GRADIENT REGULARITY FOR STRONGLY SINGULAR OR DEGENERATE ELLIPTIC AND PARABOLIC EQUATIONS

REGOLARITÀ DEL GRADIENTE PER EQUAZIONI ELLITTICHE E PARABOLICHE FORTEMENTE SINGOLARI O DEGENERI

PASQUALE AMBROSIO

Abstract. We present recent advances in the regularity theory for weak solutions to some classes of elliptic and parabolic equations with strongly singular or degenerate structure. The equations under consideration satisfy standard p-growth and p-ellipticity conditions only outside a ball centered at the origin. In the elliptic setting, we describe Besov and Sobolev regularity results for suitable nonlinear functions of the gradient of the weak solutions, covering both the subquadratic ($1 < p < 2$) and superquadratic ($p \geq 2$) regimes. Analogous results are obtained in the corresponding parabolic framework, where we address the higher spatial and temporal differentiability of the solutions under appropriate assumptions on the data.

Sunto. Presentiamo alcuni recenti sviluppi nella teoria della regolarità per soluzioni deboli di alcune classi di equazioni ellittiche e paraboliche fortemente singolari o degeneri. Le equazioni considerate soddisfano condizioni standard di crescita e di ellitticità di ordine p, tipiche degli operatori di tipo p-Laplaciano, ma soltanto all'esterno di una sfera centrata nell'origine. Nel setting ellittico, descriviamo risultati di regolarità di tipo Besov e Sobolev per opportune funzioni non lineari del gradiente delle soluzioni deboli, coprendo sia il caso subquadratico ($1 < p < 2$) sia quello superquadratico ($p \geq 2$). Risultati analoghi sono ottenuti nel corrispondente contesto parabolico, dove si analizza la maggiore regolarità delle soluzioni rispetto alla variabile spaziale e a quella temporale, sotto opportune ipotesi sui termini noti.

2020 MSC. 35B45, 35B65, 35J70, 35J75, 35K65.

Keywords. Singular elliptic equations, degenerate elliptic equations, degenerate parabolic equations, Besov spaces, Sobolev regularity.

Bruno Pini Mathematical Analysis Seminar, Vol. 16, No. 1 (2025) pp. 68–101

Dipartimento di Matematica, Università di Bologna

ISSN 2240-2829.

1. Introduction

In this note, we present some recent results proved in [1, 3, 5, 6], concerning the gradient regularity of weak solutions to certain classes of strongly singular or degenerate elliptic and parabolic equations.

As for the elliptic equation in question, this arises as the Euler-Lagrange equation of an integral functional of the Calculus of Variations. The energy density of this functional satisfies standard p-growth and p-ellipticity conditions with respect to the gradient variable, but only outside a ball with radius $\lambda > 0$ centered at the origin. More precisely, we consider the following pair of variational problems in duality[1]:

$$\text{(P1)} \qquad \inf_{\sigma \in L^{p'}(\Omega,\mathbb{R}^n)} \left\{ \int_\Omega \mathcal{H}(\sigma(x))\, dx : -\operatorname{div}\sigma = f,\ \langle \sigma, \nu\rangle = 0 \text{ on } \partial\Omega \right\}$$

and

$$\text{(P2)} \qquad \sup_{u \in W^{1,p}(\Omega)} \left\{ \int_\Omega u(x) f(x)\, dx - \int_\Omega \mathcal{H}^*(Du(x))\, dx \right\},$$

where Ω is a bounded connected open subset of $\mathbb{R}^n$ ($n \geq 2$) with Lipschitz boundary, $p \in (1,\infty)$, $p' := p/(p-1)$, f is a given function defined over $\overline{\Omega}$ with zero mean (i.e. $\int_\Omega f\, dx = 0$), ν denotes the outer normal versor of $\partial\Omega$ and the function $\mathcal{H}$ is defined by

$$\mathcal{H}(\sigma) := \frac{1}{p'}\, |\sigma|^{p'} + \lambda\, |\sigma|, \qquad \sigma \in \mathbb{R}^n.$$

With such a choice, we get

$$D\mathcal{H}^*(z) = (|z| - \lambda)_+^{p-1}\, \frac{z}{|z|}, \qquad z \in \mathbb{R}^n,$$

where $\mathcal{H}^*$ is the Legendre transform of $\mathcal{H}$ and $(\,\cdot\,)_+$ stands for the positive part (see [12]). It is well known that the Euler-Lagrange equation of the functional in (P2) is given by

[1]The equation $-\operatorname{div}\sigma = f$ in problem (P1) has to be understood as

$$\int_\Omega \langle \sigma, D\varphi\rangle\, dx = \int_\Omega f\varphi\, dx, \quad \text{for every } \varphi \in C^1(\overline{\Omega}),$$

so that it incorporates in the weak sense the homogeneous Neumann boundary condition $\langle \sigma, \nu\rangle = 0$ on $\partial\Omega$.

the PDE

$$\begin{cases} -\operatorname{div}\left(D\mathcal{H}^*(Du)\right) = f & \text{in } \Omega, \\ \langle D\mathcal{H}^*(Du), \nu\rangle = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

which has to be meant in the distributional sense. If we assume that $f \in L^{p'}(\Omega)$ and that the infimum in (P1) is finite, then problem (P1) consists in minimizing a strictly convex and coercive functional on $L^{p'}$ subject to a convex and closed constraint: therefore, a solution σ_0 to (P1) exists and must be unique. Moreover, we recall that by standard convex duality (see for instance [20]), the values inf (P1) and sup (P2) coincide and the primal-dual optimality condition characterizes the minimizer σ_0 of (P1) through the equality

$$\sigma_0(x) = D\mathcal{H}^*(Du_0(x)) \quad \text{for a.e. } x \in \Omega, \tag{1.2}$$

where $u_0 \in W^{1,p}(\Omega)$ is a solution of (P2). This is equivalent to the requirement that u_0 is a weak solution of the Euler-Lagrange equation (1.1), in the sense that

$$\int_\Omega \langle D\mathcal{H}^*(Du_0(x)), D\varphi(x)\rangle\, dx = \int_\Omega f(x)\varphi(x)\, dx, \quad \text{for every } \varphi \in W^{1,p}(\Omega).$$

Furthermore, since f has zero mean, using the direct methods of the Calculus of Variations it is not difficult to show that the dual problem (P2) admits at least one solution u_0 belonging to

$$W^{1,p}_\diamond(\Omega) := \left\{ u \in W^{1,p}(\Omega) : \int_\Omega u(x)\, dx = 0 \right\}$$

and such that (1.2) holds, so that u_0 is a distributional solution of the strongly singular or degenerate elliptic equation

$$-\operatorname{div}\left((|Du| - \lambda)_+^{p-1} \frac{Du}{|Du|} \right) = f \quad \text{in } \Omega, \tag{1.3}$$

under homogeneous Neumann boundary conditions. We also note that, in general, if one looks at the solutions u of the above equation, no more than $C^{0,1}$ regularity should be expected for them: indeed, every λ-Lipschitz function u is a solution of equation (1.3) with $f = 0$. Moreover, when $p \geq 2$ we have

$$\frac{(|Du| - \lambda)_+^{p-1}}{|Du|} |\xi|^2 \leq \langle D^2\mathcal{H}^*(Du)\,\xi, \xi\rangle \leq (p-1)(|Du| - \lambda)_+^{p-2} |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^n,$$

that is, the Hessian matrix $D^2\mathcal{H}^*(Du)$ degenerates in the region $\{|Du| \leq \lambda\}$.

One of the topics related to the regularity of solutions to equations as the one in (1.3) concerns the study of their higher differentiability of both integer and fractional order, and several results are available in this direction; see, for example, [1, 5, 10, 11, 12, 16, 17, 35]. A common aspect of nonlinear elliptic problems with growth rate $p \geq 2$ is that the higher differentiability is proved for a nonlinear function of the gradient of the weak solutions that takes into account the growth of the structure function of the equation. Indeed, already for the p-Poisson equation (which is obtained from (1.3) by setting $\lambda = 0$), the higher differentiability is established for the function

$$\mathbb{V}_p(Du) := |Du|^{\frac{p-2}{2}} Du\,,$$

as can be seen in many papers, starting from the pioneering one by Uhlenbeck [42]. In the case of equation (1.3), this phenomenon persists and higher differentiability results hold true for the function

$$H_{\frac{p}{2}}(Du) := (|Du| - \lambda)_+^{\frac{p}{2}} \,\frac{Du}{|Du|}\,. \tag{1.4}$$

In particular, Brasco, Carlier and Santambrogio [12] proved that $H_{\frac{p}{2}}(Du) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^n)$ if $p \geq 2$ and $f \in W^{1,p'}(\Omega)$.

In Section 4, we shall address the subquadratic (or *singular*) case $1 < p < 2$, which had been neglected in the literature, since extra technical difficulties arise concerning elliptic regularity whenever we are in this case. This aspect, which also occurs in the classic p-harmonic setting, has been very well explained in [8, Section 2.6]. More precisely, in the singular case we present four main results established in the paper [1].

In Section 5, we report three main results proved in the paper [5]. To be more specific, there we deal with the local $W^{1,2}$-regularity of a *novel* nonlinear function of the gradient Du of local weak solutions to (1.3), under the following assumptions:

- $f \in B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)$ if $2 < p < \infty$ (see Theorem 5.1);
- $f \in L^{\frac{np}{n(p-1)+2-p}}_{loc}(\Omega)$ if $1 < p \leq 2$ (see Theorem 5.2).

Let us recall again that, for $\lambda = 0$, equation (1.3) turns into the p-Poisson equation. For the weak solutions $u \in W^{1,p}(\Omega)$ of this last equation, Irving and Koch [25] proved that

$$\mathbb{V}_p(Du) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^n) \tag{1.5}$$

if the datum f belongs to the Besov space $B^{\frac{p-2}{p}}_{p',1}(\Omega)$, with $p > 2$ [2]. Their assumption on f is essentially sharp, in the sense that the above result is false if

$$f \in B^s_{p',1}(\Omega) \quad \text{ with } \quad s < \frac{p-2}{p}\,.$$

Indeed, Brasco and Santambrogio [13, Section 5] showed with an explicit example that condition (1.5) may not hold if f belongs to a fractional Sobolev space $W^{\sigma,p'}_{loc}(\mathbb{R}^n)$ with $0 < \sigma < (p-2)/p$, which is continuously embedded into $B^s_{p',1,loc}(\mathbb{R}^n)$ whenever $s \in (0, \sigma)$ (see Lemma 3.22 below).

