$\begin{cases} -Lu \geq u^p, & x \in \mathbb{R}^N_+, \\ u \geq 0, & x \in \mathbb{R}^N, \end{cases} \tag{1}$$

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Dipartimento di Matematica, Università di Bologna

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where $p \geq 1$ and L is a linear stable operators of order $2s$, with $s \in (0,1)$, of the form

$$Lu(x) = (1-s)\int_{\mathbb{R}^N} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}} a\left(\frac{y}{|y|}\right) dy, \tag{2}$$

with $a(\theta) \in L^\infty\left(\mathbb{S}^{N-1}\right)$ nonnegative function supposed to be positive only on a relatively open subset of the unit sphere $\mathbb{S}^{N-1}$ of $\mathbb{R}^N$.

The study of such operators, which originated in the context of Lévy-type stochastic processes (see e.g. [6]), has given rise to numerous analytical questions concerning existence of solutions, interior and boundary regularity and general qualitative properties of solutions (see e.g. [21]).

Liouville type theorems play a fundamental role in the theory of integro-differential elliptic and parabolic equations. In addition to the their intrinsic interest concerning classifications results establishing rigidity properties of the set of solutions, they are also employed to get a priori estimates and compactness results via blow-up procedure, see [23]. As a consequence, nonexistence results in the whole space and in half-spaces correspond, via topological methods, to existence results in bounded domains.

The existence of positive solutions to elliptic inequalities involving a power-like dependence in the zero-order term u has been treated and generalized in various contexts: cone-like domains for linear and quasilinear operators [2, 5, 22, 25], fully nonlinear inequalities, both uniformly elliptic and degenerate, [3, 4, 10, 17, 26], systems [7, 11, 29], Heisenberg Laplacian [8, 27]. In the context of fractional operators, various results have been established concerning solutions, see e.g [12, 13, 14, 15, 18, 30, 31, 33], while comparatively little is known about supersolutions of semilinear-type equations in half-spaces. In the recent paper [28], the existence of fundamental solutions for extremal fully nonlinear integral operators in conical domains was proved and, as application, the authors obtained the nonexistence of nontrivial supersolutions for the Lane-Emden-Fowler equation in a subcritical regime for p. In the special case where the cone is the half-space $\mathbb{R}^N_+$, they find that problem (1) with $-L \equiv (-\Delta)^s$ possesses no positive solutions for any $-1 \leq p \leq \frac{N+s}{N-s}$. It is important to note that the kernels of the operators considered in [28] are bounded from below and from above, up to ellipticity constants $0 < \lambda \leq \Lambda$, by that of the fractional Laplacian, which corresponds to the case $a(\theta)$ constant function in

(2). By contrast, the operators considered in [9] have anisotropic diffusion, since the condition $\inf_{\theta\in\mathbb{S}^{N-1}} a(\theta)=0$ is consistent with the following hypothesis: there exist positive constants $0<d\leq D$, $0<\tau_0\leq 1$ and $\nu_0\in\mathbb{S}^{N-1}$ such that

$$0\leq a(\theta)\leq D \quad \text{in } \mathbb{S}^{N-1} \tag{3}$$

and

$$a(\theta)\geq d>0 \quad \text{in } \Sigma_{\nu_0,\tau_0}(0)\cap\mathbb{S}^{N-1}, \tag{4}$$

where

$$\Sigma_{\nu_0,\tau_0}(0):=\left\{y\in\mathbb{R}^N:\ |y\cdot\nu_0|\geq(1-\tau_0)\,|y|\right\} \tag{5}$$

is the closed two fold cone centered at 0, with axis ν_0 and aperture $\arccos(1-\tau)\in\left(0,\frac{\pi}{2}\right]$. The prototype to keep in mind is

$$a(\theta)=\begin{cases}1 & \text{if } \theta\in\Sigma_{\nu_0,\tau_0}(0)\cap\mathbb{S}^{N-1}\\ 0 & \text{otherwise,}\end{cases}$$

which justifies the terminology *conical diffusion*, as the structure of the operator corresponding to this choice of $a(\theta)$

$$Lu(x)=(1-s)\int_{\Sigma_{\nu_0,\tau_0}(0)}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\,dy$$

reflects the underlying conical geometry.

Other results concerning qualitative properties of solutions to nonlocal operators with anisotropic diffusion ca be found in [19, 24, 32].

