INTRINSIC REGULARITY IN HOMOGENEOUS GROUPS

REGOLARITÀ INTRINSECA IN GRUPPI OMOGENEI

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Abstract. We collect various results about the relations among the different notions of intrinsically regular submanifolds in homogeneous groups.

Sunto. Raccogliamo vari risultati riguardo le relazioni tra le diverse nozioni di sottovarietá intrinsecamente regolare in gruppi omogenei.

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1. Introduction

Homogeneous groups, namely simply connected nilpotent Lie groups whose Lie algebra is graded, have been deeply studied in recent years from the point of view of Geometric Measure Theory. The graduation of the Lie algebra naturally equips the group with a family of anisotropic dilations. Moreover, homogeneous groups are naturally endowed with a distance, homogeneous with respect to group dilations, that is not bi-Lipschitz equivalent to the Euclidean one. These metric spaces represent a natural generalization of Euclidean spaces, which are the only commutative examples of homogeneous groups [18, Chapter 1].

Establishing suitable notions of subspace and regular submanifold is a natural goal of Geometric Measure Theory when dealing with new metric structures. Contextually, it is first necessary to agree on a suitable notion of regularity. In homogeneous groups these definitions need to be introduced by taking into account the algebraic structure

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of the group. For instance, subspaces are generalized by homogeneous subgroups and the regularity of maps acting between two homogeneous groups is generalized by Pansu differentiability [38]. In this context the concept of regularity is usually very far from the Euclidean regularity, except when the group boils down to the Euclidean one.

A strong motivation supporting the need of understanding how regular submanifolds in homogeneous groups can be considered is the necessity of stating a suitable definition of rectifiability in these spaces [16, 17]. The classical notion of regular surface in an arbitrary metric space goes back to Federer, that defined a "good" surface as the image of an open subset of an Euclidean space through a (metric) Lipschitz map. Unfortunately, this classical notion does not suit the homogeneous group geometry because, roughly speaking, open subsets of $\mathbb{R}^k$ are not always appropriate to be used as parameter spaces for k-dimensional submanifolds within a homogeneous group (see [2, Theorem 7.2] and [32]).

Various notions of regular submanifolds in homogeneous groups have been proposed in the literature (see for instance [4, 14, 22, 33, 34, 27, 40]). In these note, we focus on intrinsic graphs and $(\mathbb{G}, \mathbb{M})$-regular sets (of $\mathbb{G}$), where $\mathbb{G}$ and $\mathbb{M}$ are two stratified groups. The formers have been introduced by Franchi, Serapioni and Serra Cassano in a series of paper in the last 30 years, while $(\mathbb{G}, \mathbb{M})$-regular sets have been introduced by Magnani in [34]. The latter represents also a generalization of $\mathbb{H}$-regular surface of low codimension, introduced in [22]. We stress that all these structures are usually far from being smooth in the Euclidean sense [30].

A $(\mathbb{G}, \mathbb{M})$-regular set of $\mathbb{G}$ is a suitable non-critical level set of a Pansu differentiable map acting between two stratified groups $\mathbb{G}$ and $\mathbb{M}$ (Definition 2.7). It is immediate to observe that this definition retraces the classical one of submanifold in Euclidean spaces.

On the other side, the concept of intrinsic graph was originated by Franchi, Serapioni and Serra Cassano while proving an implicit function theorem for $\mathbb{H}$-regular surfaces of codimension one i.e. $(\mathbb{G}, \mathbb{R})$-regular sets [24, Theorem 6.5] (see also [22, Theorem 3.27] for higher codimensions). Let us consider a homogeneous group $\mathbb{G}$, a couple of complementary subgroups $(\mathbb{W}, \mathbb{V})$ and a set $A \subset \mathbb{W}$. The intrinsic graph of a map $\phi : A \to \mathbb{V}$ is the set

$$\text{graph}(\phi) = \{w\phi(w) : w \in A\}.$$

The nature of intrinsic graphs is invariant with respect to translations: if $x \in \mathbb{G}$, then there are a set $A_x \subset \mathbb{W}$ and a map $\phi_x : A_x \to \mathbb{V}$ such that

$$l_x(\text{graph}(\phi)) = \text{graph}(\phi_x).$$

An analogous property holds for group dilations. Appropriate notions of Lipschitz continuity [23, 27] and intrinsic differentiability [24, 25] have been shaped for intrinsic graphs. Roughly speaking, an intrinsic graph is intrinsically differentiable at some point if it can be suitably approximated at that point by a homogeneous subgroup in terms of the anisotropic dilations of the group. Moreover, also a stronger notion of uniform intrinsic differentiability has been introduced [4, 14]. The notion of intrinsic regular graph is an actual topic of investigation in many new interesting directions, see for instance [7, 8, 12, 13, 15, 19, 29, 35, 36, 37].

In this note, we recall the notions of intrinsic Lipschitz graph, intrinsically differentiable graph, uniformly intrinsically differentiable graph, $(\mathbb{G}, \mathbb{M})$-regular set and we gather the available results in the literature about the connections among them. The purpose of the paper is twofold: to provide guidance to the reader regarding the relation among these recently introduced objects and contextually to point out some interesting related open problems.

