FULLY NONLINEAR EQUATIONS WITH SINGULAR POTENTIALS IN PUNCTURED BALLS

EQUAZIONI COMPLETAMENTE NON LINEARI CON POTENZIALI SINGOLARI IN SFERE BUCATE

FABIANA LEONI

Abstract. We present the results contained in [6, 7], concerning radial solutions of fully nonlinear, uniformly elliptic equations posed in punctured balls, in presence of radial singular quadratic potentials. We discuss both the principal eigenvalues problem and the case of equations having also absorbing superlinear zero order terms: for the former problem, we explicitly compute the principal eigenvalues, thus obtaining an extension to the fully nonlinear framework of the Hardy-Sobolev constant; for the latter case, we provide a complete classification of solutions based on their asymptotic behavior near the singularity.

Sunto. Vengono presentati i risultati contenuti in [6, 7], riguardanti soluzioni radiali di equazioni uniformemente ellittiche completamente non lineari, in cui compaiono potenziali singolari radiali quadratici. Si considerano sia il problema agli autovalori principali sia il caso di equazioni aventi anche termini di assorbimento superlineari di ordine zero: per il primo problema, viene dato il valore esplicito dell'autovalore principale, che estende al caso completamente non lineare la cosiddetta costante di Hardy-Sobolev; per il secondo problema, viene presentato un risultato di classificazione di tutte le soluzioni radiali in base al loro comportamento asintotico nella singolarità.

2020 MSC. 35J60; 35J75; 35P15; 35P30; 35B40.

Keywords. Fully nonlinear elliptic equations; singular potential; principal eigenvalues; superlinear absorbing terms.

Bruno Pini Mathematical Analysis Seminar, Vol. 16, No. 1 (2025) pp. 27-40

Dipartimento di Matematica, Università di Bologna

ISSN 2240-2829.

1. Introduction

The aim of the present paper is to report on the recent results obtained in [6, 7] about two strongly related problems involving second order, fully nonlinear, uniformly elliptic equations posed in the punctured unit ball. Namely, we focus on radial solutions of the eigenvalue problem

$$\begin{cases} -\mathcal{M}^{\pm}(D^2u) = \bar{\lambda}\dfrac{u}{|x|^2} & \text{in } B \setminus \{0\} \\ u > 0 \text{ in } B \setminus \{0\}\,, u = 0 & \text{on } \partial B \end{cases} \tag{1}$$

and of the equations containing also a p-power superlinear zero order terms

$$\begin{cases} -\mathcal{M}^{\pm}(D^2u) + u^p = \mu\dfrac{u}{|x|^2} & \text{in } B \setminus \{0\} \\ u > 0 \text{ in } B \setminus \{0\} \end{cases} \tag{2}$$

both characterized by the presence of the singular inverse quadratic potential $|x|^{-2}$.

We use the notation

- $B = B_1(0)$ is the unit ball in $\mathbb{R}^n$;
- $\bar{\lambda} \in \mathbb{R}$ is the principal eigenvalue;
- $\mu \in \mathbb{R}$ is a parameter which will be related to $\bar{\lambda}$;
- $p > 1$ is the exponent of the superlinear absorbing zero order term;
- $\mathcal{M}^{\pm}_{\lambda,\Lambda}$ are Pucci's extremal operators with ellipticity constants $\Lambda \geq \lambda > 0$.

We recall that Pucci's extremal operators $\mathcal{M}^{-}_{\lambda,\Lambda}$ and $\mathcal{M}^{+}_{\lambda,\Lambda}$ are respectively defined as

$$\mathcal{M}^{-}_{\lambda,\Lambda}(X) = \inf_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AX) = \lambda \sum_{\mu_i > 0} \mu_i + \Lambda \sum_{\mu_i < 0} \mu_i$$

$$\mathcal{M}^{+}_{\lambda,\Lambda}(X) = \sup_{A \in \mathcal{A}_{\lambda,\Lambda}} \operatorname{tr}(AX) = \Lambda \sum_{\mu_i > 0} \mu_i + \lambda \sum_{\mu_i < 0} \mu_i$$

where $\mathcal{A}_{\lambda,\Lambda} = \{A \in \mathcal{S}_n \,:\, \lambda\, I_n \leq A \leq \Lambda\, I_n\}$, $\mathcal{S}_n$ being the space of symmetric matrices of order n and I_n the identity matrix, and $\mu_1, \ldots, \mu_n$ are the eigenvalues of any matrix $X \in \mathcal{S}_n$.