The main results reported in Section 5 are in the spirit of the ones mentioned above. Indeed, our primary goal in [5] was to find the assumptions to impose on the datum f *in the scale of local Besov or Lebesgue spaces* to obtain the $W^{1,2}$-regularity of a nonlinear function of the gradient of weak solutions to the widely degenerate or singular equation (1.3). We emphasize that, for $\lambda = 0$, our results give back those contained in [15] and [25, Remark 1.4].

A key tool in the proof of the above-mentioned Theorem 5.1 is the duality of Besov spaces (see Section 3.1). In fact, our approach has been inspired by [13], where Brasco and

[2] Owing to the identity $|\mathbb{V}_p(Du)| = |Du|^{\frac{p}{2}}$, the local $W^{1,2}$-regularity of $\mathbb{V}_p(Du)$ entails an improved local summability of Du, which, in turn, provides the foundation for the local Hölder continuity of the weak solution u itself. A pretty simple proof of the Hölder continuity of u is available when $n \geq 3$ and $p > \max\{2, n-2\}$. In this case, the reasoning is as follows. Since $D\mathbb{V}_p(Du) \in L^2_{loc}(\Omega)$, the Sobolev Embedding Theorem ensures that $\mathbb{V}_p(Du) \in L^{2n/(n-2)}_{loc}(\Omega)$, that is $Du \in L^{np/(n-2)}_{loc}(\Omega)$. Now observe that

$$\frac{np}{n-2} > n \quad \text{when} \quad p > \max\{2, n-2\}.$$

Then, using the Sobolev Embedding Theorem again, we conclude that $u \in C^{0,\alpha}_{loc}(\Omega)$, with $\alpha = 1-(n-2)/p$, since it belongs to $W^{1,\frac{np}{n-2}}_{loc}(\Omega)$ by the Poincaré-Wirtinger inequality.

As for the existence of the weak second derivatives of u in $L^2_{loc}(\Omega)$, we refer the reader to [33, Chapter 4, pages 29 and 30].

Santambrogio used a duality-based inequality in the setting of fractional Sobolev spaces, but limiting themselves to the p-Poisson equation.

In the case $1 < p \leq 2$, it is well known that, already for the p-Poisson equation, the higher differentiability of the weak solutions can be achieved without assuming any differentiability on f (neither of integer nor of fractional order), but only a suitable degree of integrability. The sharp assumption on f in the scale of Lebesgue spaces has been recently found in [15]. The aforementioned Theorem 5.2 simply tells us that a result analogous to [15, Theorem 1.1] holds when dealing with solutions of widely singular equations.

In Section 6, we move on to the parabolic version of equation (1.3), that is, we consider the strongly degenerate parabolic PDE

$$u_t - \operatorname{div}\left((|Du| - \lambda)_+^{p-1} \frac{Du}{|Du|}\right) = \widetilde{f} \quad \text{in } \Omega_T = \Omega \times (0, T)\,, \tag{1.6}$$

where $p \geq 2$, Ω is a bounded domain in $\mathbb{R}^n$ $(n \geq 2)$, $T > 0$ and λ is a positive constant. The main feature of the above equation is that the structure function

$$H_{p-1}(\xi) := \begin{cases} (|\xi| - \lambda)_+^{p-1} \frac{\xi}{|\xi|} & \text{if } \xi \in \mathbb{R}^n \setminus \{0\}, \\ 0 & \text{if } \xi = 0, \end{cases}$$

satisfies standard growth and ellipticity conditions for a growth rate $p \geq 2$, but only outside the ball $\{|\xi| < \lambda\}$.

In [6] we succeeded in proving higher differentiability results for the weak solutions of equation (1.6). These results will be exactly the subject of Section 6. For further results available in the literature, we refer the reader to [2, 4, 9, 21, 37] and the references therein.

A motivation for studying equations of the type (1.6) can be found in *gas filtration problems taking into account the initial pressure gradient* (see [6, Section 1.1] for a brief explanation).

As already pointed out in [3, 9], no more than Lipschitz regularity can be expected for solutions to (1.6). In fact, when $\widetilde{f} = 0$, any time-independent λ-Lipschitz function solves (1.6), and even more, it is a solution of the associated stationary equation (1.3) with $f = 0$.

The first result presented in Section 6 establishes the Sobolev spatial regularity of the vector field defined in (1.4), where Du now denotes the *spatial gradient* of the weak solutions to (1.6). This result, in turn, implies the Sobolev time regularity of the weak solutions, under the assumption that the datum $\widetilde{f}$ belongs to a suitable Lebesgue-Sobolev parabolic space. Such results have been obtained by adapting the techniques for the evolutionary p-Laplacian to our more degenerate context. In fact, for less degenerate parabolic problems, these issues have been widely investigated, as one can see, for example, in [19, 22] (where $\widetilde{f} = 0$) and in [36]. Moreover, establishing the Sobolev regularity of the solutions with respect to time, once the higher differentiability in space has been obtained, is a quite usual fact in these problems: see, for instance, [30, 31, 32].

The distinguishing feature of equation (1.6) is that its principal part behaves like a *p-Laplace operator only at infinity.* Let us briefly summarize a few previous results on this topic: the regularity of solutions to parabolic problems with asymptotic structure of p-Laplacian type has been studied in [26], where a BMO regularity has been proved for solutions to asymptotically parabolic systems in the case $p = 2$ and $\widetilde{f} = 0$ (see also [28], where the local Lipschitz continuity of weak solutions with respect to the spatial variable is established). In addition, we mention the work [14], where the authors consider nonhomogeneous parabolic problems involving a discontinuous nonlinearity and an asymptotic regularity in divergence form of p-Laplacian type. There, Byun, Oh and Wang establish a global Calderón-Zygmund estimate by converting a given asymptotically regular problem to a suitable regular problem.

However, it is worth noting that, in Section 6, our assumption on $\widetilde{f}$ is weaker than those considered in the works mentioned above.

Finally, in Section 7 we present three results proved in the paper [3]. More precisely, there we deal with the spatial $W^{1,2}$-regularity of a *novel* nonlinear function of the spatial gradient of local weak solutions to equation (1.6), under the following assumptions:

- $\widetilde{f} \in L^{p'}_{loc}\left(0, T; B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)\right)$ if $2 < p < \infty$ (see Theorem 7.1);
- $\widetilde{f} \in L^{2}_{loc}(\Omega_T)$ if $p = 2$ (see Theorem 7.2).

Actually, the theorems in Section 7 can somewhat be viewed as the parabolic counterpart of the elliptic results presented in Section 5, in the case $p \geq 2$.

Before stating the main assumptions and results, in Sections 2 and 3 we gather the preliminary material, including classical notations, essential definitions, basic properties of the difference quotients of Sobolev functions, and relevant facts on the function spaces involved in this note.

2. Notation and essential definitions

We start with a list of classical notations. We shall denote by C or c a general positive constant. Relevant dependencies on parameters and special constants will be suitably emphasized using parentheses or subscripts. The norm used on $\mathbb{R}^k$, $k \in \mathbb{N}$, will be the standard Euclidean one and it will be denoted by $|\cdot|$. In particular, for vectors $\xi, \eta \in \mathbb{R}^k$, we write $\langle \xi, \eta \rangle$ for the usual inner product and $|\xi| := \langle \xi, \xi \rangle^{\frac{1}{2}}$ for the corresponding Euclidean norm.

For points in space-time, we use the abbreviations $z = (x, t)$ and $z_0 = (x_0, t_0)$, for spatial variables $x, x_0 \in \mathbb{R}^n$ and times $t, t_0 \in \mathbb{R}$. We also denote by

$$B_\rho(x_0) \, = \, B(x_0, \rho) \, = \, \{x \in \mathbb{R}^n : |x - x_0| < \rho\}$$

the n-dimensional open ball with radius $\rho > 0$ and center $x_0 \in \mathbb{R}^n$. When the center is not important or is clear from the context, we shall simply write $B_\rho \equiv B_\rho(x_0)$. Different balls in the same context will be assumed to have the same center. Moreover, we use the notation

$$Q_\rho(z_0) := \, B_\rho(x_0) \times (t_0 - \rho^2, t_0), \qquad z_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}, \ \rho > 0,$$

for the backward parabolic cylinder with vertex (x_0, t_0) and width ρ. We shall sometimes omit the dependence on the vertex when all cylinders share the same vertex.

Now we introduce the auxiliary function $H_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$H_\gamma(\xi) := \begin{cases} (|\xi| - \lambda)_+^\gamma \, \frac{\xi}{|\xi|} & \text{if } \ \xi \neq 0, \\ 0 & \text{if } \ \xi = 0, \end{cases}$$

where $\lambda \geq 0$ and $\gamma > 0$ are parameters. In Sections 4 and 5, we shall deal with local weak solutions to the elliptic equation (1.3). In this framework, we define a local weak solution to (1.3) as follows:

Definition 2.1. Let $f \in L^1_{loc}(\Omega)$. A function $u \in W^{1,p}_{loc}(\Omega)$ is a *local weak solution* of equation (1.3) if and only if the condition

$$\int_\Omega \langle H_{p-1}(Du), D\varphi \rangle \, dx \;=\; \int_\Omega f\varphi \, dx$$

is satisfied for all $\varphi \in C_0^\infty(\Omega)$.

Similarly, we define a weak solution to equation (1.6) as follows:

Definition 2.2. Let $\widetilde{f} \in L^1_{loc}(\Omega_T)$. A function

$$u \;\in\; C^0\left((0,T); L^2(\Omega)\right) \cap L^p\left(0,T; W^{1,p}(\Omega)\right)$$

is a *weak solution* of equation (1.6) if and only if the condition

$$\int_{\Omega_T} \left(u \cdot \partial_t\varphi - \langle H_{p-1}(Du), D\varphi \rangle\right) \, dz \;=\; -\int_{\Omega_T} \widetilde{f}\varphi \, dz$$

is satisfied for all $\varphi \in C_0^\infty(\Omega_T)$.

2.1. **Difference quotients.** Here we recall the definition and some well-known properties of the difference quotients, which can be found, for example, in [23].