In particular, we deal with three main points. First, in Section 3 we discuss the connection between $(\mathbb{G}, \mathbb{M})$-regular sets and uniformly intrinsically differentiable graphs. We recall results in [4, 14, 10] to show that every $(\mathbb{G}, \mathbb{M})$-regular set is locally a uniformly intrinsically differentiable graph. The opposite is not known to be true.

In Section 4 we study the relation between uniformly intrinsically differentiable graphs and continuously intrinsically differentiable graphs, through the concept of intrinsic derivative. In particular we gather results in [1, 9, 12, 31, 40]. We state that every uniformly intrinsically differentiable graph is continuously intrinsically differentiable while the opposite is verified in some context but is not always true.

Eventually, in Section 5 we discuss results in [24, 25, 5, 41, 28] about the validity of a Rademacher type theorems for intrinsic Lipschitz graphs. We state that a Rademacher type theorem is available in the Heisenberg group for intrinsic Lipschitz graphs of any codimension. On the other side, we discuss some obstacle towards a generalization of the result to an arbitrary stratified group.

2. Preliminaries

A *graded group* $\mathbb{G}$ is a simply connected, nilpotent Lie group whose Lie algebra is graded, i.e. there exists a sequence of subspaces $\mathcal{V}_j$ with $j \in \mathbb{N}$, such that $\mathcal{V}_j = \{0\}$ if $j > \kappa$, $[\mathcal{V}_i, \mathcal{V}_j] \subseteq \mathcal{V}_{i+j}$ for every $i, j \geq 1$, $\mathcal{V}_\kappa \neq \{0\}$ and $\mathrm{Lie}(\mathbb{G}) = \mathcal{V}_1 \oplus \cdots \oplus \mathcal{V}_\kappa$, where

$$[\mathcal{V}_i, \mathcal{V}_j] = \mathrm{span}\{[X, Y] : X \in \mathcal{V}_i,\ Y \in \mathcal{V}_j\}.$$

The positive integer κ is the *step* of $\mathbb{G}$. If $[\mathcal{V}_i, \mathcal{V}_j] \subseteq \mathcal{V}_{i+j}$ for every $i, j \geq 1$, then $\mathbb{G}$ is called a *stratified group.*

When we deal with a graded group the exponential map $\exp : \mathrm{Lie}(\mathbb{G}) \to \mathbb{G}$ is a global diffeomorphism and this allows us to identify in a standard way $\mathbb{G}$ with $\mathrm{Lie}(\mathbb{G})$. Thus, we may consider a graded group $\mathbb{G}$ as a graded vector space $V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa$ endowed with both a Lie group and a Lie algebra structure. Moreover, we assume that $\mathbb{G}$ is equipped with a family of dilations introduced as follows

$$\delta_t(x) = \sum_{i=1}^{\kappa} t^i x_i \quad \text{if} \quad x = \sum_{i=1}^{\kappa} x_i,\ x_i \in V_i \quad \text{and} \quad t > 0. \tag{1}$$

Such a group $\mathbb{G}$ is called a *homogeneous group.* We consider $\mathbb{G}$ equipped with a homogeneous distance, hence a distance such that for every $x, y, z \in \mathbb{G}$ and $t > 0$, the conditions $d(zx, zy) = d(x, y)$ and $d(\delta_t x, \delta_t y) = t d(x, y)$ are satisfied. We also introduce the *homogeneous norm* associated with d as $\|x\| = d(x, 0)$ for every $x \in \mathbb{G}$. A Lie subgroup $\mathbb{W}$ closed with respect to dilations is called a *homogeneous subgroup.* A *couple of complementary subgroups* $(\mathbb{W}, \mathbb{V})$ of $\mathbb{G}$ is an ordered couple of homogeneous subgroups such that

$$\mathbb{G} = \mathbb{W}\mathbb{V} \qquad \mathbb{W} \cap \mathbb{V} = \{0\}.$$

When we deal with such a couple, any point $x \in \mathbb{G}$ can be written in an unique way as $x = x_{\mathbb{W}} x_{\mathbb{V}}$ with $x_{\mathbb{W}} \in \mathbb{W}$ and $x_{\mathbb{V}} \in \mathbb{V}$ so that two group projections are well defined

$$\pi_{\mathbb{W}}^{\mathbb{W},\mathbb{V}} : \mathbb{G} \to \mathbb{W},\ \pi_{\mathbb{W}}^{\mathbb{W},\mathbb{V}}(x) = x_{\mathbb{W}} \qquad \pi_{\mathbb{V}}^{\mathbb{W},\mathbb{V}} : \mathbb{G} \to \mathbb{V},\ \pi_{\mathbb{V}}^{\mathbb{W},\mathbb{V}}(x) = x_{\mathbb{V}}.$$

From now on, we assume that a couple of complementary subgroups $(\mathbb{W}, \mathbb{V})$ is fixed.