Originally introduced in [21], Pucci's extremal operators are the prototypes of second order, fully nonlinear, uniformly elliptic operators. They naturally arise in the context of nonlinear PDEs, as particular diffusion models in elasticity and fluid dynamics. We recall that they act as barriers in the whole class of uniformly elliptic operators having the same ellipticity constants, that is

$$\mathcal{M}^-_{\lambda,\Lambda}(X) \leq F(X) \leq \mathcal{M}^+_{\lambda,\Lambda}(X)$$

for any uniformly elliptic F having ellipticity constants Λ, λ and satisfying $F(O) = 0$, with O being the null matrix. As a consequence, they play a crucial role in the regularity theory for fully nonlinear elliptic equations, see [9]. Moreover, they find some applications in specific stochastic differential equations, free boundary problems and optimal stopping problems that model some financial phenomena, see e.g. [12, 18].

Pucci's extremal operators are mutually related by the identity

$$\mathcal{M}^-_{\lambda,\Lambda}(-X) = -\mathcal{M}^+_{\lambda,\Lambda}(X)$$

and they both can be seen as a generalization of Laplace operator

$$\mathcal{M}^-_{\lambda,\lambda}(X) = \mathcal{M}^+_{\lambda,\lambda}(X) = \lambda\,\mathrm{tr}(X)$$

Let us emphasize that, for $\lambda < \Lambda$, operators $\mathcal{M}^\pm_{\lambda,\lambda}$ are neither linear nor in divergence form. As an example, we observe that the homogeneous plane equation

$$\mathcal{M}^-_{\lambda,\Lambda}(D^2u) = 0 \quad \text{in } \mathbb{R}^2$$

is equivalent to

$$\Delta u = \left(\sqrt{\frac{\Lambda}{\lambda}} - \sqrt{\frac{\lambda}{\Lambda}}\right)\sqrt{-\det(D^2u)} \quad \text{in } \mathbb{R}^2 .$$

However, uniform ellipticity is preserved. Moreover, when evaluated on smooth radial functions, Pucci's operators take an explicit expression which resembles that of the Laplacian and leads to ordinary differential equations with piecewise constant coefficients. Indeed, for smooth, radial functions $u(x) = u(|x|)$, Pucci's operators can be written as

$$\begin{aligned}\mathcal{M}^+(D^2u) = {} & \Lambda(u''(r))^+ - \lambda(u''(r))^- \\ & +\Lambda(n-1)\left(\frac{u'(r)}{r}\right)^+ - \lambda(n-1)\left(\frac{u'(r)}{r}\right)^- ,\end{aligned}$$

as well as

$$\mathcal{M}^-(D^2u) = \lambda(u''(r))^+ - \Lambda(u''(r))^- \\ +\lambda(n-1)\left(\frac{u'(r)}{r}\right)^+ - \Lambda(n-1)\left(\frac{u'(r)}{r}\right)^- .$$

Thus, the ODEs satisfied by radial solutions u of Pucci's extremal equations have coefficients jumping at the points where u changes monotonicity and/or convexity. For fixed monotonicity/convexity regime, the obtained ODEs are analogous to the ones associated with Laplace operator. The coefficients are replaced by new ones depending on the dimension like parameters

$$\tilde{n}_+ = \frac{\lambda}{\Lambda}(n-1)+1\,, \quad \tilde{n}_- = \frac{\Lambda}{\lambda}(n-1)+1\,.$$

We will refer to $\tilde{n}_\pm$ as the effective dimension associated with $\mathcal{M}^\pm$ respectively. Note that one has always

$$\tilde{n}_- \geq n \geq \tilde{n}_+\,,$$

with equalities holding true if and only if $\Lambda = \lambda$. We will assume always that $\tilde{n}_+ > 2$.