Inspired by the pioneering proof used by Berestycki, Capuzzo Dolcetta and Nirenberg in [5] to prove semilinear Liouville-type results in cones, we obtain the following

Theorem 1.1 ([9, Theorem 1.1])**.** *Let $s\in(0,1)$ and let L be any operator of the form* (2) *satisfying* (3)-(4)*. If $1\leq p\leq\frac{N+s}{N-s}$ and $u\in C^2\left(\mathbb{R}^N_+\right)\cap\mathcal{L}_s$ is a solution of* (1)*, then $u\equiv 0$.*

In the above theorem, $\mathcal{L}_s$ denotes the functional space

$$\mathcal{L}_s = \left\{ u \in L^1_{loc}\left(\mathbb{R}^N\right) : \limsup_{|x|\to+\infty} \frac{|u(x)|}{|x|^{2s-\delta}} < +\infty \text{ for some } \delta \in (0, 2s] \right\}$$

and the requirement $u \in C^2\left(\mathbb{R}^N_+\right) \cap \mathcal{L}_s$ is a sufficient condition ensuring both that $Lu(x)$ is well defined for every $x \in \mathbb{R}^N_+$ and that the integration by parts formula stated in Proposition 2.1 holds.

It is worth emphasizing that the key ingredients used in [5] need to be completely reconsidered, due to both the nonlocal character of the operators we consider and their weak diffusion.

A Liouville result in the whole space can be produced by means of the arguments used in the proof of Theorem 1.1, in a simplified form due to the absence of the boundary of the domain which, by contrast, introduce extra challenges in the half-space setting.

Theorem 1.2. *Let $N \geq 2$ and $1 \leq p \leq \frac{N}{N-2s}$. If $u \in C^2\left(\mathbb{R}^N\right) \cap \mathcal{L}_s$ is a nonnegative solution of*

$$-Lu \geq u^p \quad \text{in } \mathbb{R}^N, \tag{6}$$

then $u \equiv 0$.

The upper bound on p in Theorem 1.2 is sharp for the fractional Laplacian and extends to the fully nonlinear extremal operators $\mathcal{M}^{\pm}$, which are the integro-differential analog of Pucci's operators. In [20] it is proved that the equation $-\mathcal{M}^{\pm}u = u^p$, with $p > 1$, has no entire positive (viscosity) supersolutions if, and only if, $p < \frac{N^{\pm}}{N^{\pm}-2s}$, $N^{\pm}$ being dimensional-like numbers reducing to the dimension N when $-\mathcal{M}^{\pm} = (-\Delta)^s$. The exponent $p = \frac{N}{N-2s}$, which serves as a threshold number separating the existence from the nonexistence regimes, is the nonlocal analogue of the celebrated Serrin's exponent $\frac{N}{N-2}$.
As far as the half-space $\mathbb{R}^N_+$ is concerned, the bound on p in Theorem 1.1 is sharp for $-L \equiv (-\Delta)^s$, as the following theorem shows.

Theorem 1.3 ([9, Theorem 4.1])**.** *For any* $p > \frac{N+s}{N-s}$*, the problem*

$$\begin{cases} (-\Delta)^s u \geq u^p, & x \in \mathbb{R}^N_+, \\ u > 0, & x \in \mathbb{R}^N_+, \\ u = 0, & x \in \mathbb{R}^N_- := \left\{ x \in \mathbb{R}^N \,:\, x_N < 0 \right\} \end{cases} \tag{7}$$

admits solutions $u \in C^2\left(\mathbb{R}^N_+\right) \cap \mathcal{L}_s$*.*

In accordance with a stability principle, one finds that $\frac{N+s}{N-s}$ converges, as $s \to 1^-$, to the critical exponent $\frac{N+1}{N-1}$ associated to the inequality $-\Delta u \geq u^p$ in $\mathbb{R}^N_+$.

An explanation of the main results will be provided in the next section, with a focus on the underlying ideas of the proofs and on the differences encountered in the transition from the local to the nonlocal framework.

We shall conclude this note by presenting a new result (Proposition 2.2), not contained in [9], concerning the existence of positive one-dimensional solutions to problem (1) in the case $p > \frac{1+s}{1-s}$, for general linear operators of the form (2) satisfying the assumptions (3)-(4).

2. Description of the results

The arguments used in the proofs of Theorems 1.1 and 1.2 rely on a rescaled test function method, suitably adapted to our nonlocal setting with weak diffusion.

2.1. **Nonexistence of supersolutions in $\mathbb{R}^N$.** Denote by $B_r = \left\{ x \in \mathbb{R}^N \,:\, |x| < r \right\}$ the open ball of radius $r > 0$ centered at the origin. Let $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 < \varphi \leq 1$ in B_1 and

$$\varphi = \begin{cases} 1 & \text{if } x \in B_{\frac{1}{2}} \\ 0 & \text{if } x \notin B_1. \end{cases} \tag{8}$$

The idea is to use the "conical diffusion" to prove the following inequality: there exists $M > 0$ such that

$$-L\varphi(x) \leq M\varphi(x) \qquad \forall x \in \mathbb{R}^N. \tag{9}$$

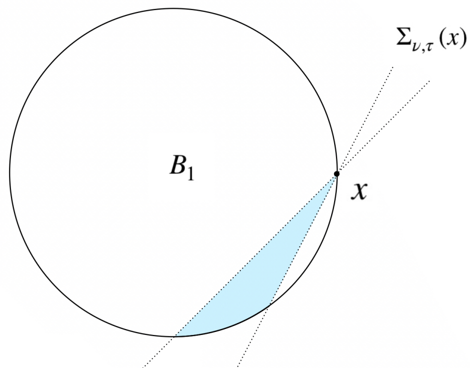

FIGURE 1.