We are in the position to recall some notions about the theory of intrinsic graphs, introduced and deeply studied by Franchi, Serapioni and Serra Cassano in a series of papers, among which for instance [25, 27, 40]. Let $A \subset \mathbb{W}$ and consider a function $\phi : A \to \mathbb{V}$, the *intrinsic graph* of ϕ is the set

$$\mathrm{graph}(\phi) := \{w\phi(w) : w \in A\}.$$

The following group parametrization map Φ is said the *graph map* of ϕ

$$\Phi : A \to \mathbb{G},\ \Phi(w) = w\phi(w).$$

The word intrinsic in this context is meant to individuate concepts whose nature is invariant with respect to group dilations and translations. For instance, if $x \in \mathbb{G}$, there exist well defined a set $A_x \subset \mathbb{W}$ and a map $\phi_x : A_x \subset \mathbb{W} \to \mathbb{V}$ such that

$$l_x(\mathrm{graph}(\phi)) = \mathrm{graph}(\phi_x).$$

Thus, translations preserve the nature of intrinsic graphs. A similar statement holds also for dilations (see [27, Proposition 2.21]).

Various intrinsic notions of regularity have been introduced. We recall below the notions of intrinsic Lipschitz continuity and intrinsic differentiability.

Definition 2.1 (Intrinsic Lipschitz continuity)**.** *We say that $\phi : A \to \mathbb{V}$, $A \subset \mathbb{W}$, is intrinsic Lipschitz if there exists some $L \geq 0$ such that for every $w, w' \in A$ we have*

$$\|\pi_{\mathbb{V}}(\Phi(w')^{-1}\Phi(w))\| \leq L\|\pi_{\mathbb{W}}(\Phi(w')^{-1}\Phi(w))\|.$$

The following characterizations holds.

Proposition 2.1. [27, Proposition 3.3] *Let $A \subset \mathbb{W}$ and $\phi : A \to \mathbb{V}$, then ϕ is intrinsic Lipschitz if and only if $\|\phi_{x^{-1}}(w)\| \leq L\|w\|$ for some $L \leq 0$ and for every $x \in \mathrm{graph}(\phi)$ and $w \in A_{x^{-1}}$.*

Intrinsic Lipschitz maps can be equivalently defined by cones constructed by the homogeneous distance, mimicking the Euclidean definition (see [27]).

In order to introduce intrinsic differentiability, we first need to introduce the analogue of linear maps in this context: a map $L : \mathbb{W} \to \mathbb{V}$ is said *intrinsically linear* if its intrinsic graph is a homogeneous subgroup.

We are now ready to introduce the notion of intrinsic differentiability.

Definition 2.2 (Intrinsic differentiability)**.** *Let $A \subset \mathbb{W}$ open, $\phi : A \subset \mathbb{W} \to \mathbb{V}$, $u \in A$ and set $x = \Phi(u)$. The map ϕ is said intrinsically differentiable at u if and only if there exists an intrinsic linear map $L : \mathbb{W} \to \mathbb{V}$ such that*

$$\|L(w)^{-1}\phi_{x^{-1}}(w)\| = o(\|w\|) \quad \text{as } \|w\| \to 0, w \in A_{x^{-1}}. \tag{2}$$

Then the map L is denoted by $d\phi_u$ and it is called the intrinsic differential at ϕ at u. The homogeneous subgroup $\mathbb{T}_x := \mathrm{graph}(d\phi_u)$ is called the tangent subgroup of $\mathrm{graph}(\phi)$ *at x.*

Roughly speaking, ϕ is intrinsically differentiable at u if there exists the blow-up of $l_{x^{-1}}(\mathrm{graph}(\phi))$ at 0 with respect to the group dilations, it is unique, and it is a homogeneous subgroup $\mathbb{T}_x$ (see also [25, Definition 3.2.6, Theorem 3.2.8]).

A stronger notion of pointwise intrinsic differentiability is also available (see [4, Definition 3.16] and [14, Definition 3.3]).

Definition 2.3 (Uniform intrinsic differentiability)**.** *Let $A \subset \mathbb{W}$ be an open set and let $\phi : A \to \mathbb{V}$ be a function. The map ϕ is* uniformly intrinsically differentiable *at a point $\bar{w} \in A$ if there exists an intrinsically linear map $L : \mathbb{W} \to \mathbb{V}$ such that*

$$\lim_{r \to 0^+} \sup_{\|\bar{w}^{-1}w'\| < r} \sup_{0 < \|w\| < r} \frac{\|L(w)^{-1}\phi_{\Phi(w')^{-1}}(w)\|}{\|w\|} = 0, \tag{3}$$

where Φ denotes the graph map of ϕ, $w' \in A$ and $w \in A_{\Phi(w')^{-1}}$. The map ϕ is said uniformly intrinsically differentiable on A if it is uniformly intrinsically differentiable at w for every $w \in A$.

We may easily introduce notions of intrinsic regular graphs as follows.