For problem (1), we are interested in the existence of constant sign solutions. In this case, the value $\bar{\lambda}$ is referred to as the principal eigenvalue. The main difficulty we have to face is the proof of existence and comparison results for unbounded solutions. We will reach the conclusion by using a sort of variational characterization and an approximation argument by considering subquadratic singular potentials $|x|^{-\gamma}$, with $0 < \gamma < 2$. In case of Laplace operator, the principal eigenvalue $\bar{\lambda}$ is explicitly known, and it is the inverse of the best constant in Hardy's inequality. For Pucci's operators, we obtain here an analogous explicit expression which extends to the fully nonlinear setting the best constant in Hardy's inequality. The results are presented in Section 2.

For problem (2), we are concerned with the classification of all possible radial solutions based on their asymptotic behavior around the origin. In case of Laplace operator, the problem has been extensively studied, see [10], but, once again, the non variational structure of Pucci's operators makes previously developed energy based methods not directly applicable to the fully nonlinear setting. We will use then ad hoc techniques: construction of local barriers, new comparison results for singular solutions and use of the Emden–Fowler transformation in order to apply the theory of planar autonomous dynamical systems to the equation satisfied by the new unknown. The results are described in Section 3.

2. The principal eigenvalues problem.

In the linear case $\Lambda = \lambda$, up to a constant factor, problem (1) becomes

$$\begin{cases} -\Delta u = \bar{\lambda}\dfrac{u}{|x|^2} & \text{in } B \setminus \{0\} \\ \\ u > 0 \text{ in } B \setminus \{0\}\,,\ u = 0 & \text{on } \partial B \end{cases}$$

which can be solved by considering the minimum problem

$$\bar{\lambda} := \inf_{u \in H_0^1(B), \int_B \frac{|u(x)|^2}{|x|^2} = 1} \int_B |\nabla u|^2 \, dx$$

i.e. by minimizing the normalized related Rayleigh quotient.

The problem is well studied and $\bar{\lambda}$ is nothing but the inverse of the best constant in Hardy's inequality, that is

$$\bar{\lambda} = \frac{(n-2)^2}{4}\,.$$

However, due to the lack of compactness of the inclusion of $H_0^1(B)$ into $L^2\left(B, \frac{1}{|x|^2}\right)$, $\bar{\lambda}$ is not a minimum and the problem has no finite energy solutions. An explicit radial solution, not belonging to $H_0^1(B)$ and usually called *virtual minimizer*, is

$$u(x) = \frac{(-\ln|x|)}{|x|^{\frac{n-2}{2}}}\,.$$

The relevant literature in the linear case is extremely extensive. We quote here only the monographs [10, 20] for Hardy type inequalities and related elliptic (and parabolic) problems involving inverse squared potential.

In the fully nonlinear case, we can follow the same approach used to study the principal eigenvalue problem for regular potentials.

Let us define the eigenvalue on the model of [1], i.e. by the optimization formula

$$\begin{aligned} \bar{\lambda} := \sup\{\mu \,:\, \exists\, u \in C(B \setminus \{0\})\,, \\ u > 0 \text{ in } B \setminus \{0\}, \mathcal{M}^{\pm}(D^2u) + \mu\frac{u}{|x|^2} \le 0\} \end{aligned}$$

This is the standard approach used to study eigenvalues problems in the fully nonlinear framework, see e.g. [19, 8, 17, 4, 5, 22, 2].