Definition 2.3. For every vector-valued function $F : \mathbb{R}^n \to \mathbb{R}^k$ the *finite difference operator* in the direction x_j is defined by

$$\tau_{j,h}F(x) = F(x + he_j) - F(x)\,,$$

where $h \in \mathbb{R}$, e_j is the unit vector in the direction x_j and $j \in \{1, \ldots, n\}$.

The *difference quotient* of F with respect to x_j is defined for $h \in \mathbb{R} \setminus \{0\}$ by

$$\Delta_{j,h}F(x) \;=\; \frac{\tau_{j,h}F(x)}{h}\,.$$

When no confusion arises, we shall omit the index j and simply write τ_h or Δ_h instead of $\tau_{j,h}$ or $\Delta_{j,h}$, respectively.

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^n$ be an open set and let $F \in W^{1,q}(\Omega)$, with $q \geq 1$. Moreover, let $G : \Omega \to \mathbb{R}$ be a measurable function and consider the set*

$$\Omega_{|h|} := \{x \in \Omega : \text{dist}\,(x, \partial\Omega) > |h|\}\,.$$

Then:

(i) $\Delta_h F \in W^{1,q}(\Omega_{|h|})$ *and* $\partial_{x_i}(\Delta_h F) = \Delta_h(\partial_{x_i} F)$ *for every* $i \in \{1, \ldots, n\}$.

(ii) *If at least one of the functions F or G has support contained in $\Omega_{|h|}$, then*

$$\int_\Omega F\, \Delta_h G\, dx \;=\; -\int_\Omega G\, \Delta_{-h} F\, dx\,.$$

(iii) *We have*

$$\Delta_h(FG)(x) = F(x + he_j)\Delta_h G(x) \;+\; G(x)\Delta_h F(x)\,.$$

The next result about the finite difference operator is a kind of integral version of the Lagrange theorem and can be obtained by combining [23, Lemma 8.1] with [34, theorem on page 3].

Lemma 2.5. *If $0 < \rho < R$, $|h| < \frac{R-\rho}{2}$, $1 < q < \infty$ and $F \in L^1_{loc}(B_R, \mathbb{R}^k)$ is such that $DF \in L^q(B_R, \mathbb{R}^{k\times n})$, then*

$$\int_{B_\rho} |\tau_h F(x)|^q\, dx \;\leq\; c^q\, |h|^q \int_{B_R} |DF(x)|^q\, dx\,,$$

where c is a positive constant depending only on n. Moreover, if $F \in L^q(B_R, \mathbb{R}^k)$, then we have

$$\int_{B_\rho} |F(x + he_j)|^q\, dx \;\leq\; \int_{B_R} |F(x)|^q\, dx\,.$$

Finally, we recall the following fundamental result, whose proof can be found in [23, Lemma 8.2].

Lemma 2.6. *Let $F : \mathbb{R}^n \to \mathbb{R}^k$ be a function in $L^q(B_R, \mathbb{R}^k)$, with $1 < q < \infty$. Assume that there exist $\rho \in (0, R)$ and a constant $M > 0$ such that*

$$\sum_{j=1}^{n} \int_{B_\rho} |\tau_{j,h} F(x)|^q\, dx \;\leq\; M^q\, |h|^q$$

for all $h \in \mathbb{R}$ *satisfying* $|h| < \frac{R-\rho}{2}$. *Then* $F \in W^{1,q}(B_\rho, \mathbb{R}^k)$ *and*

$$\|DF\|_{L^q(B_\rho)} \leq M\,.$$

Moreover, for each $j \in \{1, \dots, n\}$,

$$\Delta_{j,h} F \to \partial_{x_j} F \quad \text{in } L^q_{loc}(B_R, \mathbb{R}^k) \quad \text{as } h \to 0\,.$$

3. Function spaces

In this section, we recall the definitions and basic properties of some function spaces that will be used throughout this note. We begin with Besov spaces, and then move on to fractional Sobolev spaces (also known as *Sobolev-Slobodeckij spaces*).

3.1. **Besov spaces.**

We denote by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ the Schwartz space and the space of tempered distributions on $\mathbb{R}^n$, respectively. If $v \in \mathcal{S}(\mathbb{R}^n)$, then

$$\hat{v}(\xi) = (\mathscr{F} v)(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\,\langle x, \xi\rangle}\, v(x)\, dx, \qquad \xi \in \mathbb{R}^n, \tag{3.1}$$

denotes the Fourier transform of v. As usual, $\mathscr{F}^{-1} v$ and $v^\vee$ stand for the inverse Fourier transform, given by the right-hand side of (3.1) with i in place of $-i$. Both $\mathscr{F}$ and $\mathscr{F}^{-1}$ are extended to $\mathcal{S}'(\mathbb{R}^n)$ in the standard way.

Now, let $\Gamma(\mathbb{R}^n)$ be the collection of all sequences $\varphi = \{\varphi_j\}_{j=0}^\infty \subset \mathcal{S}(\mathbb{R}^n)$ such that

$$\begin{cases} \operatorname{supp} \varphi_0 \subset \{x \in \mathbb{R}^n : |x| \leq 2\} \\ \operatorname{supp} \varphi_j \subset \{x \in \mathbb{R}^n : 2^{j-1} \leq |x| \leq 2^{j+1}\} \quad \text{if } j \in \mathbb{N}, \end{cases}$$

for every multi-index β there exists a positive number c_β such that

$$2^{j|\beta|}\, |D^\beta \varphi_j(x)| \leq c_\beta\,, \qquad \forall\, j \in \mathbb{N}_0\,,\ \forall\, x \in \mathbb{R}^n$$

and

$$\sum_{j=0}^\infty \varphi_j(x) = 1\,, \qquad \forall\, x \in \mathbb{R}^n.$$

Then, it is well known that $\Gamma(\mathbb{R}^n)$ is not empty (see [40, Section 2.3.1, Remark 1]). Moreover, if $\{\varphi_j\}_{j=0}^\infty \in \Gamma(\mathbb{R}^n)$, the entire analytic functions $(\varphi_j\, \hat{v})^\vee(x)$ make sense pointwise in $\mathbb{R}^n$ for any $v \in \mathcal{S}'(\mathbb{R}^n)$. Therefore, the following definition makes sense:

Definition 3.1. Let $s \in \mathbb{R}$, $1 \le p, q \le \infty$ and $\varphi = \{\varphi_j\}_{j=0}^{\infty} \in \Gamma(\mathbb{R}^n)$. We define the *Besov space* $B^s_{p,q}(\mathbb{R}^n)$ as the set of all $v \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|v\|_{B^s_{p,q}(\mathbb{R}^n)} := \left(\sum_{j=0}^{\infty} 2^{jsq}\, \|(\varphi_j\, \hat{v})^{\vee}\|^q_{L^p(\mathbb{R}^n)}\right)^{\frac{1}{q}} < +\infty \qquad \text{if } q < \infty, \tag{3.2}$$

and

$$\|v\|_{B^s_{p,q}(\mathbb{R}^n)} := \sup_{j\,\in\,\mathbb{N}_0} \, 2^{js}\, \|(\varphi_j\, \hat{v})^{\vee}\|_{L^p(\mathbb{R}^n)} < +\infty \qquad \text{if } q = \infty. \tag{3.3}$$

Remark 3.2. The space $B^s_{p,q}(\mathbb{R}^n)$ defined above is a Banach space with respect to the norm $\|\cdot\|_{B^s_{p,q}(\mathbb{R}^n)}$. Obviously, this norm depends on the chosen sequence $\varphi \in \Gamma(\mathbb{R}^n)$, but this is not the case for the spaces $B^s_{p,q}(\mathbb{R}^n)$ themselves, in the sense that two different choices for the sequence φ give rise to equivalent norms (see [40, Sections 2.3.2 and 2.3.3]). This justifies our omission of the dependence on φ in the left-hand side of (3.2)$-$(3.3) and in the sequel.

The norms of the *classical Besov spaces* $B^s_{p,q}(\mathbb{R}^n)$ with $s \in (0,1)$, $1 \le p < \infty$ and $1 \le q \le \infty$ can be characterized via differences of the functions involved (cf. [40, Section 2.5.12, Theorem 1]). More precisely, for $h \in \mathbb{R}^n$ and a measurable function $v : \mathbb{R}^n \to \mathbb{R}$, let us define

$$\delta_h v(x) := \, v(x+h) - v(x)\,.$$

Then we have the equivalence

$$\|v\|_{B^s_{p,q}(\mathbb{R}^n)} \,\approx\, \|v\|_{L^p(\mathbb{R}^n)} \,+\, [v]_{B^s_{p,q}(\mathbb{R}^n)}\,,$$

where

$$[v]_{B^s_{p,q}(\mathbb{R}^n)} := \left(\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{|\delta_h v(x)|^p}{|h|^{sp}}\,dx\right)^{\frac{q}{p}} \frac{dh}{|h|^n}\right)^{\frac{1}{q}}, \qquad \text{if } 1 \le q < \infty, \tag{3.4}$$

and

$$[v]_{B^s_{p,\infty}(\mathbb{R}^n)} := \sup_{h\,\in\,\mathbb{R}^n}\left(\int_{\mathbb{R}^n} \frac{|\delta_h v(x)|^p}{|h|^{sp}}\,dx\right)^{\frac{1}{p}}. \tag{3.5}$$

In (3.4), if one simply integrates for $|h| < r$ for a fixed $r > 0$, then an equivalent norm is obtained, since

$$\left(\int_{\{|h|\geq r\}}\left(\int_{\mathbb{R}^n}\frac{|\delta_h v(x)|^p}{|h|^{sp}}\,dx\right)^{\frac{q}{p}}\frac{dh}{|h|^n}\right)^{\frac{1}{q}}\leq c(n,s,p,q,r)\,\|v\|_{L^p(\mathbb{R}^n)}\,.$$

Similarly, in (3.5) one can simply take the supremum over $|h|\leq r$ and obtain an equivalent norm. By construction, $B^s_{p,q}(\mathbb{R}^n)\subset L^p(\mathbb{R}^n)$.

Remark 3.3. For $s\in(0,1)$ and $1\leq p,q<\infty$, we can simply say that $v\in B^s_{p,q}(\mathbb{R}^n)$ if and only if $v\in L^p(\mathbb{R}^n)$ and $\frac{\delta_h v}{|h|^s}\in L^q\left(\frac{dh}{|h|^n};L^p(\mathbb{R}^n)\right)$.