The above inequality is trivially satisfied for $|x| \geq 1$, independently of M, due to the sign of the quantities involved in the definition of the operator $-L$ and the definition of φ. For $|x| < 1$, the inequality (9) is equivalent to prove that

$$\inf_{x \in B_1} \frac{L\varphi(x)}{\varphi(x)} > -\infty.$$

By a contradiction argument, taking a sequence $(x_n)_{n\in\mathbb{N}}$ of points of B_1 such that

$$\lim_{n\to+\infty} \frac{L\varphi(x_n)}{\varphi(x_n)} = -\infty \quad \text{and} \quad \lim_{n\to+\infty} x_n = x \in \overline{B}_1,$$

it is easy to exclude the case $|x| < 1$ since

$$\lim_{n\to+\infty} \frac{L\varphi(x_n)}{\varphi(x_n)} = \frac{L\varphi(x)}{\varphi(x)}.$$

On the other hand, if $|x| = 1$, using (4) we infer that

$$\begin{aligned} \lim_{n\to+\infty} L\varphi(x_n) &= \int_{\mathbb{R}^N} \frac{\varphi(x+y)+\varphi(x-y)}{|y|^{N+2s}} a\left(\frac{y}{|y|}\right) dy \\ &\geq d \int_{\Sigma_{\nu_0,\tau_0}(0)} \frac{\varphi(x+y)+\varphi(x-y)}{|y|^{N+2s}} dy. \end{aligned} \tag{10}$$

The last integral in (10) is positive, since

$$y \in \Sigma_{\nu_0,\tau_0}(0) \quad \Rightarrow \quad x \pm y \in \Sigma_{\nu_0,\tau_0}(x) := x + \Sigma_{\nu_0,\tau_0}(0)$$

and the cone $\Sigma_{\nu_0,\tau_0}(x)$ has a full measure intersection with the support of φ, see FIGURE 1.

This leads to the contradiction

$$\lim_{n\to+\infty}\frac{L\varphi(x_n)}{\varphi(x_n)}=+\infty.$$

We now consider the rescaled functions $\varphi_R(x)=\varphi\left(\frac{x}{R}\right)$ for $R>0$. Multiplying both sides of the inequality $-Lu\geq u^p$ by φ_R, using Fubini's theorem and the inequality (9), we obtain

$$\int_{\mathbb{R}^N}u^p\varphi_R\,dx\leq-\int_{\mathbb{R}^N}uL\varphi_R\,dx\leq MR^{-2s}\int_{\mathbb{R}^N}u\varphi_R\,dx, \tag{11}$$

the term R^{-2s} being a consequence of the $2s$-homogeneity of the operator $-L$.
Since $\varphi_R\to1$ locally uniformly as $R\to+\infty$, we immediately get $u\equiv0$ from (11) in the case $p=1$. If instead $p>1$, an application of the Hölder inequality yields

$$\int_{\mathbb{R}^N}u^p\varphi_R\,dx\leq C\,R^{N-\frac{2sp}{p-1}}, \tag{12}$$

with C a positive constant independent on R. Then $u\equiv0$ in $\mathbb{R}^N$ by letting $R\to+\infty$.
At this point, we wish to emphasize that the inequality (9) has allowed us to work directly with the test function φ in the application of the rescaled test function method, without the need to consider suitable powers of φ_R, as is usually required in the local setting.
As far as the critical case $p=\frac{N}{N-2s}$ is concerned, then (12) implies

$$u^p\in L^1(\mathbb{R}^N). \tag{13}$$

An additional difference, due to the nonlocal nature of the problem we are considering, emerges when compared to the local case. If $-L\equiv-\Delta$, one has $L\varphi(x)=0$ for $|x|\leq\frac{1}{2}$ and for $|x|\geq1$. Hence, the corresponding integrals on the right hand side of (11) is in fact computed only in the spherical shell $\frac{R}{2}\leq|x|\leq R$. Then Hölder inequality yields

$$\int_{\mathbb{R}^N}u^p\varphi_R\,dx\leq C\left(\int_{\frac{R}{2}\leq|x|\leq R}u^p\varphi_R\,dx\right)^{\frac{1}{p}}$$

and, in view of (13), the integral on the right hand side of the above inequality vanishes as $R\to+\infty$.