Having in mind the results in the linear case, one immediately realizes that also for the operators $\mathcal{M}^\pm$ there are explicit solutions of the eigenvalue problem. Namely,

$$u(r) = \frac{-\ln r}{r^{\frac{\tilde{n}_+-2}{2}}} \Longrightarrow -\mathcal{M}^+(D^2u) = \Lambda\left(\frac{\tilde{n}_+-2}{2}\right)^2 \frac{u}{r^2}$$

$$u(r) = \frac{-\ln r}{r^{\frac{\tilde{n}_--2}{2}}} \Longrightarrow -\mathcal{M}^-(D^2u) = \lambda\left(\frac{\tilde{n}_--2}{2}\right)^2 \frac{u}{r^2}$$

Then, by definition, one has

$$\Lambda\left(\frac{\tilde{n}_+-2}{2}\right)^2 \leq \bar{\lambda}(\mathcal{M}^+)\,, \quad \lambda\left(\frac{\tilde{n}_--2}{2}\right)^2 \leq \bar{\lambda}(\mathcal{M}^-)\,.$$

In case of regular coefficients and bounded solutions, the inequalities above are usually sufficient in order to establish equalities. Indeed, a usual argument by contradiction is: assume that the inequalities above are strict. Then, there exist positive supersolutions at values $\mu = \Lambda\left(\frac{\tilde{n}_+-2}{2}\right)^2$ and $\mu = \lambda\left(\frac{\tilde{n}_--2}{2}\right)^2$ respectively, and, as a consequence, maximum principle holds true for subsolutions of problem (1) with these values of μ. This yields a contradiction with the existence of positive solutions vanishing on the boundary. The above argument fails in the present case because of the lacking of the maximum principle for unbounded singular solutions, even if there exist positive supersolutions.

In order to establish equalities, we pursue the analogy with the linear case, fully taking advantage of the radial symmetry. We explain the argument used for $\mathcal{M}^+$, the case of $\mathcal{M}^-$ being completely analogous.

Let us introduce the space of functions

$$\mathcal{V} = \left\{u \in C^2([0,1]) \,:\, u'(0) = 0\,,\ \mathrm{supp}(u) \text{ compact in } [0,1)\right\}\,,$$

endowed with the norm

$$\|u\| = \left(\int_0^1 |u'|^2 r^{\tilde{n}_+-1} dr\right)^{1/2}\,,$$

and let us denote by $\mathcal{H}_0^1$ the closure of $\mathcal{V}$. Then, for all $\gamma \leq 2$, we can consider the minimum problem

$$\lambda_{\gamma,var} := \inf_{\substack{u\in\mathcal{H}_0^1 \\ \int_0^1 u^2 r^{\tilde{n}_+-1-\gamma} dr = 1}} \int_0^1 |u'|^2 r^{\tilde{n}_+-1} dr\,.$$

Arguing exactly as in the linear case, one easily proves that

$$\lambda_{2,var} = \left(\frac{\tilde{n}_+ - 2}{2}\right)^2 .$$

Furthermore, one has

$$\lambda_{\gamma,var} \to \lambda_{2,var} \qquad \text{as } \gamma \to 2 .$$

Indeed, one can first observe that, by their own definition, $\lambda_{2,var} \leq \lambda_{\gamma,var}$.

On the other hand, for any $\epsilon > 0$ there exists $v \in \mathcal{H}_0^1$ such that

$$\int_0^1 |v'|^2 r^{\tilde{n}_+-1} dr \leq (\lambda_{2,var} + \epsilon) \int_0^1 |v|^2 r^{\tilde{n}_+-3} dr .$$

Moreover, there exists γ_0 sufficiently close to 2 in order that, for $\gamma \geq \gamma_0$,

$$\int_0^1 |v|^2 r^{\tilde{n}_+-1-\gamma} dr \geq (1-\epsilon) \int_0^1 |v|^2 r^{\tilde{n}_+-3} dr .$$