Let Ω be an arbitrary open set in $\mathbb{R}^n$. As usual, $\mathcal{D}(\Omega)=C^\infty_0(\Omega)$ stands for the space of all infinitely differentiable functions in $\mathbb{R}^n$ with compact support in Ω. Let $\mathcal{D}'(\Omega)$ be the dual space of all distributions in Ω and let $g\in\mathcal{S}'(\mathbb{R}^n)$. Then we denote by $g|_\Omega$ its restriction to Ω, i.e.

$$g|_\Omega\in\mathcal{D}'(\Omega):\quad (g|_\Omega)(\phi)=g(\phi)\quad \text{for } \phi\in\mathcal{D}(\Omega).$$

Definition 3.4. Let Ω be an arbitrary domain in $\mathbb{R}^n$ with $\Omega\neq\mathbb{R}^n$ and let $s\in\mathbb{R}$, $1\leq p\leq\infty$ and $1\leq q\leq\infty$. Then

$$B^s_{p,q}(\Omega):=\left\{v\in\mathcal{D}'(\Omega):\ v=g|_\Omega\ \text{ for some } g\in B^s_{p,q}(\mathbb{R}^n)\right\}$$

and

$$\|v\|_{B^s_{p,q}(\Omega)}:=\ \inf\ \|g\|_{B^s_{p,q}(\mathbb{R}^n)}\,,$$

where the infimum is taken over all $g\in B^s_{p,q}(\mathbb{R}^n)$ such that $g|_\Omega=v$.

For a bounded C^∞-domain $\Omega\subset\mathbb{R}^n$, the classical Besov spaces $B^s_{p,q}(\Omega)$ with $s\in(0,1)$, $1\leq p<\infty$ and $1\leq q\leq\infty$ can be characterized via differences of the functions involved. More precisely, we have the following result (see [41, Section 5.2.2]).

Theorem 3.5. *Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$. Let $s\in(0,1)$, $1\leq p<\infty$ and $1\leq q<\infty$. Then, a function $v:\Omega\to\mathbb{R}$ belongs to the Besov space $B^s_{p,q}(\Omega)$ if and only if $v\in L^p(\Omega)$ and*

$$[v]_{B^s_{p,q}(\Omega)}:=\left(\int_{\mathbb{R}^n}\left(\int_\Omega\frac{|\delta_h v(x)|^p}{|h|^{sp}}\cdot\mathbb{1}_\Omega(x+h)\,dx\right)^{\frac{q}{p}}\frac{dh}{|h|^n}\right)^{\frac{1}{q}}<+\infty. \tag{3.6}$$

Moreover, the Besov space $B^s_{p,\infty}(\Omega)$ consists of all functions $v \in L^p(\Omega)$ such that

$$[v]_{B^s_{p,\infty}(\Omega)} := \sup_{h \in \mathbb{R}^n} \left(\int_\Omega \frac{|\delta_h v(x)|^p}{|h|^{sp}} \cdot \mathbb{1}_\Omega(x+h)\, dx \right)^{\frac{1}{p}} < +\infty. \tag{3.7}$$

For $s \in (0,1)$, $1 \le p < \infty$ and $1 \le q \le \infty$, we also have the equivalence

$$\|v\|_{B^s_{p,q}(\Omega)} \approx \|v\|_{L^p(\Omega)} + [v]_{B^s_{p,q}(\Omega)}\,.$$

If one replaces $\mathbb{R}^n$ in (3.6) by a ball $B(0,r)$ for a fixed $r > 0$, then an equivalent norm is obtained. Similarly, in (3.7) one can simply take the supremum over $|h| \le r$ and obtain an equivalent norm.

If $s \in \mathbb{R}$, $1 \le p < \infty$ and $1 \le q < \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is a dense subset of $B^s_{p,q}(\mathbb{R}^n)$ (cf. [40, Theorem 2.3.3]). Consequently, in that case, a continuous linear functional on $B^s_{p,q}(\mathbb{R}^n)$ can be interpreted in the usual way as an element of $\mathcal{S}'(\mathbb{R}^n)$. More precisely, $g \in \mathcal{S}'(\mathbb{R}^n)$ belongs to the dual space $(B^s_{p,q}(\mathbb{R}^n))'$ of the space $B^s_{p,q}(\mathbb{R}^n)$ if and only if there exists a positive number c such that

$$|g(\phi)| \le c\, \|\phi\|_{B^s_{p,q}(\mathbb{R}^n)} \qquad \text{for all } \phi \in \mathcal{S}(\mathbb{R}^n)\,.$$

We endow $(B^s_{p,q}(\mathbb{R}^n))'$ with the natural dual norm, defined by

$$\|g\|_{(B^s_{p,q}(\mathbb{R}^n))'} = \sup \left\{ |g(\phi)| : \phi \in \mathcal{S}(\mathbb{R}^n) \text{ and } \|\phi\|_{B^s_{p,q}(\mathbb{R}^n)} \le 1 \right\}, \qquad g \in (B^s_{p,q}(\mathbb{R}^n))'.$$

Now we recall the following duality formula, which has to be meant as an isomorphism of normed spaces (see [40, Theorem 2.11.2]).

Theorem 3.6. *Let $s \in \mathbb{R}$, $1 \le p < \infty$ and $1 \le q < \infty$. Then*

$$(B^s_{p,q}(\mathbb{R}^n))' = B^{-s}_{p',q'}(\mathbb{R}^n)\,,$$

where $p' = \infty$ if $p = 1$ (similarly for q').

Remark 3.7. The restrictions $p < \infty$ and $q < \infty$ in Theorem 3.6 are natural, since, if either $p = \infty$ or $q = \infty$, then $\mathcal{S}(\mathbb{R}^n)$ is not dense in $B^s_{p,q}(\mathbb{R}^n)$, and the density of $\mathcal{S}(\mathbb{R}^n)$ in $B^s_{p,q}(\mathbb{R}^n)$ is the basis of our interpretation of the dual space $(B^s_{p,q}(\mathbb{R}^n))'$.

Definition 3.8. For $s \in \mathbb{R}$, $1 \le p \le \infty$ and $1 \le q \le \infty$, we define $\mathring{B}^s_{p,q}(\mathbb{R}^n)$ as the completion of $\mathcal{S}(\mathbb{R}^n)$ in $B^s_{p,q}(\mathbb{R}^n)$ with respect to the norm

$$v \mapsto \|v\|_{B^s_{p,q}(\mathbb{R}^n)}\,.$$

Of course, in the definition above, only the limit cases $\max\{p,q\} = \infty$ are of interest. We shall denote by $(\mathring{B}^s_{p,q}(\mathbb{R}^n))'$ the topological dual of $\mathring{B}^s_{p,q}(\mathbb{R}^n)$, which is endowed with the natural dual norm

$$\|g\|_{(\mathring{B}^s_{p,q}(\mathbb{R}^n))'} = \sup\left\{|g(\phi)| : \phi \in \mathcal{S}(\mathbb{R}^n) \text{ and } \|\phi\|_{B^s_{p,q}(\mathbb{R}^n)} \le 1\right\}, \qquad g \in (\mathring{B}^s_{p,q}(\mathbb{R}^n))'.$$

The following duality result can be found in [40, Section 2.11.2, Remark 2] (see also [39, pages 121 and 122]).

Theorem 3.9. *Let $s \in \mathbb{R}$, $1 \le p \le \infty$ and $1 \le q \le \infty$. Then*

$$(\mathring{B}^s_{p,q}(\mathbb{R}^n))' \;=\; B^{-s}_{p',q'}(\mathbb{R}^n)\,,$$

where $p' = 1$ if $p = \infty$ (similarly for q').

One also has the following version of Sobolev embeddings (a proof can be found in [24, Proposition 7.12]).

Lemma 3.10. *Suppose that $0 < \alpha < 1$.*
(a) If $1 < p < \frac{n}{\alpha}$ and $1 \le q \le p^*_\alpha := \frac{np}{n-\alpha p}$, then there exists a continuous embedding $B^\alpha_{p,q}(\mathbb{R}^n) \hookrightarrow L^{p^*_\alpha}(\mathbb{R}^n)$.
(b) If $p = \frac{n}{\alpha}$ and $1 \le q \le \infty$, then there exists a continuous embedding $B^\alpha_{p,q}(\mathbb{R}^n) \hookrightarrow$ BMO$(\mathbb{R}^n)$, where BMO *denotes the space of functions with bounded mean oscillations* [23, Chapter 2].

We now recall the following inclusions between Besov spaces (see [24, Proposition 7.10 and Formula (7.35)]).

Lemma 3.11. *Suppose that $0 < \beta < \alpha < 1$.*
(a) If $1 < p \le \infty$ and $1 \le q \le r \le \infty$, then $B^\alpha_{p,q}(\mathbb{R}^n) \subset B^\alpha_{p,r}(\mathbb{R}^n)$.
(b) If $1 < p \le \infty$ and $1 \le q, r \le \infty$, then $B^\alpha_{p,q}(\mathbb{R}^n) \subset B^\beta_{p,r}(\mathbb{R}^n)$.
(c) If $1 \le q \le \infty$, then $B^\alpha_{\frac{n}{\alpha},q}(\mathbb{R}^n) \subset B^\beta_{\frac{n}{\beta},q}(\mathbb{R}^n)$.

Combining Lemmas 3.10 and 3.11, we obtain the following Sobolev-type embedding theorem for Besov spaces $B^{\alpha}_{p,\infty}(\mathbb{R}^n)$ that are excluded from assumptions (a) in Lemma 3.10.

Lemma 3.12. *Suppose that $0 < \beta < \alpha < 1$. If $1 < p < \frac{n}{\alpha}$, then there exists a continuous embedding $B^{\alpha}_{p,\infty}(\mathbb{R}^n) \hookrightarrow L^{p^*_\beta}(\mathbb{R}^n)$.*

We can also define local Besov spaces as follows. Given a domain $\Omega \subset \mathbb{R}^n$, we say that a function $v : \Omega \to \mathbb{R}$ belongs to the *local Besov space* $B^s_{p,q,loc}(\Omega)$ if $\phi\, v \in B^s_{p,q}(\mathbb{R}^n)$ whenever $\phi \in C^\infty_0(\Omega)$. It is worth noticing that one can prove suitable local versions of Lemmas 3.10 and 3.11, by using local Besov spaces.

The following lemma is a consequence of Remark 3.3 and its proof can be found in [7, Lemma 7].