In the nonlocal setting, a more refined analysis is required, since $L\varphi(x) < 0$ for all $|x| \leq \frac{1}{2}$. It turns out that

$$\int_{\mathbb{R}^N} u^p \varphi_R dx \leq C \left[R^{-s} \left(\int_{\mathbb{R}^N} u^p dx \right)^{\frac{1}{p}} + \left(\int_{\sqrt{R} \leq |x| \leq R} u^p dx \right)^{\frac{1}{p}} \right].$$

Using (13) and $s > 0$, we still conclude that $u \equiv 0$ by letting $R \to +\infty$.

2.2. **Nonexistence of supersolutions in $\mathbb{R}^N_+$.** In the local setting (see e.g. [5]), the test function used in the half-space case is typically of the form $\Phi(x) = w(x)\varphi(x)$, with $w(x) = x_N$. The function w, which is a positive harmonic function in $\mathbb{R}^N_+$ vanishing on the boundary $\partial\mathbb{R}^N_+$, is a natural choice, suggested both by the geometry of the domain in which the problem is posed and by the integration by parts. The following lemma is useful in our setting.

Lemma 2.1 ([9, Lemma 2.4])**.** *Let $0 < \alpha < 2s$ and $w_\alpha(x) = (x_N)^\alpha_+$. For any $x \in \mathbb{R}^N_+$,*

$$Lw_\alpha(x) = C_\alpha x_N^{\alpha - 2s}, \tag{14}$$

where

$$C_\alpha \begin{cases} < 0, & 0 < \alpha < s, \\ = 0, & \alpha = s, \\ > 0, & s < \alpha < 2s. \end{cases}$$

As far as the integration by parts is concerned, it is worth stressing that this step is not straightforward, given that the test functions used in the proof of Theorem 1.3 lack regularity on $\partial\mathbb{R}^N_+$, and the operator $-L$ involved may be regarded as singular there.

Proposition 2.1 ([9, Proposition 2.2])**.** *Let $(2s-1)_+ < \alpha < 2s$, $\psi \in C^2_0\left(\mathbb{R}^N\right)$ and $v_\alpha(x) = (x_N)^\alpha_+ \psi(x)$. Then for any $u \in C^2\left(\mathbb{R}^N_+\right) \cap \mathcal{L}_s$ satisfying*

$$u = 0 \quad \text{in } \overline{\mathbb{R}^N_-} \quad \text{and} \quad \|u\|_{C^2\left(K \cap \mathbb{R}^N_+\right)} < +\infty, \quad \forall \text{ compact } K \subseteq \overline{\mathbb{R}^N_+},$$

we have

$$\int_{\mathbb{R}^N} uLv_\alpha \, dx = \int_{\mathbb{R}^N} v_\alpha Lu \, dx.$$

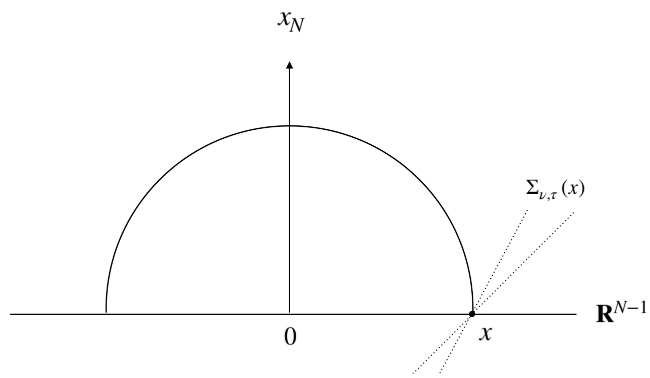

FIGURE 2.

In analogy with the case of $\mathbb{R}^N$, our goal is to derive an inequality of the form

$$-L\Phi_s(x) \leq M\Phi_s(x) \quad \forall x \in \mathbb{R}^N_+, \tag{15}$$

where $\Phi_s(x) = (x_N)^s_+\varphi(x)$, M is a positive constant and $\varphi \in C_0^\infty(\mathbb{R}^N)$ is a suitable cut-off function. Then, we apply a rescaled test function argument with the use of Lemma 2.1 and Proposition 2.1.

A crucial property in deriving such estimate is that, for each $x \in \partial B_1 \cap \overline{\mathbb{R}^N_+}$, the cone $\Sigma_{\nu_0,\tau_0}(x)$ should intersects the support of the function Φ_s in a set of full measure. But differently from the case of $\mathbb{R}^N$ and φ as in (8), in the half-space $\mathbb{R}^N_+$ there might that for some $x \in \partial\mathbb{R}^N_+ \cap \partial B_1$ then

$$\left|\Sigma_{\nu_0,\tau_0}(x) \cap B_1 \cap \mathbb{R}^N_+\right| = 0,$$

see FIGURE 2.