Thus, one has

$$\int_0^1 |v'|^2 r^{\tilde{n}_+-1} dr \leq (\lambda_{2,var} + \epsilon)(1-\epsilon)^{-1} \int_0^1 |v|^2 r^{\tilde{n}_+-1-\gamma} dr$$

which yields

$$\lambda_{\gamma,var} \leq (\lambda_{2,var} + \epsilon)(1-\epsilon)^{-1} .$$

For $1 \leq \gamma < 2$ the variational formulation produces a solution of the ODE which is always decreasing and convex, so that it is a radial solution for Pucci's operators, i.e. a solution to

$$(1) \qquad \begin{cases} -\mathcal{M}^{\pm}(D^2 u) = \Lambda \, \lambda_{\gamma,var} \dfrac{u}{|x|^\gamma} & \text{in } B \setminus \{0\} \\ u > 0 \text{ in } B \setminus \{0\} \, , u = 0 & \text{on } \partial B \end{cases}$$

For *subquadratic potentials* we have a comparison principle for smooth, radial, *bounded* sub and super-solutions which does not require any condition at the origin.

Theorem 2.1. *Let $f \in C(B)$ and $u, v \in C\left(\overline{B}\right) \cap C^2(B \setminus \{0\})$ be radial functions satisfying*

$$F(D^2 u) \geq f(r) r^{-\gamma} \geq F(D^2 v) \qquad \text{in } B \setminus \{0\} ,$$

where F is any second order fully nonlinear uniformly elliptic operator. Then, $u \leq v$ on ∂B implies $u \leq v$ in $\overline{B}$.

We emphasize that sub and super solutions may be really singular at the origin, so that the result is not due to the removability of the singularity. For a proof of the above theorem we refer to [6].

The analysis we performed in the subquadratic case $\gamma < 2$ guarantees the validity of the maximum principle "below" the eigenvalue, and we can conclude e.g. for operator $\mathcal{M}^+_{\lambda,\Lambda}$

$$1 \leq \gamma < 2 \Longrightarrow \bar{\lambda}_\gamma = \Lambda\, \lambda_{\gamma,var}\,.$$

Hence, by using also the monotonicity of the eigenvalue with respect to the potential, we deduce

$$\bar{\lambda} \leq \bar{\lambda}_\gamma = \Lambda\, \lambda_{\gamma,var} \stackrel{\gamma\to 2}{\to} \Lambda\, \lambda_{2,var} = \Lambda \left(\frac{\tilde{n}_+ - 2}{2}\right)^2 \leq \bar{\lambda}\,.$$

Perfectly analogous results can be obtained for the operator $\mathcal{M}^-_{\lambda,\Lambda}$, yielding the following theorem.

Theorem 2.2. *For the eigenvalue problem (1) one has:*

(i) $\bar{\lambda}(\mathcal{M}^+) = \Lambda \left(\frac{\tilde{n}_+ - 2}{2}\right)^2, \quad \bar{\lambda}(\mathcal{M}^-) = \lambda \left(\frac{\tilde{n}_- - 2}{2}\right)^2$ *and the functions* $u^\pm(x) = \frac{-\ln r}{r^{\frac{\tilde{n}_\pm - 2}{2}}}$ *are related explicit eigenfunctions;*

(ii) $\bar{\lambda}(\mathcal{M}^\pm)$ *are stable with respect to potential and domain approximations:*

$$\bar{\lambda}(\mathcal{M}^\pm) = \lim_{\gamma\to 2} \bar{\lambda}_\gamma(\mathcal{M}^\pm)$$

$$\bar{\lambda}(\mathcal{M}^\pm) = \lim_{\delta\to 0} \bar{\lambda}(\mathcal{M}^\pm, B \setminus \overline{B_\delta})$$

$$\bar{\lambda}(\mathcal{M}^\pm) = \lim_{\epsilon\to 0} \bar{\lambda}\left(\mathcal{M}^\pm, \frac{1}{r^2 + \epsilon^2}\right)$$

(iii) *for any uniformly elliptic operator F having ellipticity constants $\lambda \leq \Lambda$, one has*

$$\Lambda \left(\frac{\tilde{n}_+ - 2}{2}\right)^2 \leq \bar{\lambda}(F) \leq \lambda \left(\frac{\tilde{n}_- - 2}{2}\right)^2$$

A complete proof can be found in [6].