Lemma 3.13. *Let $s \in (0,1)$ and $1 \leq p, q < \infty$. A function $v \in L^p_{loc}(\Omega)$ belongs to the local Besov space $B^s_{p,q,loc}(\Omega)$ if and only if*

$$\left\| \frac{\delta_h v}{|h|^s} \right\|_{L^q\left(\frac{dh}{|h|^n}\,;\, L^p(B_r)\right)} < +\infty$$

for any ball $B_r \subset B_{2r} \Subset \Omega$. Here, the measure $\frac{dh}{|h|^n}$ is restricted to the ball $B(0,r)$ on the h-space.

The next result represents the local counterpart of Lemma 3.12:

Lemma 3.14. *On any domain $\Omega \subset \mathbb{R}^n$ we have the continuous embedding $B^{\alpha}_{p,\infty,loc}(\Omega) \hookrightarrow L^r_{loc}(\Omega)$ for all $r \in \left[1, \frac{np}{n-\alpha p}\right)$, provided $\alpha \in (0,1)$ and $1 < p < \frac{n}{\alpha}$.*

We refer to [38, Sections 30-32] for a proof of this lemma. In fact, the above statement follows by localizing the corresponding result proved for functions defined on $\mathbb{R}^n$ in [38], by simply using a smooth cut-off function.

For the treatment of parabolic equations, we now give the following definition.

Definition 3.15. Let Ω be a bounded C^∞-domain in $\mathbb{R}^n$. Let $s \in (0,1)$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Then, we say that a map $g \in L^p(\Omega \times (t_0, t_1))$ belongs to the space

$L^p\left(t_0, t_1; B^s_{p,q}(\Omega)\right)$ if and only if

$$\int_{t_0}^{t_1} [g(\cdot,t)]^p_{B^s_{p,q}(\Omega)}\, dt \leq \infty\,,$$

where $[\,\cdot\,]_{B^s_{p,q}(\Omega)}$ is defined by (3.6) or (3.7).

In the sequel, we shall use the corresponding local version of the space defined above, which will be denoted by the subscript "*loc*". More precisely, given a bounded domain $\Omega \subset \mathbb{R}^n$, we write $g \in L^p_{loc}\left(0, T; B^s_{p,q,loc}(\Omega)\right)$ if and only if $g \in L^p\left(t_0, t_1; B^s_{p,q}(\Omega')\right)$ for all domains $\Omega' \times (t_0, t_1) \Subset \Omega_T := \Omega \times (0, T)$, with Ω' being a (bounded) C^∞-domain. We shall also use the following notation, which is typical of Bochner spaces:

$$\|g\|_{L^p(t_0,\, t_1\,;\, B^s_{p,q}(\Omega'))} := \left(\int_{t_0}^{t_1} \|g(\cdot,t)\|^p_{B^s_{p,q}(\Omega')}\, dt\right)^{\frac{1}{p}}.$$

3.2. **Fractional Sobolev spaces.** Here we recall the definition and some properties of the fractional Sobolev spaces that appear in the statement of Theorem 7.1 below (for more details, we refer to [18]).

Let Ω be a general, possibly nonsmooth, open set in $\mathbb{R}^n$. For any $s \in (0,1)$ and for any $q \in [1, +\infty)$, we define the fractional Sobolev space $W^{s,q}(\Omega, \mathbb{R}^k)$ as follows:

$$W^{s,q}(\Omega, \mathbb{R}^k) := \left\{ v \in L^q(\Omega, \mathbb{R}^k) : \frac{|v(x) - v(y)|}{|x-y|^{\frac{n}{q}+s}} \in L^q(\Omega \times \Omega) \right\},$$

i.e. an intermediate Banach space between $L^q(\Omega, \mathbb{R}^k)$ and $W^{1,q}(\Omega, \mathbb{R}^k)$, endowed with the norm

$$\|v\|_{W^{s,q}(\Omega)} := \left(\int_\Omega |v|^q\, dx\, + \int_\Omega \int_\Omega \frac{|v(x)-v(y)|^q}{|x-y|^{n+sq}}\, dx\, dy\right)^{\frac{1}{q}},$$

where the term

$$[v]_{W^{s,q}(\Omega)} := \left(\int_\Omega \int_\Omega \frac{|v(x)-v(y)|^q}{|x-y|^{n+sq}}\, dx\, dy\right)^{\frac{1}{q}} \tag{3.8}$$

is the so-called *Gagliardo seminorm* of v.

Remark 3.16. For every $s \in (0,1)$ and every $q \in [1, \infty)$, we have $B^s_{q,q}(\mathbb{R}^n) = W^{s,q}(\mathbb{R}^n)$. In fact, using the change of variable $y = x + h$ in (3.8) with $\Omega = \mathbb{R}^n$, one gets the seminorm (3.4) with $p = q$.

As in the classic case with s being an integer, the space $W^{\sigma,q}(\Omega)$ is continuously embedded in $W^{s,q}(\Omega)$ when $s \leq \sigma$, as shown by the next result (see [18, Proposition 2.1]).

Proposition 3.17. *Let Ω be an open set in $\mathbb{R}^n$, let $q \in [1,+\infty)$ and let $0 < s \leq \sigma < 1$. Then there exists a constant $C = C(n,s,q) \geq 1$ such that, for any $v \in W^{\sigma,q}(\Omega)$, we have*

$$\|v\|_{W^{s,q}(\Omega)} \leq C \|v\|_{W^{\sigma,q}(\Omega)} .$$

In particular, $W^{\sigma,q}(\Omega) \subseteq W^{s,q}(\Omega)$.

As is well known when $s \in \mathbb{N}$, under certain regularity assumptions on the open set $\Omega \subset \mathbb{R}^n$, any function in $W^{s,q}(\Omega)$ can be extended to a function in $W^{s,q}(\mathbb{R}^n)$. Extension results are needed to improve some embedding theorems, in the classic case as well as in the fractional one. In this regard, we now give the following definition.

Definition 3.18. For any $s \in (0,1)$ and any $q \in [1,\infty)$, we say that an open set $\Omega \subseteq \mathbb{R}^n$ is an *extension domain* for $W^{s,q}$ if there exists a positive constant $C = C(n,q,s,\Omega)$ such that for every function $v \in W^{s,q}(\Omega)$ there exists $\widetilde{v} \in W^{s,q}(\mathbb{R}^n)$ with $\widetilde{v} = v$ on Ω and

$$\|\widetilde{v}\|_{W^{s,q}(\mathbb{R}^n)} \leq C \|v\|_{W^{s,q}(\Omega)} .$$

In general, an arbitrary open set may not be an extension domain for $W^{s,q}$. However, the following result ensures that every open Lipschitz set Ω with bounded boundary is an extension domain for $W^{s,q}$ (a proof can be found in [18, Theorem 5.4]).

Theorem 3.19. *Let $q \in [1,+\infty)$, let $s \in (0,1)$ and let $\Omega \subseteq \mathbb{R}^n$ be an open set of class $C^{0,1}$ with bounded boundary. Then $W^{s,q}(\Omega)$ is continuously embedded in $W^{s,q}(\mathbb{R}^n)$, namely for any $v \in W^{s,q}(\Omega)$ there exists $\widetilde{v} \in W^{s,q}(\mathbb{R}^n)$ such that $\widetilde{v} = v$ on Ω and*

$$\|\widetilde{v}\|_{W^{s,q}(\mathbb{R}^n)} \leq C \|v\|_{W^{s,q}(\Omega)}$$

for some positive constant $C = C(n,q,s,\Omega)$.

For more information on the problem of characterizing the class of sets that are extension domains for $W^{s,q}$, we refer the interested reader to [43], where an answer to this question has been given (see also [27] and [29, Chapters 11 and 12]).

We now recall the following Sobolev-type embedding theorem, whose proof can be found in [18, Theorem 6.7].

Theorem 3.20. *Let $s \in (0,1)$ and $q \in [1,+\infty)$ be such that $sq < n$. Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain for $W^{s,q}$. Then there exists a positive constant $C = C(n,q,s,\Omega)$ such that, for any $v \in W^{s,q}(\Omega)$, we have*

$$\|v\|_{L^r(\Omega)} \leq C\,\|v\|_{W^{s,q}(\Omega)}$$

for any $r \in [q,q^*]$; that is, the space $W^{s,q}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $r \in [q,q^*]$, where $q^* := nq/(n-sq)$ is the so-called "fractional critical exponent".

Moreover, if Ω is bounded, then the space $W^{s,q}(\Omega)$ is continuously embedded in $L^r(\Omega)$ for any $r \in [1,q^*]$.

Remark 3.21. In the critical case $r = q^*$, the constant C in Theorem 3.20 does not depend on Ω (see [18, Remark 6.8]).

The next embedding result can be obtained by combining [40, Section 2.2.2, Remark 3] with [40, Section 2.3.2, Proposition 2(ii)].

Lemma 3.22. *Let $s \in (0,1)$ and $q \geq 1$. Then, for every $\sigma \in (0,1-s)$ we have the continuous embedding $W^{s+\sigma,q}_{loc}(\mathbb{R}^n) \hookrightarrow B^s_{q,1,loc}(\mathbb{R}^n)$.*

For the treatment of parabolic equations, we now give the following definition.

Definition 3.23. Let $q \in [1,+\infty)$ and let $s \in (0,1)$. We say that a map $g \in L^q(\Omega \times (t_0,t_1),\mathbb{R}^k)$ belongs to the space $L^q\left(t_0,t_1;W^{s,q}(\Omega,\mathbb{R}^k)\right)$ if and only if

$$\int_{t_0}^{t_1}\int_\Omega\int_\Omega \frac{|g(x,t)-g(y,t)|^q}{|x-y|^{n+sq}}\,dx\,dy\,dt < \infty.$$

In the sequel, we will use the corresponding local version of the space defined above, which will be denoted by the subscript "*loc*". More precisely, we write $g \in L^q_{loc}\left(0,T;W^{s,q}_{loc}(\Omega,\mathbb{R}^k)\right)$ if and only if $g \in L^q\left(t_0,t_1;W^{s,q}(\Omega',\mathbb{R}^k)\right)$ for all domains $\Omega' \times (t_0,t_1) \Subset \Omega_T := \Omega \times (0,T)$.

4. Besov estimates for a class of elliptic equations

In this section, we are concerned with the regularity properties of the local weak solutions to equation (1.3). All the results that we report here are proved in the paper [1].