Definition 2.4 (Intrinsic Lipschitz graph)**.** *A subset* $\Sigma \subset \mathbb{G}$ *is an intrinsic Lipschitz graph if there exist a couple of complementary subgroups* $(\mathbb{W}, \mathbb{V})$*, a subset* $A \subset \mathbb{W}$ *and an intrinsic Lipschitz map* $\phi : A \subset \mathbb{W} \to \mathbb{V}$ *such that* $\Sigma = \mathrm{graph}(\phi)$*.*

Definition 2.5 (Continuously intrinsically differentiable graph)**.** *A subset* $\Sigma \subset \mathbb{G}$ *is an continuously intrinsically differentiable graph if there exist a couple of complementary subgroups* $(\mathbb{W}, \mathbb{V})$*, a subset* $A \subset \mathbb{W}$ *and a map* $\phi : A \subset \mathbb{W} \to \mathbb{V}$ *such that* $\Sigma = \mathrm{graph}(\phi)$*,* ϕ *is intrinsically differentiable at every* $u \in A$ *and the intrinsic differential* $A \ni u \to d\phi_u$ *is continuous on* A*.*

Definition 2.6 (Uniformly intrinsically differentiable graph)**.** *A subset* $\Sigma \subset \mathbb{G}$ *is an uniformly intrinsically differentiable graph if there exist a couple of complementary subgroups* $(\mathbb{W}, \mathbb{V})$*, a subset* $A \subset \mathbb{W}$ *and a uniformly intrinsically differentiable map* $\phi : A \subset \mathbb{W} \to \mathbb{V}$ *such that* $\Sigma = \mathrm{graph}(\phi)$*.*

Now, we briefly recall Pansu differentiability in order to introduce $(\mathbb{G}, \mathbb{M})$-regular sets (of $\mathbb{G}$). First, we recall that if $\mathbb{G}$ and $\mathbb{M}$ are stratified groups, then a map $L : \mathbb{G} \to \mathbb{M}$ is called *h-homomorphism* if $L(xy) = L(x)L(y)$ for every $x, y \in \mathbb{G}$ and $\delta_t(L(x)) = L(\delta_t(x)$ for every $x \in \mathbb{G}$, $t > 0$. Let $\Omega \subset \mathbb{G}$ be an open set and let $\mathbb{M}$ be an other stratified group. Consider a map $f : \Omega \to \mathbb{M}$ and a point $x \in \Omega$. We say that f is Pansu differentiable at x if there exists a h-homomorphism $L : \mathbb{G} \to \mathbb{M}$ such that

$$\|L(x^{-1}y)^{-1} f(x)^{-1} f(y)\| = \|x^{-1}y\| \quad \text{as} \quad y \to x. \tag{4}$$

The map L is called the *Pansu differential* of f at x and it is denoted by $Df(x)$. Moreover, we say that $f \in C^1_h(\Omega, \mathbb{M})$ if it is everywhere Pansu differentiable and the Pansu differential is continuous on Ω.

The notion of Pansu differentiability has been used to mimic the Euclidean notion of submanifold in stratified groups [34, Definition 10.2]. In order to recall this notion we need to introduce also the definition of h-epimorphism. A h-homomorphism $L : \mathbb{G} \to \mathbb{M}$ is called an *h-epimorphism* if it is surjective and there exists a homogeneous subgroup $\mathbb{V}$ of $\mathbb{G}$ complementary to $\ker(Df(x))$ (see also [34, Definition 2.5, Proposition 7.14]). Notice that equivalently one may assume that there exists a homogeneous subgroup $\mathbb{V} \subset \mathbb{G}$ such that $L|_{\mathbb{V}} : \mathbb{V} \to \mathbb{M}$ is an h-isomorphism (see [12, Proposition 5.4]).

Definition 2.7 (($\mathbb{G}, \mathbb{M}$)-regular set)**.** *Let $\mathbb{G}$ and $\mathbb{M}$ be two stratified groups. We say that a subset $\Sigma \subset \mathbb{G}$ is a $(\mathbb{G}, \mathbb{M})$-regular set if for every point $\bar{x} \in \Sigma$, there exist*

i) *an open neighbourhood $\Omega \subset \mathbb{G}$ of $\bar{x}$*

ii) *$f \in C^1_h(\Omega, \mathbb{M})$ such that*

$$\Sigma \cap \Omega = f^{-1}(0)$$

and for every $x \in \Omega$, $Df(x)$ is an h-epimorphism.

The notion of $(\mathbb{G}, \mathbb{M})$-regular set is a generalization of the previous notion of $\mathbb{H}$-regular surface or $\mathcal{C}^1_{\mathbb{H}}$ surface in Heisenberg groups (see [22, Definition 5.2]).

3. Uniformly intrinsically differentiable graphs VS $(\mathbb{G}, \mathbb{M})$-regular sets

The aim of this section is to trace the relation between the given notions of intrinsic regular graphs and of $(\mathbb{G}, \mathbb{M})$-regular sets.

Every $(\mathbb{G}, \mathbb{M})$-regular set is locally a uniformly intrinsically differentiable graph. This is proved by [10, Theorem 4.3.7], where the author improves the regularity of the parametrizing map provided by the implicit function theorem [34, Theorem 1.4].