3. Equations with superlinear absorbing terms.

For problem (2), we are interested in existence of positive radial solutions and their asymptotic behavior as $r \to 0$. Let us state here the results we obtained in the case $0 < \mu < \bar{\lambda}$ and for the operator $\mathcal{M}^+$.

We start by observing that there are three growth exponents naturally associated with the equation

$$-\mathcal{M}^+(D^2u) + u^p = \mu \frac{u}{|x|^2} \quad \text{in } B \setminus \{0\}$$

The first one is the exponent of the *scaling invariance* of the equation, namely

$$\frac{2}{p-1}$$

Indeed, if u is a solution, then the same holds true for

$$u_\alpha(r) = \alpha^{\frac{2}{1-p}} u\left(\frac{r}{\alpha}\right) \qquad \text{for all } \alpha > 0\,.$$

The other two exponents are the growth exponents of the power functions which are solutions of the equation

$$-\mathcal{M}^+(D^2u) = \mu \frac{u}{|x|^2} \quad \text{in } B \setminus \{0\}\,,$$

namely the roots of the algebraic equation

$$\gamma^2 - (\tilde{n}_+ - 2)\gamma + \frac{\mu}{\Lambda} = 0$$

which has the two positive solutions

$$\tau^\pm = \frac{\tilde{n}_+ - 2}{2} \pm \sqrt{\left(\frac{\tilde{n}_+ - 2}{2}\right)^2 - \frac{\mu}{\Lambda}}\,,$$

clearly satisfying

$$0 < \tau^- < \tau^+\,.$$

The functions $\frac{1}{|x|^{\tau^+}}$ and $\frac{1}{|x|^{\tau^-}}$ are sometimes referred to as *fundamental solutions* of the equation

$$-\mathcal{M}^+(D^2u) = \mu \frac{u}{|x|^2} \quad \text{in } B \setminus \{0\}\,.$$

The three exponents $\frac{2}{p-1}$, τ^+ and τ^- describe all the possible behaviors of solutions of equation (2) around the origin, according to the following result.

Theorem 3.1. *Let u be a classical radial positive solution of (2). Then, for some $c>0$, as $r\to 0$ one has*

(i) *if $1<p<1+\frac{2}{\tau^+}$, i.e. when $\tau^-<\tau^+<\frac{2}{p-1}$, then*

$$\text{either } u(r)r^{\tau^-}\to c \text{ or } u(r)r^{\tau^+}\to c \text{ or } u(r)r^{\frac{2}{p-1}}\to c$$

(ii) *if $1+\frac{2}{\tau^+}\leq p<1+\frac{2}{\tau^-}$, i.e. when $\tau^-<\frac{2}{p-1}\leq\tau^+$, then*

$$u(r)r^{\tau^-}\to c$$

(iii) *if $p=1+\frac{2}{\tau^-}$, i.e. when $\tau^-=\frac{2}{p-1}$, then*

$$u(r)r^{\tau^-}|\ln r|^{\frac{\tau^-}{2}}\to c$$

(iv) *if $p>1+\frac{2}{\tau^-}$, i.e. when $\frac{2}{p-1}<\tau^-<\tau^+$, then*

$$u(r)r^{\frac{2}{p-1}}\to c$$

For the proof, we refer to [7]. Let us recall here its basic ingredients.

The first essential tool in our proof is a new comparison principle facing the problem of unbounded solutions. It relies on the absorbing capability of the superlinear zero order term and, even if we state here the result for the operator $\mathcal{M}^+$, it holds true for any second order degenerate elliptic operator. See [7] for the proof.

Theorem 3.2. *Let $\mu>0$, $p>1$ and let u and v be two positive radial functions in $C^2(B\setminus\{0\})$ satisfying*

$$\mathcal{M}^+(D^2u)+\mu\frac{u}{r^2}-u^p\geq 0\geq\mathcal{M}^+(D^2v)+\mu\frac{v}{r^2}-v^p \qquad \text{in } B\setminus\{0\}\,.$$

Assume that $u\,,\ v\in C\left(\overline{B}\setminus\{0\}\right)$ and that there exist positive constants c_1,c_2 such that

$$c_1r^{-\frac{2}{p-1}}\leq u(r)\,,v(r)\leq c_2r^{-\frac{2}{p-1}} \qquad \text{for } \ 0<r\leq r_0\leq 1\,.$$

If $u(1)\leq v(1)$, then $u\leq v$ in $\overline{B}\setminus\{0\}$.