As already anticipated in the introduction, we are interested in the higher differentiability of the vector field

$$H_{\frac{p}{2}}(Du) := (|Du| - \lambda)_+^{\frac{p}{2}} \, \frac{Du}{|Du|}\,, \qquad \text{for } p \in (1,2)\,,$$

where $u \in W^{1,p}_{loc}(\Omega)$ is a local weak solution of (1.3). In [1], we have established a Besov regularity result for the function $H_{p/2}(Du)$ under suitable assumptions on the right-hand side f of (1.3). As an easy consequence of this result, we have also deduced the Besov regularity of the solution σ_0 to (P1), under the same hypotheses on f.

More precisely, the main results of this section are the following theorems. For notation and definitions we refer to Sections 2 and 3.1.

Theorem 4.1 ([1, Theorem 1.1])**.** *Let $n \geq 2$, $p \in (1,2)$ and $\alpha \in (0,1)$. Moreover, let $u \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of equation (1.3). Then, the following implications hold:*

$$f \in W^{1,p'}_{loc}(\Omega) \;\;\Rightarrow\;\; H_{\frac{p}{2}}(Du) \in B^{\frac{1}{3-p}}_{2,\infty} \;\; \textit{locally in } \Omega,$$

$$f \in B^{\alpha}_{p',\infty,loc}(\Omega) \;\;\Rightarrow\;\; H_{\frac{p}{2}}(Du) \in B^{\min\left\{\frac{\alpha+1}{2},\,\frac{1}{3-p}\right\}}_{2,\infty} \;\; \textit{locally in } \Omega.$$

Furthermore:

(a) if $f \in W^{1,p'}_{loc}(\Omega)$, then for any ball $B_R \Subset \Omega$ the following estimate

$$\int_{B_{R/2}} \left|\tau_{j,h} H_{\frac{p}{2}}(Du)\right|^2 dx \,\leq C_1 \left(\|Df\|_{L^{p'}(B')} \, \|Du\|_{L^p(B')} \, |h|^2 + \, \|Du\|^p_{L^p(B')} \, |h|^{\frac{2}{3-p}} \right)$$

holds true for every $j \in \{1, \ldots, n\}$, for every $h \in \mathbb{R}$ such that $|h| \leq r_0 < \frac{1}{2}\,\mathrm{dist}\,(B_R, \partial\Omega)$, for $B' = B_R + B(0, r_0)$ and a positive constant $C_1 = C_1(n,p,R)$.

(b) If, on the other hand, $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$*, then for any ball* $B_R \Subset \Omega$ *the following estimate*

$$\int_{B_{R/2}} \left|\tau_{j,h} H_{\frac{p}{2}}(Du)\right|^2 dx \leq C_2 \left(\|f\|_{B^{\alpha}_{p',\infty}(B')} \|Du\|_{L^p(B')} |h|^{\alpha+1} + \|Du\|^p_{L^p(B')} |h|^{\frac{2}{3-p}} \right)$$

holds true for every $j \in \{1, \dots, n\}$*, for every* $h \in \mathbb{R}$ *such that* $|h| \leq r_0 < \frac{1}{2} \operatorname{dist}(B_R, \partial\Omega)$*, for* $B' = B_R + B(0, r_0)$ *and a positive constant* $C_2 = C_2(n, p, R)$*.*

Theorem 4.2 ([1, Theorem 1.2])**.** *Let* $n \geq 2$*,* $p \in (1, 2)$ *and* $\alpha \in (0, 1)$*. Moreover, let* $\sigma_0 \in L^{p'}(\Omega, \mathbb{R}^n)$ *be the solution of* (P1)*. Then, the following implications hold:*

$$f \in W^{1,p'}_{loc}(\Omega) \;\Rightarrow\; \sigma_0 \in B^{\frac{2}{p'(3-p)}}_{p',\infty} \text{ locally in } \Omega,$$

$$f \in B^{\alpha}_{p',\infty,loc}(\Omega) \;\Rightarrow\; \sigma_0 \in B^{\min\left\{\frac{\alpha+1}{p'}, \frac{2}{p'(3-p)}\right\}}_{p',\infty} \text{ locally in } \Omega.$$

Furthermore:

(a) if $f \in W^{1,p'}_{loc}(\Omega)$*, then for every solution* u *of problem* (P2) *and every ball* $B_R \Subset \Omega$*, the following estimate*

$$\int_{B_{R/2}} |\tau_{j,h}\sigma_0|^{p'} dx \leq C_1 \left(\|Df\|_{L^{p'}(B')} \|Du\|_{L^p(B')} |h|^2 + \|Du\|^p_{L^p(B')} |h|^{\frac{2}{3-p}} \right)$$

holds true for every $j \in \{1, \dots, n\}$*, for every* $h \in \mathbb{R}$ *such that* $|h| \leq r_0 < \frac{1}{2} \operatorname{dist}(B_R, \partial\Omega)$*, for* $B' = B_R + B(0, r_0)$ *and a positive constant* $C_1 = C_1(n, p, R)$*.*

(b) If, on the other hand, $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$*, then for every solution* u *of problem* (P2) *and every ball* $B_R \Subset \Omega$*, the following estimate*

$$\int_{B_{R/2}} |\tau_{j,h}\sigma_0|^{p'} dx \leq C_2 \left(\|f\|_{B^{\alpha}_{p',\infty}(B')} \|Du\|_{L^p(B')} |h|^{\alpha+1} + \|Du\|^p_{L^p(B')} |h|^{\frac{2}{3-p}} \right)$$

holds true for every $j \in \{1, \dots, n\}$*, for every* $h \in \mathbb{R}$ *such that* $|h| \leq r_0 < \frac{1}{2} \operatorname{dist}(B_R, \partial\Omega)$*, for* $B' = B_R + B(0, r_0)$ *and a positive constant* $C_2 = C_2(n, p, R)$*.*

Remark 4.3. If we take $p = 2$ and $f \in W^{1,2}_{loc}(\Omega)$ in the statements (a) of Theorems 4.1 and 4.2, then we get back the full Sobolev regularity results of [12, Theorem 4.2 and

Corollary 4.3], i.e. there is no discrepancy between our results and the ones contained in [12].

As a consequence of Theorem 4.2 and Lemma 3.14, we obtain the following higher integrability result for σ_0.

Corollary 4.4 ([1, Corollary 1.4]). *Under the assumptions of Theorem 4.2, we obtain the following implications:*

$$f \in W^{1,p'}_{loc}(\Omega) \;\;\Rightarrow\;\; \sigma_0 \in L^r_{loc}(\Omega, \mathbb{R}^n) \quad \textit{for all } r \in \left[1, \frac{n(3-p)\,p'}{n(3-p)-2}\right),$$

$$f \in B^{\alpha}_{p',\infty,loc}(\Omega) \;\;\Rightarrow\;\; \sigma_0 \in L^s_{loc}(\Omega, \mathbb{R}^n)\,,$$

where

$$s = \begin{cases} \textit{any value in } \left[1, \frac{np'}{n-\alpha-1}\right), & \textit{if } \; 0 < \alpha < \frac{p-1}{3-p}\,, \\[2ex] \textit{any value in } \left[1, \frac{n(3-p)\,p'}{n(3-p)-2}\right), & \textit{if } \; \frac{p-1}{3-p} \le \alpha < 1. \end{cases}$$

As a direct consequence of the previous results, we also get a gain of integrability for the gradient of the local weak solutions of (1.3). Indeed, we have the following result.

Corollary 4.5 ([1, Corollary 1.5]). *Under the assumptions of Theorem 4.1, we obtain the following implications:*

$$f \in W^{1,p'}_{loc}(\Omega) \;\;\Rightarrow\;\; Du \in L^r_{loc}(\Omega, \mathbb{R}^n) \quad \textit{for all } r \in \left[1, \frac{np\,(3-p)}{n(3-p)-2}\right),$$

$$f \in B^{\alpha}_{p',\infty,loc}(\Omega) \;\;\Rightarrow\;\; Du \in L^s_{loc}(\Omega, \mathbb{R}^n)\,,$$

where

$$s = \begin{cases} \textit{any value in } \left[1, \frac{np}{n-\alpha-1}\right), & \textit{if } \; 0 < \alpha < \frac{p-1}{3-p}\,, \\[2ex] \textit{any value in } \left[1, \frac{np\,(3-p)}{n(3-p)-2}\right), & \textit{if } \; \frac{p-1}{3-p} \le \alpha < 1. \end{cases}$$

In [1], we have also analyzed the non-singular case $p \geq 2$, by proving that, unlike what may occur in the singular one, the Besov regularity of the datum f translates into a Besov regularity for the function $H_{\frac{p}{2}}(Du)$ with no loss in the order of differentiation. More specifically, we have established the following result.

Theorem 4.6 ([1, Theorem 4.1])**.** *Let $n \geq 2$, $p \in [2, \infty)$, $\alpha \in (0,1)$ and $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$. Moreover, let $u \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of equation (1.3). Then*

$$H_{\frac{p}{2}}(Du) \,\in\, B^{\frac{\alpha+1}{2}}_{2,\infty,loc}(\Omega, \mathbb{R}^n)\,.$$

Furthermore, for any ball $B_R \Subset \Omega$, the following estimate

$$\int_{B_{R/2}} \left|\tau_{j,h} H_{\frac{p}{2}}(Du)\right|^2 dx \,\leq c\left(\|f\|_{B^{\alpha}_{p',\infty}(B')}\,\|Du\|_{L^p(B')}\,|h|^{\alpha+1} +\, \|Du\|^p_{L^p(B')}\,|h|^2\right)$$

holds true for every $j \in \{1, \ldots, n\}$, for every $h \in \mathbb{R}$ such that $|h| \leq r_0 < \frac{1}{2}\operatorname{dist}(B_R, \partial\Omega)$, for $B' = B_R + B(0, r_0)$ and a positive constant $c = c(n, p, R)$.

The Besov regularity of $H_{\frac{p}{2}}(Du)$ established by the above theorem allows us to get a gain of integrability for Du. More precisely, we have the following result.