Accordingly, we introduce a test function $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$ such that $0 < \varphi \leq 1$ in $B_1\left(\left(1-\gamma_0\right)e_N\right)$ and

$$\varphi(x) = \begin{cases} 1, & x \in B_{1-\frac{\gamma_0}{2}}\left(\left(1-\gamma_0\right)e_N\right), \\ 0, & x \notin B_1\left(\left(1-\gamma_0\right)e_N\right), \end{cases}$$

where $\gamma_0 = \gamma(\nu_0, \tau_0)$ is chosen in such a way that for any $x \in \partial B_1\left(\left(1-\gamma_0\right)e_N\right) \cap \overline{\mathbb{R}^N_+}$ it holds

$$\left|\Sigma_{\nu_0,\tau_0}(x) \cap B_1\left(\left(1-\gamma_0\right)e_N\right) \cap \mathbb{R}^N_+\right| > 0,$$

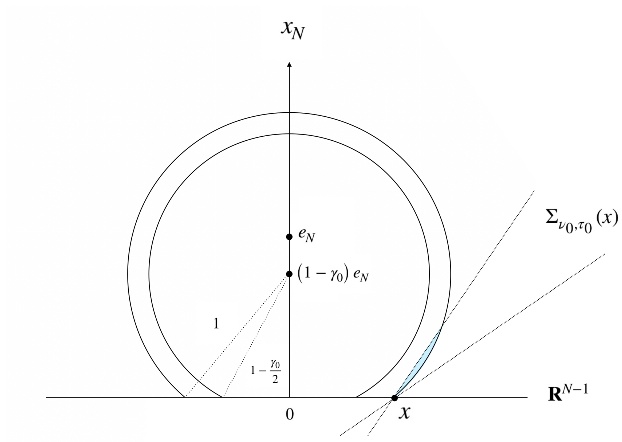

FIGURE 3.

see FIGURE 3.

Then inequality (15) follows. As a matter of fact, in the proof of Theorem 1.3, a stronger variant of (15) is used, in particular to deal with the critical case $p = \frac{N+s}{N-s}$: for $\alpha_0 \in (s, \min\{1, 2s\})$ then

$$-L\Phi_{\alpha_0}(x) - L\Phi_s(x) \leq M\Phi_s(x) \quad \forall x \in \mathbb{R}^N_+. \tag{16}$$

After establishing this inequality, then the rescaled test function method, together with a careful application of the integration by parts formula, yields the proof of Theorem 1.1. All the details can be found in [9, Theorems 3.4 and 1.1].

2.3. **Optimality of the exponent $\frac{N+s}{N-s}$.** The existence of nontrivial solutions to problem (7) in the supercritical regime $p > \frac{N+s}{N-s}$ is established by applying the fractional Kelvin transform to the one-dimensional functions of Lemma 2.1 (see [9, Proof of Theorem 4.1]). The question if the exponent $p = \frac{N+s}{N-s}$ is optimal for any linear operator L in the class (2)-(3)-(4) is an open question.

On the other hand, we establish that one-dimensional solutions of

$$\begin{cases} -Lu = u^p, & x \in \mathbb{R}^N_+, \\ u = 0, & x \in \mathbb{R}^N_-, \end{cases} \tag{17}$$

for general L, appear as soon as $p > \frac{1+s}{1-s}$. This result therefore shows that the critical exponent associated with the operator L, which separates the range of existence from that of nonexistence of supersolutions to (17), can be bounded from above by $\frac{1+s}{1-s}$, independently of the choice of the function $a \in L^\infty\left(\mathbb{S}^{N-1}\right)$ satisfying assumptions (3)-(4).

Proposition 2.2. *Let $s \in (0,1)$ and let L be any operator of the form (2) satisfying the assumptions (3)-(4). Then, for any $p \in \left(\frac{1+s}{1-s}\,,+\infty\right)$, problem (17) has a positive solution $u \in C^2\left(\mathbb{R}^N_+\right)$.*

Remark 2.1. *It is worth emphasizing that the solutions provided by the above proposition do not belong to the functional space $\mathcal{L}_s$. Nevertheless, we point out that the operator L is still well defined on such functions, as will become clear from the proof.*
It remains an open question whether solutions (or supersolutions) in Proposition 2.2 can be found in $\mathcal{L}_s$, and whether the assumption $u \in \mathcal{L}_s$ in Theorems 1.1 and 1.2 can be replaced by the more general condition $u \in L^1_s(\mathbb{R}^N)$ (see, e.g., [1, (1.15)] for the definition of $L^1_s(\mathbb{R}^N)$).