Another tool we employ in the proof of Theorem 3.1 is the change of variable given by the classical Emden-Fowler transformed, see [13], which is given by

$$x(t)=e^{\frac{2}{p-1}t}u(e^t) \qquad t\leq 0\,.$$

The Emden-Fowler transformed is classically used to study semilinear homogeneous equations. In the fully nonlinear setting, it has been firstly used for radial solutions of equations involving Pucci's extremal operators in [11], and subsequently applied in [3, 14, 15].

In order to explain the usefulness of such a change of variable, let us show the computation for semilinear equations. We observe that if u is a radial solution of

$$-\Delta u + u^p = \mu \frac{u}{r^2},$$

then the new unknown $x(t)$ satisfies the autonomous equation

$$x'' + \left(N - 2 - \frac{4}{p-1}\right) x' + \left(\frac{4}{(p-1)^2} - (N-2)\frac{2}{p-1} + \mu\right) x = x^p$$

or, equivalently,

$$x'' + 2\left(\frac{N-2}{2} - \frac{2}{p-1}\right) x' + \left(\frac{2}{p-1} - \tau^-\right)\left(\frac{2}{p-1} - \tau^+\right) x = x^p$$

The asymptotic analysis for $t \to -\infty$ of solutions to the above equation above equation can be performed by applying the theory of planar autonomous dynamical systems, in particular the Poincaré-Bendixson theorem, see [16], yielding that a trajectory $(x(t), x'(t))$ converges to an equilibrium point of the given equation as $t \to -\infty$.

In the considered case, the equilibria are

$$x = 0,$$

and

$$x = c^* = \left(\left(\frac{2}{p-1} - \tau^+\right)\left(\frac{2}{p-1} - \tau^-\right)\right)^{\frac{1}{p-1}},$$

the latter existing provided that either $\frac{2}{p-1} < \tau^-$ or $\frac{2}{p-1} > \tau^+$.

Now, one has

$$x(t) \to c^* \text{ as } t \to -\infty \iff u(r) r^{\frac{2}{p-1}} \to c^* \text{ as } r \to 0,$$

whereas if $x(t) \to 0$ as $t \to -\infty$, then $x(t)$ is asymptotic to one of the solutions of the linearized equation exiting from $(0, 0)$, provided that they exist.

The homogeneous equation

$$x'' + 2\left(\frac{N-2}{2} - \frac{2}{p-1}\right) x' + \left(\frac{2}{p-1} - \tau^-\right)\left(\frac{2}{p-1} - \tau^+\right) x = 0$$

has the solutions

$$e^{\lambda_1 t}\,, \qquad e^{\lambda_2 t}$$

with

$$\lambda_1 = \frac{2}{p-1} - \tau^+\,, \quad \lambda_2 = \frac{2}{p-1} - \tau^-$$

so that

$$x(t)e^{-\lambda_1 t} \to c \text{ as } t \to -\infty \Longleftrightarrow u(r) r^{\tau^+} \to c \text{ as } r \to 0$$

as well as

$$x(t)e^{-\lambda_2 t} \to c \text{ as } t \to -\infty \Longleftrightarrow u(r) r^{\tau^-} \to c \text{ as } r \to 0$$

The possible different behavior of $u(r)$ as $r \to 0$ then can be explained as consequences of the possibilities

$$x(t) \to c^*$$

or

$$x(t)e^{-\lambda_1 t} \to c$$

or

$$x(t)e^{-\lambda_2 t} \to c$$

as $t \to -\infty$, which can occur only if $c^* > 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$. We refer for details to the proofs given in [7].