Corollary 4.7 ([1, Corollary 4.2])**.** *Under the assumptions of Theorem 4.6, we get*

$$H_{\frac{p}{2}}(Du) \in L^r_{loc}(\Omega, \mathbb{R}^n) \quad \text{for all } r \in \left[1, \frac{2n}{n-\alpha-1}\right)$$

and

$$Du \in L^s_{loc}(\Omega, \mathbb{R}^n) \quad \text{for all } s \in \left[1, \frac{np}{n-\alpha-1}\right).$$

Again, if we come back to the variational problem (P1), we obtain the following higher integrability result for its solution:

Corollary 4.8 ([1, Corollary 4.4])**.** *Let $n \geq 2$, $p \in [2, \infty)$, $\alpha \in (0,1)$ and $f \in B^{\alpha}_{p',\infty,loc}(\Omega)$. Moreover, let $\sigma_0 \in L^{p'}(\Omega, \mathbb{R}^n)$ be the solution of (P1). Then*

$$\sigma_0 \in L^r_{loc}(\Omega, \mathbb{R}^n) \quad \text{for all } r \in \left[1, \frac{np'}{n-\alpha-1}\right).$$

5. Second-order regularity for a class of elliptic equations

In this section, we consider again the local $W^{1,p}$ solutions of (1.3). All the results that we report here are proved in [5]. There, we have established the local $W^{1,2}$-regularity of a nonlinear function of the gradient Du of local weak solutions to equation (1.3), by assuming that

- $f \in B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)$ if $2 < p < \infty$ (see Theorem 5.1);
- $f \in L^{\frac{np}{n(p-1)+2-p}}_{loc}(\Omega)$ if $1 < p \leq 2$ (see Theorem 5.2).

The above-mentioned theorems, in turn, imply the local higher integrability of Du under the same hypotheses on the function f (cf. Corollary 5.4).

The main results of this section are in the spirit of those contained in [13, 15, 25], which we have already discussed in Section 1. In order to state these results, we introduce the auxiliary function

$$\mathcal{G}_\lambda(t) := \int_0^t \frac{\omega^{\frac{p}{2}+\frac{1}{p-1}}}{(\omega+\lambda)^{1+\frac{1}{p-1}}}\, d\omega \qquad \text{for } t \geq 0\,.$$

Moreover, for $\xi \in \mathbb{R}^n$ we define the following vector-valued function:

$$\mathcal{V}_\lambda(\xi) := \begin{cases} \mathcal{G}_\lambda((|\xi|-\lambda)_+)\,\frac{\xi}{|\xi|} & \text{if } |\xi| > \lambda, \\ 0 & \text{if } |\xi| \leq \lambda. \end{cases}$$

Notice that, for $\lambda = 0$, we have

$$\mathcal{V}_0(\xi) = \frac{2}{p}\,\mathbb{V}_p(\xi) := \frac{2}{p}\,|\xi|^{\frac{p-2}{2}}\xi\,. \tag{5.1}$$

At this point, our main results read as follows.

Theorem 5.1 ([5, Theorem 1.1])**.** *Let $n \geq 2$, $p > 2$, $\lambda \geq 0$ and $f \in B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)$. Moreover, let $u \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of equation (1.3). Then*

$$\mathcal{V}_\lambda(Du) \in W^{1,2}_{loc}(\Omega, \mathbb{R}^n)\,.$$

Furthermore, for every pair of concentric balls $B_r \subset B_R \Subset \Omega$ we have

$$\int_{B_{r/4}} |D\mathcal{V}_\lambda(Du)|^2\, dx$$

$$\leq \left(C + \frac{C}{r^2}\right)\left[1 + \lambda^p + \|Du\|^p_{L^p(B_R)} + \|f\|^{p'}_{L^{p'}(B_R)}\right] + C\,\|f\|^{p'}_{B^{\frac{p-2}{p}}_{p',1}(B_R)}$$

for a positive constant C depending only on n, p and R.

Theorem 5.2 ([5, Theorem 1.4])**.** *Let $n \geq 2$, $1 < p \leq 2$, $\lambda \geq 0$ and $f \in L^{\frac{np}{n(p-1)+2-p}}_{loc}(\Omega)$. Moreover, let $u \in W^{1,p}_{loc}(\Omega)$ be a local weak solution of equation (1.3). Then*

$$\mathcal{V}_\lambda(Du) \,\in\, W^{1,2}_{loc}(\Omega, \mathbb{R}^n)\,.$$

Furthermore, for every pair of concentric balls $B_r \subset B_R \Subset \Omega$ we have

$$\int_{B_{r/4}} |D\mathcal{V}_\lambda(Du)|^2\,dx$$
$$\leq \frac{C}{r^2}\left[1 + \lambda^p \,+\, \|Du\|^p_{L^p(B_R)} \,+ \|f\|^{p'}_{L^{\frac{np}{n(p-1)+2-p}}(B_R)}\right] + C\,\|f\|^{p'}_{L^{\frac{np}{n(p-1)+2-p}}(B_R)}$$

for a positive constant C depending only on n, p and R.

Remark 5.3. Looking at (5.1), one can easily understand that Theorem 5.1 extends the result established in [25, Remark 1.4] to a class of widely degenerate elliptic equations with standard growth, under a sharp assumption on the order of differentiation of f.

As an easy consequence of the higher differentiability results in Theorems 5.1 and 5.2, since the gradient of the solution is bounded in the region $\{|Du| \leq \lambda\}$ and $\mathcal{G}_\lambda(t) \approx t^{p/2}$ for large values of t (see [5, Lemma 2.8]), we are able to establish the following higher integrability result for the gradient of local weak solutions of (1.3):

Corollary 5.4 ([5, Corollary 1.5])**.** *Under the assumptions of Theorem 5.1 or Theorem 5.2, we have*

$$Du \in L^q_{loc}(\Omega, \mathbb{R}^n),$$

where

$$q = \begin{cases} \text{any value in } [1, \infty) & \text{if } n = 2, \\ \frac{np}{n-2} & \text{if } n \geq 3. \end{cases}$$

6. Sobolev regularity for a class of parabolic equations: part I

In this section, we are interested in the regularity properties of the weak solutions $u : \Omega_T \to \mathbb{R}$ to the strongly degenerate parabolic equation (1.6). All the results presented here are proved in [6]. The first one, given below, can be considered as the parabolic counterpart of [12, Theorem 4.2]. For notation and definitions we refer to Section 2.

Theorem 6.1 ([6, Theorem 1.1]). *Let $n \geq 2$, $p \in [2, \infty)$, $\lambda > 0$, $\frac{np+4}{np+4-n} \leq \vartheta < \infty$ and $\widetilde{f} \in L^{\vartheta}\left(0, T; W^{1,\vartheta}(\Omega)\right)$. Moreover, assume that*

$$u \in C^0\left((0,T); L^2(\Omega)\right) \cap L^p\left(0,T; W^{1,p}(\Omega)\right)$$

is a weak solution of equation (1.6). Then

$$H_{\frac{p}{2}}(Du) \in L^2_{loc}\left(0,T; W^{1,2}_{loc}(\Omega, \mathbb{R}^n)\right).$$

Furthermore, the estimate

$$\int_{Q_{\varrho/2}(z_0)} |DH_{\frac{p}{2}}(Du)|^2 \, dz \leq c\left(\lambda \, \|D\widetilde{f}\|_{L^{\vartheta}(Q_{R_0})} + \|D\widetilde{f}\|_{L^{\vartheta}(Q_{R_0})}^{\frac{np+4}{np+2-n}}\right) + \frac{c}{R^2}\left(\|Du\|^p_{L^p(Q_{R_0})} + \|Du\|^2_{L^p(Q_{R_0})} + \lambda^p + \lambda^2\right)$$

holds true for any parabolic cylinder $Q_\varrho(z_0) \subset Q_R(z_0) \subset Q_{R_0}(z_0) \Subset \Omega_T$ and a positive constant c depending at most on n, p, ϑ and R_0.

As anticipated in Section 1, from Theorem 6.1 we can deduce that any weak solution u of (1.6) admits a weak time derivative u_t, which belongs to the local Lebesgue space $L^{\min\{\vartheta, p'\}}_{loc}(\Omega_T)$. The idea is roughly as follows. Consider equation (1.6); since the above theorem tells us that in a certain pointwise sense the second spatial derivatives of u exist, we may develop the expression under the divergence symbol. This will give us an expression that equals u_t, from which we get the desired summability of the time derivative. Such an argument must be made more rigorous. Furthermore, we also need to make explicit *a priori* local estimates. These are provided in the following theorem.

Theorem 6.2 ([6, Theorem 1.2]). *Under the assumptions of Theorem 6.1, the time derivative of the solution exists in the weak sense and satisfies*

$$\partial_t u \in L^{\min\{\vartheta,\, p'\}}_{loc}(\Omega_T).$$

Furthermore, the estimate

$$\left(\int_{Q_{\varrho/2}(z_0)} |\partial_t u|^{\min\{\vartheta,\, p'\}}\, dz \right)^{\frac{1}{\min\{\vartheta,\, p'\}}}$$
$$\leq c\,\|\widetilde{f}\|_{L^\vartheta(Q_{R_0})} + c\,\|Du\|^{\frac{p-2}{2}}_{L^p(Q_{R_0})} \left(\lambda\,\|D\widetilde{f}\|_{L^\vartheta(Q_{R_0})} + \|D\widetilde{f}\|^{\frac{np+4}{np+2-n}}_{L^\vartheta(Q_{R_0})} \right)^{\frac{1}{2}}$$
$$+ \frac{c}{R} \left(\|Du\|^{2p-2}_{L^p(Q_{R_0})} + \|Du\|^{p}_{L^p(Q_{R_0})} + (\lambda^p + \lambda^2)\,\|Du\|^{p-2}_{L^p(Q_{R_0})} \right)^{\frac{1}{2}}$$

holds true for any parabolic cylinder $Q_\varrho(z_0) \subset Q_R(z_0) \subset Q_{R_0}(z_0) \Subset \Omega_T$ and a positive constant c depending on n, p, ϑ and R_0.

Remark 6.3. It is worth pointing out that, starting from the weaker assumption

$$\widetilde{f} \in L^{\frac{np+4}{np+4-n}} \left(0, T; W^{1,\frac{np+4}{np+4-n}}(\Omega)\right), \tag{6.1}$$

Sobolev regularity results such as those of Theorems 6.1 and 6.2 seemed not to have been established yet for weak solutions to parabolic PDEs that are far less degenerate than equation (1.6). In particular, the results of Theorems 6.1 and 6.2 can be easily extended to the case $\lambda = 0$, i.e. to the evolutionary p-Poisson equation

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = \widetilde{f} \quad \text{ in } \Omega_T, \tag{6.2}$$

under the assumption (6.1). Therefore, our results have improved the existing literature, already for equations of the form (6.2), which exhibit a milder degeneracy.