Theorem 3.1. [10, Corollary 4.3.8] *Let $\mathbb{G}$ and $\mathbb{M}$ be two stratified groups and let $\Omega \subset \mathbb{G}$ be an open set. Let $f \in C^1_h(\Omega, \mathbb{M})$ be a function and set $\Sigma = f^{-1}(0)$. Assume that there exists a homogeneous subgroup $\mathbb{V}$ of $\mathbb{G}$ such that $Df(x)|_{\mathbb{V}}$ is a h-isomorphism for every $x \in \Sigma$. Then, for any homogeneous subgroup $\mathbb{W}$ complementary to $\mathbb{V}$, Σ is parametrized by a uniformly intrinsically differentiable map $\phi : A \to \mathbb{V}$ with $A \subset \mathbb{W}$ open set, i.e. $\Sigma = \mathrm{graph}(\phi)$.*

Let us give some details of the proof. First we set a point $x = \Phi(u) \in \Sigma$, $u \in A$ and we consider the splitting $(\ker Df(x), \mathbb{V})$. We apply the implicit function theorem [34, Theorem 1.4] with respect to this splitting so that to obtain the implicit parametrization $\psi : A' \to \mathbb{V}$, with $A' \subset \ker Df(x)$ open, such that $\Psi(0) = x$. Then, we directly verify the definition of uniform intrinsic differentiability of the map ψ at 0 and we show that

$\ker Df(x)$ is the tangent subgroup of ψ at 0 ([10, Theorem 4.3.3]). We conclude the proof by proving a geometrical characterization of the uniform intrinsic differentiability in terms of the existence of a uniform tangent subgroup [10, Theorem 4.3.6]. This characterization shows that the uniform intrinsic differentiability of an intrinsic graph is independent of the couple of complementary subgroups with respect to which the graph parametrization is written.

On the other side, an interesting open problem in the general setting is to answer if any uniformly intrinsically differentiable graph is a $(\mathbb{G}, \mathbb{M})$-regular set.

Some results in particular contexts are available. In the setting of the Heisenberg group $\mathbb{H}^n$ the equivalence between the two families of submanifolds was proved by Arena and Serapioni in [4, Theorem 4.5].

When $\mathbb{M}$ is the Euclidean space $\mathbb{R}^k$, the proof by Arena and Serapioni has been generalized by Di Donato in [14, Theorem 4.1], so that we know that $(\mathbb{G}, \mathbb{R}^k)$-regular sets of $\mathbb{G}$ are locally the intrinsic graph of a uniformly intrinsically differentiable map $\phi : A \to \mathbb{V}$, with $A \subset \mathbb{W}$, $(\mathbb{W}, \mathbb{V})$ a couple of complementary subgroups of $\mathbb{G}$ and $\mathbb{V}$ horizontal and k-dimensional, and vice versa.

Actually, the main relevant obstacle towards a complete answer to the question is the lack of a suitable Whitney-type extension theorem for functions acting from a closed subset of a stratified group $\mathbb{G}$ to a second stratified group $\mathbb{M}$. In fact, if we consider the intrinsic graph of a uniformly intrinsically differentiable map $\phi : A \to \mathbb{V}$ with $A \subset \mathbb{W}$ and $(\mathbb{W}, \mathbb{V})$ a couple of complementary subgroups of $\mathbb{G}$, in order to retrace the argument used to prove [14, Theorem 4.1], we would need a suitable Whitney-type theorem to prove that the intrinsic graph of ϕ is contained in the zero-level set of a map $f \in C^1_h(\Omega, \mathbb{M})$, with $\Omega \subset \mathbb{G}$ an open set and $\mathbb{M}$ a suitable stratified group, such that $Df(y)$ is a h-epimorphism for every $y \in \Omega \cap \operatorname{graph}(\phi)$.

Thus, up to our knowledge there could exist $(\mathbb{G}, \mathbb{M})$-regular sets that are not uniformly intrinsically differentiable graphs.

4. Uniformly intrinsically differentiable graphs VS Continuously intrinsically differentiable graphs

Aim of this section is to resume results about the relations between uniformly intrinsically differentiable graphs and continuously intrinsically differentiable graphs.

An immediate comparison between Definitions 2.2 and 2.3 shows that a uniformly intrinsically differentiable map is everywhere intrinsically differentiable. Moreover, it is true that a uniformly intrinsically differentiable graph is continuously intrinsically differentiable.

Proposition 4.1. [12, Proposition 3.12] *Let* $(\mathbb{W}, \mathbb{V})$ *be a couple of complementary subgroups of* $\mathbb{G}$*. Let* $A \subset \mathbb{W}$ *be an open set and let* $\phi : A \to \mathbb{V}$ *be uniformly intrinsically differentiable on* A*. Then the intrinsic differential* $w \ni A \to d\phi_w$ *is continuous on* A*.*

The converse instead is very delicate. The more general available result is [1, Theorem 1.6], which provides a characterization of uniformly intrinsically differentiable maps with horizontal target subgroups.

Let us introduce some notation due to discuss this result. From now on we assume that $(\mathbb{W}, \mathbb{V})$ is a couple of complementary subgroups of a stratified group $\mathbb{G}$. Moreover, we assume that $\mathbb{V}$ is a *horizontal subgroup*, i.e. $\mathbb{V}$ is a homogeneous subgroup contained in V_1. We denote by n_1 the dimension of V_1.