Since Proposition 2.2 establishes a new result not covered in [9], we include a detailed proof.

Proof of Proposition 2.2. For $\alpha \in (-1,0)$, let $u_\alpha : \mathbb{R}^N \mapsto \mathbb{R}$ be the function defined by

$$u_\alpha(x) := h_\alpha(x_N) := \begin{cases} x_N^\alpha & \text{if } x_N > 0 \\ 0 & \text{if } x_N \leq 0. \end{cases}$$

We first note that the operator $Lu_\alpha(x)$ is well defined for any $x \in \mathbb{R}^N_+$. For this, it is sufficient to consider the case $x = e_N = (0,\dots,0,1)$, since by the homogeneity properties of L and u_α one has

$$Lu_\alpha(x) = x_N^{\alpha-2s} Lu_\alpha(e_N). \tag{18}$$

The map $y \mapsto u_\alpha(e_N+y) + u_\alpha(e_N-y) - 2u_\alpha(e_N)$ is smooth in $\overline{B_{\frac12}}$, hence

$$\left|\frac{u_\alpha(e_N+y) + u_\alpha(e_N-y) - 2u_\alpha(e_N)}{|y|^{N+2s}} a\left(\frac{y}{|y|}\right)\right| \leq \frac{C}{|y|^{N-2(1-s)}} \qquad \forall y \in B_{\frac12} \setminus \{0\}\,, \tag{19}$$

where C is a positive constant depending only on α and D, with D defined in (3).
Moreover, the integrals

$$\int_{\mathbb{R}^N\setminus B_{\frac{1}{2}}} \frac{|u_\alpha(e_N \pm y)|}{|y|^{N+2s}} \left|a\left(\frac{y}{|y|}\right)\right| dy \quad \text{and} \quad \int_{\mathbb{R}^N\setminus B_{\frac{1}{2}}} \frac{1}{|y|^{N+2s}} \left|a\left(\frac{y}{|y|}\right)\right| dy \tag{20}$$

are finite. This is evident for the last one, since $s > 0$ and by (3). As far as the first one is concerned, denoting by

$$\Omega_1 = \left\{ y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, -1 < y_N < -\frac{1}{2} \right\}$$

and by

$$\Omega_2 = \left\{ y = (y', y_N) \in \mathbb{R}^{N-1} \times \mathbb{R} \,:\, y_N \geq -\frac{1}{2}\,,\ |y| \geq \frac{1}{2} \right\},$$

we have

$$\int_{\mathbb{R}^N\setminus B_{\frac{1}{2}}} \frac{|u_\alpha(e_N + y)|}{|y|^{N+2s}} \left|a\left(\frac{y}{|y|}\right)\right| dy \leq D \left(\underbrace{\int_{\Omega_1} \frac{(1+y_N)^\alpha}{|y|^{N+2s}}\, dy}_{=:I_1} + 2^{-\alpha} \underbrace{\int_{\Omega_2} \frac{1}{|y|^{N+2s}}\, dy}_{=:I_2} \right). \tag{21}$$

The integral I_2 is finite since $s > 0$. As for I_1, using first the Fubini-Tonelli theorem for nonnegative functions and then the change of variables $z' = \frac{y'}{|y_N|}$, we obtain

$$\begin{aligned} I_2 &= \int_{-1}^{-\frac{1}{2}} (1+y_N)^\alpha \left(\int_{\mathbb{R}^{N-1}} \frac{dy'}{(|y'|^2 + y_N^2)^{\frac{N+2s}{2}}} \right) dy_N \\ &= \int_{\mathbb{R}^{N-1}} \frac{dz'}{(1+|z'|^2)^{\frac{N+2s}{2}}} \int_{-1}^{-\frac{1}{2}} \frac{(1+y_N)^\alpha}{|y_N|^{1+2s}}\, dy_N \\ &\leq 2^{1+2s} \int_{\mathbb{R}^{N-1}} \frac{dz'}{(1+|z'|^2)^{\frac{N+2s}{2}}} \int_{-1}^{-\frac{1}{2}} (1+y_N)^\alpha\, dy_N < +\infty, \end{aligned} \tag{22}$$

since $\alpha \in (-1, 0)$ and $N + 2s > N - 1$.
A symmetric computation yields

$$\int_{\mathbb{R}^N\setminus B_{\frac{1}{2}}} \frac{|u_\alpha(e_N - y)|}{|y|^{N+2s}} \left|a\left(\frac{y}{|y|}\right)\right| dy < +\infty. \tag{23}$$