7. Sobolev regularity for a class of parabolic equations: part II

In this final section, we are interested in the local weak solutions $u : \Omega_T \to \mathbb{R}$ of the strongly degenerate parabolic equation (1.6). All the results presented here are proved in

[3]. There, we have established the spatial $W^{1,2}$-regularity of a nonlinear function of the spatial gradient Du of the local weak solutions to (1.6), by assuming that

- $\widetilde{f} \in L^{p'}_{loc}\left(0,T; B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)\right)$ if $p > 2$ (see Theorem 7.1);
- $\widetilde{f} \in L^2_{loc}(\Omega_T)$ if $p = 2$ (see Theorem 7.2).

The above-mentioned theorems, in turn, imply the Sobolev time regularity of the local weak solutions to the evolutionary p-Poisson equation, under the same hypotheses on the function $\widetilde{f}$ (cf. Theorem 7.4, where we only address the case $p > 2$, since the Sobolev time regularity is well known for the heat equation with source term in $L^2_{loc}(\Omega_T)$).

As already noted in Section 1, the first two theorems of this section can somewhat be considered as the parabolic analog of the elliptic results presented in Section 5, in the case $p \geq 2$. In order to state these theorems, we introduce the auxiliary function

$$\mathcal{G}_{\alpha,\lambda}(t) := \int_0^t \frac{\omega^{\frac{p-1+2\alpha}{2}}}{(\omega+\lambda)^{\frac{1+2\alpha}{2}}}\, d\omega \qquad \text{for } t \geq 0\,, \tag{7.1}$$

where $\alpha \geq 0$. Moreover, for $\xi \in \mathbb{R}^n$ we define the following vector field:

$$\mathcal{V}_{\alpha,\lambda}(\xi) := \begin{cases} \mathcal{G}_{\alpha,\lambda}((|\xi|-\lambda)_+)\,\frac{\xi}{|\xi|} & \text{if } |\xi| > \lambda, \\ 0 & \text{if } |\xi| \leq \lambda. \end{cases} \tag{7.2}$$

At this point, our main results read as follows.

Theorem 7.1 ([3, Theorem 1.1]). *Let $n \geq 2$, $p > 2$, $\lambda \geq 0$ and $\widetilde{f} \in L^{p'}_{loc}\left(0,T; B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)\right)$. Moreover, let*

$$\alpha = \begin{cases} 0 & \textit{if } \lambda = 0, \\ \textit{any value in } \left[\frac{p+1}{2(p-1)}, \infty\right) & \textit{if } \lambda > 0, \end{cases} \tag{7.3}$$

and assume that

$$u \in C^0\left((0,T); L^2(\Omega)\right) \cap L^p_{loc}\left(0,T; W^{1,p}_{loc}(\Omega)\right)$$

is a local weak solution of equation (1.6). Then

$$\mathcal{V}_{\alpha,\lambda}(Du) \,\in\, L^2_{loc}\left(0,T; W^{1,2}_{loc}(\Omega,\mathbb{R}^n)\right),$$

where the function $\mathcal{V}_{\alpha,\lambda}$ is defined according to (7.1)−(7.2). Furthermore, for any parabolic cylinder $Q_r(z_0) \subset Q_\rho(z_0) \subset Q_R(z_0) \Subset \Omega_T$ we have

$$\int_{Q_{r/2}(z_0)} |D_x \mathcal{V}_{\alpha,\lambda}(Du)|^2 \, dz \le \left(C + \frac{C}{\rho^2} \right) \left[\|Du\|^p_{L^p(Q_R)} + \|Du\|^2_{L^p(Q_R)} + \lambda^p + \lambda^2 + 1 \right] + C \, \|\widetilde{f}\|^{p'}_{L^{p'}\left(t_0 - R^2, t_0; B^{\frac{p-2}{p}}_{p',1}(B_R(x_0)) \right)}$$

for a positive constant C depending only on n, p and R in the case $\lambda = 0$, and additionally on α if $\lambda > 0$. Besides, if $\lambda = 0$ we get

$$Du \in L^p_{loc}(0, T; W^{\sigma,p}_{loc}(\Omega, \mathbb{R}^n)) \qquad \text{for all } \sigma \in \left(0, \frac{2}{p} \right).$$

Theorem 7.2 ([3, Theorem 1.3])**.** *Let $n \ge 2$, $\lambda \ge 0$ and $\widetilde{f} \in L^2_{loc}(\Omega_T)$. Moreover, let*

$$\alpha = \begin{cases} 0 & \text{if } \lambda = 0, \\ \text{any value in } \left[\frac{3}{2}, \infty\right) & \text{if } \lambda > 0, \end{cases} \tag{7.4}$$

and assume that

$$u \in C^0\left((0,T); L^2(\Omega) \right) \cap L^2_{loc}\left(0, T; W^{1,2}_{loc}(\Omega) \right)$$

is a local weak solution of the equation

$$u_t - \operatorname{div}\left((|Du| - \lambda)_+ \frac{Du}{|Du|} \right) = \widetilde{f} \quad \text{in } \Omega_T.$$

Then

$$\mathcal{V}_{\alpha,\lambda}(Du) \in L^2_{loc}\left(0, T; W^{1,2}_{loc}(\Omega, \mathbb{R}^n) \right).$$

Furthermore, for any parabolic cylinder $Q_r(z_0) \subset Q_\rho(z_0) \subset Q_R(z_0) \Subset \Omega_T$ we have

$$\int_{Q_{r/2}(z_0)} |D_x \mathcal{V}_{\alpha,\lambda}(Du)|^2 \, dz \le \frac{C}{\rho^2} \left(\|Du\|^2_{L^2(Q_R)} + \lambda^2 + 1 \right) + C \left(\|\widetilde{f}\|^2_{L^2(Q_R)} + 1 \right)$$

for a positive constant C depending only on n and R in the case $\lambda = 0$, and additionally on α if $\lambda > 0$.

Remark 7.3. Notice that, for every $\alpha \geq 0$, we have

$$\mathcal{V}_{\alpha,0}(\xi) = \frac{2}{p}\,\mathbb{V}_p(\xi) := \frac{2}{p}\,|\xi|^{\frac{p-2}{2}}\xi\,, \tag{7.5}$$

i.e. $\mathcal{V}_{\alpha,0}$ is actually independent of the parameter α, which explains the choices $(7.3)_1$ and $(7.4)_1$ in the statements of the above theorems. The conditions $(7.3)_2$ and $(7.4)_2$ are instead needed to carry out the proofs of Theorems 7.1 and 7.2 (see [3] for more details). Moreover, looking at (7.5), one can easily understand that, on the one hand, Theorem 7.1 extends the result established in [19, Lemma 5.1] for the parabolic p-Laplace equation to a widely degenerate parabolic setting. On the other hand, it extends the aforementioned result to the case of data in a suitable Lebesgue-Besov parabolic space, which turns out to be optimal, as can be seen by appropriately adapting the example in [13, Section 5] to the parabolic context (in this regard, see also [5, page 3]).

For $p > 2$, we now consider the evolutionary p-Poisson equation

$$u_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = \widetilde{f} \quad \text{in } \Omega_T\,, \tag{7.6}$$

which is obtained from equation (1.6) by setting $\lambda = 0$. From Theorem 7.1, one can easily deduce that the local weak solutions of (7.6) admit a weak time derivative which belongs to the local Lebesgue space $L^{p'}_{loc}(\Omega_T)$. The idea is essentially the same as in the proof of Theorem 6.2 and leads us to the following result.

Theorem 7.4 ([3, Theorem 1.5])**.** *Let $n \geq 2$, $p > 2$ and $\widetilde{f} \in L^{p'}_{loc}\left(0,T; B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)\right)$. Moreover, assume that*

$$u \in C^0\left((0,T); L^2(\Omega)\right) \cap L^p_{loc}\left(0,T; W^{1,p}_{loc}(\Omega)\right)$$

is a local weak solution of equation (7.6). Then, the time derivative of u exists in the weak sense and satisfies

$$\partial_t u \,\in\, L^{p'}_{loc}(\Omega_T).$$

Furthermore, for any parabolic cylinder $Q_r(z_0) \subset Q_\rho(z_0) \subset Q_R(z_0) \Subset \Omega_T$ we have

$$\left(\int_{Q_{r/2}(z_0)} |\partial_t u|^{p'}\,dz\right)^{\frac{1}{p'}} \leq \left(C + \frac{C}{\rho}\right)\left[\|Du\|^{p-1}_{L^p(Q_R)} + \|Du\|^{\frac{p}{2}}_{L^p(Q_R)} + \|Du\|^{\frac{p-2}{2}}_{L^p(Q_R)}\right]$$

$$+\ C\,\|Du\|_{L^p(Q_R)}^{\frac{p-2}{2}}\,\|\widetilde{f}\|_{L^{p'}\left(t_0-R^2,t_0;B^{\frac{p-2}{p}}_{p',1}(B_R(x_0))\right)}^{\frac{p'}{2}}+\|\widetilde{f}\|_{L^{p'}(Q_R)}$$

for a positive constant C depending only on n, p and R.

Remark 7.5. It is worth pointing out that, starting from the weaker assumption

$$\widetilde{f}\in L^{p'}_{loc}\left(0,T;B^{\frac{p-2}{p}}_{p',1,loc}(\Omega)\right)\qquad\text{with }\ p>2,$$

Sobolev regularity results such as those of Theorems 7.1 and 7.4 seemed not to have been established yet for weak solutions to parabolic PDEs that are far less degenerate than equation (1.6) with $\lambda>0$. In particular, the results of Theorems 7.1 and 7.4 permit to improve the existing literature, already for the evolutionary p-Poisson equation (7.6), which exhibits a milder degeneracy. Moreover, as far as we know, for $\lambda>0$ the result of Theorem 7.2 is completely new.

Acknowledgments. The present note is based on a talk given by the author at the University of Bologna in October 2025, in the series *Seminari di Analisi Matematica Bruno Pini*. The author is a member of the Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni (GNAMPA) of the Istituto Nazionale di Alta Matematica (INdAM), and is partially supported by the INdAM–GNAMPA 2025 Project "Regolarità ed esistenza per operatori anisotropi" (CUP E5324001950001). The author also acknowledges financial support from the IADE_CITTI_2020 Project "Intersectorial applications of differential equations" (CUP J34I20000980006).

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Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna, Italy.

Email address: pasquale.ambrosio@unibo.it