Let us fix coordinates: we denote by p the topological dimension of $\mathbb{V}$ and we consider $(v_1, \dots, v_p, w_{p+1}, \dots, w_q)$ an orthonormal graded basis of $\mathbb{G}$ such that $(v_1, \dots, v_p)$ is a basis of $\mathbb{V}$ and $(w_{p+1}, \dots, w_q)$ is a basis of $\mathbb{W}$. We consider $A \subset \mathbb{W}$ open set and a map $\phi : A \to \mathbb{V}$ and we denote by $\widetilde{\phi} : \widetilde{A} \subset \mathbb{R}^{q-p} \to \mathbb{R}^p$ the map read in coordinates with respect to the fixed bases. For every $j = p+1, \dots q$ we denote by $X_j \in \mathrm{Lie}(\mathbb{G})$ the left-invariant vector field such that $X_j(0) = w_j$.

Under our hypotheses, if $\phi : A \to \mathbb{V}$, with $A \subset \mathbb{W}$ open set, is intrinsically differentiable at $w \in A$, then its intrinsic differential $d\phi_w$ is a linear map depending only on the coordinates in the first layer [14, Proposition 3.4]. Hence, we denote by $\nabla^{\phi}\widetilde{\phi}(w) \in \mathbb{R}^{p\times(n_1-p)}$ the *intrinsic Jacobian matrix* which is the matrix representing the intrinsic differential $d\phi_w$ with respect to the bases $(w_{p+1}, \dots, w_{n_1})$ and $(v_1, \dots, v_p)$.

For $j = p+1, \dots, q$ we define the continuous *projected vector field* $D^{\phi}_{X_j}$ on $\mathbb{W}$ as

$$\big(D^{\phi}_{X_j}\big)_w = d(\pi^{\mathbb{W},\mathbb{V}}_{\mathbb{W}})_{\Phi(w)}\Big(\big(X_j\big)_{\Phi(w)}\Big),$$

(see [1, Definition 1.3] and [31, Definition 4.2.12]). Through these projected vector fields we introduce a notion of intrinsic partial derivatives.

Definition 4.1 (Intrinsic partial derivatives)**.** *Let $\widetilde{w} \in \widetilde{A}$. Given $j \in \{p+1, \dots, n_1\}$, we say that $\widetilde{\phi}$ has $D^{\phi}_{X_j}$-derivative at $\widetilde{w}$ if and only if there exists a vector $\begin{pmatrix}\alpha_{1,j} & \dots & \alpha_{p,j}\end{pmatrix} \in \mathbb{R}^p$ such that for any integral curve $\widetilde{\gamma} : (-\delta, \delta) \to \widetilde{A}$ of $(i_{\mathbb{W}})_*(D^{\phi}_{X_j})$ with $\widetilde{\gamma}(0) = \widetilde{w}$ the equality*

$$\lim_{s\to 0} \frac{\widetilde{\phi}(\widetilde{\gamma}(s)) - \widetilde{\phi}(\widetilde{w})}{s} = \begin{pmatrix}\alpha_{1,j} & \dots & \alpha_{p,j}\end{pmatrix}^T$$

holds. We introduce the notation

$$D^{\phi}_{X_j}\widetilde{\phi}(\widetilde{w}) = \begin{pmatrix} D^{\phi}_{X_j}\widetilde{\phi}_1(\widetilde{w}) \\ \dots \\ D^{\phi}_{X_j}\widetilde{\phi}_p(\widetilde{w}) \end{pmatrix} = \begin{pmatrix} \alpha_{1,j} \\ \dots \\ \alpha_{p,j} \end{pmatrix}$$

for $j = p+1, \dots, n_1$.

In [1, Definition 1.5] the authors also consider the notion of *vertically broad* hölder regularity* for the map ϕ (refer also to [31, Theorem 4.3.1]). We report it below.

The map ϕ is said *vertically broad* Hölder* if for every point $\widetilde{w}_0 \in \widetilde{A}$ there exist a neighbourhood $U_{\widetilde{w}_0}$ of $\widetilde{w}_0$ and a positive $\delta > 0$ such that for every $\widetilde{w} \in U_{\widetilde{w}_0}$ and for every projected vector field $D^{\phi}_{X_j}$ with $X_j \in \mathrm{Lie}(\mathbb{W}) \cap V_d$ for every $d > 1$, there exists an integral curve $\gamma : [-\delta, \delta] \to \widetilde{A}$ of $D^{\phi}_{X_j}$ starting at $\widetilde{w}$, such that

$$\lim_{r\to 0} \left\{ \frac{|\phi(\gamma(t)) - \phi(\gamma(s))|}{|t-s|^{\frac{1}{d}}} : t, s \in [-\delta, \delta],\ 0 < |t-s| \le r \right\} = 0, \tag{5}$$

In the light of our notation, we can state [1, Theorem 1.6, b),d)] as follows.