Using (18)-(23) we obtain that, for any $x \in \mathbb{R}^N_+$,

$$y \mapsto \frac{u_\alpha(x+y) + u_\alpha(x-y) - 2u_\alpha(x)}{|y|^{N+2s}} a\left(\frac{y}{|y|}\right) \in L^1(\mathbb{R}^N).$$

Passing to polar coordinates

(24)
$$Lu_\alpha(x) = (1-s)\left(\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty}\frac{h_\alpha(1+\theta_N r)+h_\alpha(1-\theta_N r)-2h_\alpha(1)}{r^{1+2s}}\,dr\right)a(\theta)\,d\theta\right)x_N^{\alpha-2s}.$$

Applying the change of variable $|\theta_N|r = t$, with $\theta_N \neq 0$, we then obtain

$$\int_0^{+\infty}\frac{h_\alpha(1+\theta_N r)+h_\alpha(1-\theta_N r)-2h_\alpha(1)}{r^{1+2s}}\,dr = |\theta_N|^{2s}\int_0^{+\infty}\frac{h_\alpha(1+t)+h_\alpha(1-t)-2h_\alpha(1)}{t^{1+2s}}\,dt.$$

The above equality still holds in the case $\theta_N = 0$. By (24) we have

$$Lu_\alpha(x) = C_\alpha x_N^{\alpha-2s} \tag{25}$$

with

$$C_\alpha = (1-s)\left(\int_{\mathbb{S}^{N-1}} a(\theta)|\theta_N|^{2s}\,d\theta\right)\left(\int_0^{+\infty}\frac{h_\alpha(1+t)+h_\alpha(1-t)-2h_\alpha(1)}{t^{1+2s}}\,dt\right). \tag{26}$$

Using the assumptions (3)-(4), the sign of C_α is given by latter integral in (26), which coincides, up to a positive multiplicative constant, to the one-dimensional fractional Laplacian $-(-\Delta)^s_{\mathbb{R}} h_\alpha(1)$. By [1, Lemma 2.1] we then infer that

$$C_\alpha \begin{cases} <0 & \text{for } \alpha\in(s-1,0) \\ =0 & \text{for } \alpha = s-1 \\ >0 & \text{for } \alpha\in(-1,s-1). \end{cases} \tag{27}$$

For k positive constant to be fixed, using (25), we obtain that for any $x\in\mathbb{R}^N_+$

$$L\left(ku_\alpha\right)(x) + (ku_\alpha)^p(x) = kx_N^{\alpha-2s}\left(C_\alpha + k^{p-1}x_N^{\alpha(p-1)+2s}\right). \tag{28}$$

Taking $\alpha = -\frac{2s}{p-1}$, by (28) we have

$$L\left(ku_\alpha\right)(x) + (ku_\alpha)^p(x) = kx_N^{\alpha-2s}\left(C_\alpha + k^{p-1}\right) \qquad \forall x\in\mathbb{R}^N_+. \tag{29}$$

Moreover, $C_\alpha < 0$ if, and only if, $p > \frac{1+s}{1-s}$. Hence, choosing $k = |C_\alpha|^{\frac{1}{p-1}}$ in (29), the result follows. □

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References

[1] N. Abatangelo, S. Dipierro, E. Valdinoci. A gentle invitation to the fractional world. Unitext 176, La Mat. per il 3+2, Springer Cham, pp. x+257, ©2025.

[2] S. Armstrong, B. Sirakov. *Nonexistence of positive supersolutions of elliptic equations via the maximum principle.* Comm. Partial Differential Equation **36** (2011), 2011-2047.

[3] S. Armstrong, B. Sirakov. *Sharp Liouville results for fully nonlinear equations with power-growth nonlinearities.* Ann. Sc. Norm. Super. Pisa Cl. Sci. **10** (2011), 711-728.

[4] M. Bardi, A. Cesaroni. *Liouville properties and critical value of fully nonlinear elliptic operators.* J. Differential Equations **261** (2016), 3775-3799.

[5] H. Berestycki, I. Capuzzo Dolcetta, L. Nirenberg. *Superlinear indefinite elliptic problems and nonlinear Liouville theorems.* Topol. Methods Nonlinear Anal. **4** (1994), 59-78.

[6] J. Bertoin. Lévy Processes. Cambridge Tracts in Mathematics, vol.121, Cambridge University Press, Cambridge, (1996).

[7] M. F. Bidaut-Véron, S. Pohozaev. *Nonexistence results and estimates for some nonlinear elliptic problems.* J. Anal. Math. **84** (2001), 1-49.

[8] I. Birindelli, I. Capuzzo Dolcetta, A. Cutrì. *Liouville theorems for semilinear equations on the Heisenberg group.* Ann. Inst. H. Poincaré C Anal. Non Linéaire **14** (1997), 295-308.