Theorem 4.1. [1, Theorem 1.6, (b),(d)] *Let $A \subset \mathbb{W}$ be an open set and let $\phi : A \to \mathbb{V}$. Then the following conditions are equivalent.*

a) *ϕ is uniformly intrinsically differentiable on A.*

b) *The following three conditions are realized* b1) *ϕ is intrinsically differentiable everywhere on A;*

b2) *for $j = p+1, \dots n_1 - p$ there exists the intrinsic partial derivative $D^{\phi}_{X_j}\widetilde{\phi}(\widetilde{w})$ at every $\widetilde{w} \in \widetilde{A}$ and the map $\widetilde{A} \ni \widetilde{w} \to D^{\phi}_{X_j}\widetilde{\phi}(\widetilde{w}) \in \mathbb{R}$ is continuous;*

b3) *$\widetilde{\phi}$ is vertically broad* Hölder.*

By taking into account the following proposition (see [1, Proposition 3.27] and [1, Proposition 3.19]) and Proposition 4.1, one realizes that if condition b3) would be removed from Theorem 4.1, then every continuously intrinsically differentiable graph would be uniformly intrinsically differentiable.

Proposition 4.2. *Assume that ϕ is intrinsically differentiable at every $w \in A$ and assume that the intrinsic differential is continuous. Then, for every $i \in \{1, \dots, p\}$, $j \in \{p+1, \dots, n_1\}$ and for every $\widetilde{w} \in \widetilde{A}$ there exists the intrinsic partial derivative $D^{\phi}_{X_j}\widetilde{\phi}_i(\widetilde{w})$ and*

$$D^{\phi}_{X_j}\widetilde{\phi}_i(\widetilde{w}) = [\nabla^{\phi}\widetilde{\phi}(\widetilde{w})]_{i,j}. \tag{6}$$

Moreover, the map $D^{\phi}_{X_j}\widetilde{\phi}_i : \widetilde{A} \to \mathbb{R}$ is continuous.

In some settings the vertically broad* Hölder condition can be removed. We refer the reader to [40, Theorem 4.95] for the case of $\mathbb{H}^1$ with $\mathbb{V}$ of dimension 1, to [9, Theorem 1.4] for the case of $\mathbb{H}^n$, $n > 1$ with $1 \leq \dim(\mathbb{V}) \leq n$ and to [1, Theorem 6.1.7] for the case of groups $\mathbb{G}$ of step 2 with $\mathbb{V}$ of dimension 1. The core of the three proofs lies in proving that, in the three settings, condition b2) implies condition b3) of Theorem 4.1. Thus, in these three settings continuously intrinsically differentiable graphs are uniformly intrinsically differentiable.

On the other side, in general in the statement of Theorem 4.1 the vertically broad* Hölder condition cannot be removed. In particular, in [31, Example 4.5.1] Kozhenvnikov provides a counterexample in the easiest step-3 group, namely the Engel group (see also [1, Remark 4.18]). The author shows that there exists a one codimensional continuously intrinsically differentiable graph that is not uniformly intrinsically differentiable.

Up to our knowledge, in the literature there is not any related result for the case when $\mathbb{V}$ is not horizontal.

5. Intrinsic Lipschitz graphs VS intrinsic differentiability: Rademacher-type results

In this section we deal with a natural question arising from the notions of intrinsic Lipschitz continuity and intrinsic differentiability: establishing the validity of a Rademacher type theorem.

The first result in this direction was [24, Theorem 4.29]. The authors focused on the Heisenberg group $\mathbb{H}^n$ and on a couple of complementary subgroups $(\mathbb{W}, \mathbb{V})$ with $\mathbb{V}$ horizontal and one dimensional. We recall below some details about their result. It deeply relies on the theory of sets of locally finite H-perimeter. For the definitions about sets of locally finite H-perimeter and H-boundary we refer the reader to [40, Sections 3.5 and 5.1]. Consider an open subset $A \subset \mathbb{W}$, a function $\phi : A \to \mathbb{V}$ and a vector $v \in V_1$ such that $\mathbb{V} = \operatorname{span}(v)$. The map ϕ can be identified with the unique real-valued mapping $\varphi : U \to \mathbb{R}$ that satisfies

$$\phi(w) = \varphi(w)v$$

for every $w \in U$. In this context the subgraph of an intrinsic Lipschitz map with one dimensional target space is a set of locally finite H-perimeter.

Theorem 5.1. [25, Theorem 4.2.9] *Let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups, with $\mathbb{V}$ of dimension one. If $\phi : \mathbb{W} \to \mathbb{V}$ is an intrinsic Lipschitz map, its subgraph*

$$E_\phi^- = \{w(tv) : w \in U \ , \ t < \varphi(w)\}$$

is a set of locally finite H-perimeter.

Theorem 5.1 was the key starting point to prove the Rademacher-type Theorem 5.2 for intrinsic Lipschitz maps. It has been proved in [24, Theorem 4.29] in the setting of the Heisenberg group and then it has been generalized to groups of type $\star$ in [25].

Definition 5.1. *A stratified group $\mathbb{G} = V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa$ is of type $\star$ if there exists a basis $(v_1, \dots, v_{m_1})$ of V_1 such that*

$$[v_j, [v_j, v_i]] = 0 \quad \text{for every } i, j = 1, \dots, m_1.$$

All step-2 stratified groups are of type $\star$. One can prove that there exist stratified groups of type $\star$ of any step.