[9] I. Birindelli, L. Du, G. Galise. *Liouville results for semilinear integral equations with conical diffusion.* Calc. Var. Partial Differential Equations **64** (2025), no. 3, Paper No. 86, 26 pp.

[10] I. Birindelli, G. Galise, F. Leoni. *Liouville theorems for a family of very degenerate elliptic nonlinear operators.* Nonlinear Anal. **161** (2017), 198-211.

[11] I. Birindelli, E. Mitidieri. *Liouville theorems for elliptic inequalities and applications.* Proc. Roy. Soc. Edinburgh Sect. A **128** (1998), 1217-1247.

[12] K. Bogdan, T. Kulczycki, A. Nowak. *Gradient estimates for harmonic and q-harmonic functions of symmetric stable processes.* Illinois J. Math. —bf 46 (2002), 541-556.

[13] X. Cabré, Y. Sire. *Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates.*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **31** (2014), no. 1, 23-53.

[14] W. Chen, L. D'Ambrosio, Y. Li. *Some Liouville theorems for the fractional Laplacian.* Nonlinear Anal. **121** (2015), 370-381.

[15] W. Chen, Y. Fang, R. Yang. *Liouville theorems involving the fractional Laplacian on a half space.* Adv. Math. **274** (2015), 167-198.

[16] W. Chen, C. Li, B. Ou. *Classification of solutions for an integral equation.* Comm. Pure Appl. Math. **59** (2006), 330-343.

[17] A. Cutrì, F. Leoni. *On the Liouville property for fully nonlinear equations.* Ann. Inst. H. Poincaré C Anal. Non Linéaire **17** (2000), 219-245.

[18] M. M. Fall. *Entire s-harmonic functions are affine.* Proc. Amer. Math. Soc. **144** (2016), 2587-2592.

[19] M. M. Fall, T. Weth. *Liouville theorems for a general class of nonlocal operators.* Potential Anal. **45** (2016), 187-200.

[20] P. Felmer, A. Quaas. *Fundamental solutions and Liouville type theorems for nonlinear integral operators..* Adv. Math. **226** (2011), 2712-2738.

[21] X. Fernández-Real, X. Ros-Oton. Integro-differential elliptic equations. Progr. Math., 350 Birkhäuser/Springer, Cham, (2024), xvi+395 pp.

[22] J. García-Melián, A. Quaas, B. Sirakov. *Liouville theorems for nonlinear elliptic equations in half-spaces.* J. Anal. Math. **139** (2019), 559-583.

[23] B. Gidas, J. Spruck. *Apriori bounds of positive solutions of nonlinear elliptic equations.* Commun.Partial Differ. Equ. **6** (1981) 883–901.

[24] M. Kassmann, M. Rang, R. W. Schwab. *Integro-differential equations with nonlinear directional dependence.* Indiana Univ. Math. J. **63** (2014), 1467-1498.

[25] V. Kondratiev, V. Liskevich, V. Moroz. *Positive solutions to superlinear second-order divergence type elliptic equations in cone-like domains.* Ann. Inst. H. Poincaré C Anal. Non Linéaire **22** (2005), 25-43.

[26] F. Leoni. *Explicit subsolutions and a Liouville theorem for fully nonlinear uniformly elliptic inequalities in halfspaces.* J. Math. Pures Appl. **98** (2012), 574-590.

[27] X.-N. Ma, Q. Ou. *A Liouville theorem for a class semilinear elliptic equations on the Heisenberg group.* Adv. Math. **413** (2023), 20 pp.

[28] G. Nornberg, D. dos Prazeres, A. Quaas. *Fundamental solutions and critical Lane-Emden exponents for nonlinear integral operators in cones.* J. Funct. Anal. **287** (2024), 1-32.

[29] A. Quaas, B. Sirakov. *Existence and non-existence results for fully nonlinear elliptic systems.* Indiana Univ. Math. J. **58** (2009), 751-788.

[30] X. Ros-Oton, J. Serra. *Regularity theory for general stable operators.* J. Differential Equations **260** (2016), 8675-8715.

[31] X. Ros-Oton, J. Serra. *Boundary regularity for fully nonlinear integro-differential equations.* Duke Math. J. **165** (2016), 2079-2154.

[32] X. Ros-Oton, J. Serra, E. Valdinoci. *Pohozaev identities for anisotropic integrodifferential operators.* Comm. Partial Differential Equations **42** (2017), no. 8, 1290-1321.

[33] H. Yang, W. Zou. *Symmetry of components and Liouville-type theorems for semilinear elliptic systems involving the fractional Laplacian.* Nonlinear Anal. **180** (2019), 208-224.

Dipartimento di Matematica Guido Castelnuovo, Sapienza Università di Roma, Piazzale Aldo Moro 5, 00185 Rome, Italy

Email address: galise@mat.uniroma1.it