Theorem 5.2. [25, Theorem 4.3.5] *Let $\mathbb{G}$ be a stratified group of type $\star$ of topological dimension q and let $\mathbb{G} = \mathbb{W}\mathbb{V}$ be the product of two complementary subgroups, with $\mathbb{V}$ of dimension one. Let $A \subset \mathbb{W}$ be an open set and let $\phi : A \to \mathbb{V}$ be an intrinsic Lipschitz function, then ϕ is intrinsically differentiable $(\mathcal{L}^{q-1} \llcorner \mathbb{W})$-almost everywhere on A.*

We briefly describe the proof of this theorem. One of the main tools used to prove Theorem 5.2 is a blow-up result valid at almost every point of the intrinsic graph of the considered intrinsic Lipschitz map ϕ. In particular, the authors prove that the blow-up of the subgraph of ϕ, E^-_ϕ, at each point of its H-reduced boundary exists, is unique and is a vertical half-space i.e. an half-space whose boundary is a vertical subgroup. (A homogeneous subgroup $\mathbb{W}$ is called *vertical* if for some $1 \leq \ell \leq \kappa$, $\mathbb{W} = N_\ell \oplus V_{\ell+1} \oplus \cdots \oplus V_\kappa$, where N_ℓ is a linear subspace of V_ℓ.) This permits to deduce the intrinsic differentiability of ϕ at all the points of the form $\pi_{\mathbb{W}}(x) \in \mathbb{W}$ for some point $x \in \partial^\star_H E^-_\phi$. Finally, the proof is substantially completed by observing that almost every point of the graph of ϕ belongs to the H-reduced boundary of the subgraph and observing that the group projection $\pi_{\mathbb{W}}$ on $\mathbb{W}$ preserves full-measure sets.
Theorem 5.2 cannot be easily extended to a generic stratified group. For instance, a well-known counterexample [21, Example 3.2] describes a locally finite H-perimeter set E in the Engel group (that is the simplest step-3 stratified group) such that $0 \in \partial^\star_H E$ but the blow-up at 0 fails, in the sense that the blow-up of E at 0 is not a vertical half-space.

Recently in [6, Theorem 5.2], the authors proposed an alternative proof of the intrinsic Rademacher's theorem in $\mathbb{H}^1$, still for one codimensional intrinsic Lipschitz graphs. The proof is based on the characterization of intrinsic Lipschitz graphs in the Heisenberg group in terms of a system of non linear first order PDEs, proved in [5]. Thus, the authors provide a completely PDEs based proof of the Rademacher's theorem in $\mathbb{H}^1$. Since in this case the intrinsic differentiability is not proved at the points of the H-reduced boundary, [6, Theorem 5.2] opens to the possibility of studying the problem for the one codimensional case in the Engel group (and in more general stratified groups), avoiding the counterexample [21, Example 3.2].

Vittone proved a generalization in the Heisenberg group $\mathbb{H}^n$ of Theorem 5.2 to intrinsic Lipschitz maps with target space a horizontal subgroup $\mathbb{V}$ with $1 \leq \dim(\mathbb{V}) \leq n$; see [41, Theorem 1.1]. His proof relies on three main ingredients: an approximation result for intrinsic Lipschitz graphs by (Euclidean) smooth uniformly intrinsic Lipschitz maps whose domain is an entire homogeneous subgroup [41, Theorem 1.6], the first original result available for one-codimensional intrinsic Lipschitz graphs, [24, Theorem 4.29] and some delicate tools coming from the theory of currents in the Heisenberg group. Currents are defined through the complex of differential forms introduced by Rumin in [39]. The involvement of the language of Heisenberg currents makes the proof quite technical. For good introductions to this theme one can also refer to [22, Section 5] and [26].

For the sake of completeness, we recall that a Rademacher-type theorem was proved also by Antonelli and Merlo in [3]. It holds for intrinsic Lipschitz functions $\phi : A \subset \mathbb{W} \to \mathbb{V}$, where $\mathbb{W}$ and $\mathbb{V}$ are complementary subgroups of a stratified group $\mathbb{G}$, such that $\mathbb{V}$ is a normal homogeneous subgroup. Its proof substantially relies on the Pansu-Rademacher's theorem, that in this context is applied to the graph map $\Phi : A \to \mathbb{G}$. For example, [3, Theorem 1.1] permits to deduce that if $(\mathbb{W}, \mathbb{V})$ is a couple of complementary subgroups of $\mathbb{H}^n$ with $\mathbb{W}$ horizontal of dimension $1 \leq \dim(\mathbb{W}) \leq n$, then any intrinsic Lipschitz map $\phi : A \to \mathbb{V}$, where $A \subset \mathbb{W}$ is an open set, is $\mathcal{L}^{\dim(\mathbb{W})}$-a.e. intrinsically differentiable.

Eventually, Julia, Nicolussi Golo and Vittone in [28] showed explicitly that it is possible to build nowhere intrinsically differentiable intrinsic Lipschitz maps in suitable stratified groups, among which suitable stratified groups of step 2. In particular, it is possible to build a 2-codimensional intrinsic Lipschitz graph in $\mathbb{H}^1 \times \mathbb{R}$ such that, at any of its points, there exist infinite blow-ups, and none of them is a homogeneous subgroup. Thus, actually the Rademacher's theorem for intrinsic Lipschitz functions does not hold on an arbitrary stratified group, even of step 2.

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