References

[1] H. Berestycki, L. Nirenberg, S.R.S. Varadhan, *The principal eigenvalue and maximum principle for second-order elliptic operators in general domains.* Comm. Pure Appl. Math. **47** (1) (1994), 47–92.

[2] H. Berestycki, I. Capuzzo Dolcetta, A. Porretta, L. Rossi, *Maximum Principle and generalized principal eigenvalue for degenerate elliptic operators*, J. Math. Pures Appl. **103** (5) (2015), 1276–1293.

[3] I. Birindelli, G. Galise, F. Leoni, F. Pacella, *Concentration and energy invariance for a class of fully nonlinear elliptic equations*, Calc. Var. Partial Differential Equations **57** (2018), 22 pp.

[4] I. Birindelli, F. Demengel, *Eigenvalue, maximum principle and regularity for fully nonlinear homogeneous operators*, Commun. Pure Appl. Anal. **6** (2007), 335–366.

[5] I. Birindelli, F. Demengel, *Regularity and uniqueness of the first eigenfunction for singular fully nonlinear operators*, J. Differential Equations **249** (2010), 1089–1110.

[6] I. Birindelli, F. Demengel, F. Leoni. *Principal eigenvalues and eigenfunctions for fully nonlinear equations in punctured balls.* J. Math. Pures Appl. **186** (2024) 74–102.

[7] I. Birindelli, F. Demengel, F. Leoni. *Radial singular solutions of fully nonlinear equations in punctured balls.* submitted paper, arXiv:2408.13550 (2024)

[8] J. Busca, M. Esteban, A. Quaas, *Nonlinear eigenvalues and bifurcation problems for Pucci's operator*, Ann. Inst. H. Poincarè Anal. Non Lineaire **22** (2) (2005), 187–206.

[9] X. Cabré, L. Caffarelli, Fully nonlinear elliptic equations, American Mathematical Society Colloquium Publications vol. 43, 1995.

[10] F.C. Cirstea, A complete classification of the isolated singularities for nonlinear elliptic equations with inverse square potentials, Memoirs of AMS, vol. 227 (1068), 2014.

[11] P. L. Felmer, A. Quaas, *On critical exponents for the Pucci's extremal operators*, Ann. Inst. H. Poincaré Anal. Non Linéaire **20** (2003), no. 5, 843–865.

[12] W. Fleming, H. Soner, Controlled Markov Processes and Viscosity Solutions, Springer-Verlag, New York, 1992.

[13] R.H. Fowler, *Further studies on Emden's and similar differential equations*, Q. J. Math. **2** (1931), 259–288.

[14] G. Galise, A. Iacopetti, F. Leoni, *Liouville-type results in exterior domains for radial solutions of fully nonlinear equations*, J. Differential Equations **269**, no. 6 (2020), 5034–506.

[15] G. Galise, A. Iacopetti, F. Leoni, F. Pacella, *New concentration phenomena for a class of radial fully nonlinear equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire **37**, no. 5 (2020), 1109–1141.

[16] P. Hartman, Ordinary Differential Equations, 2nd ed., Birkhäuser, Boston, 1982.

[17] H. Ishii, Y. Yoshimura, *Demi-eigenvalues for uniformly elliptic Isaacs operators*, unpublished preprint (2004).

[18] P.L. Lions, *Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations, part 2*, Comm. Partial Differential Equations **8** (1983), 1229–1276.

[19] P. L. Lions, *Bifurcation and optimal stochastic control*, Nonl. Anal. Th. and Appl. **7** (2) (1983), 177–207.

[20] I. Peral, F. Soria, Elliptic and Parabolic Equations Involving the Hardy-Leray Potential, Berlin, Boston: De Gruyter, 2021.

[21] C. Pucci, *Operatori Ellittici Estremanti*, Ann. Mat. Pura Appl. **72** (1966), 141–170.

[22] A. Quaas, B. Sirakov, *Principal eigenvalues and the Dirichlet problem for fully nonlinear elliptic operators*, Adv. Math. **218** (2008), 105–135.

Dipartimento di Matematica, Sapienza Università di Roma, P.le A. Moro 2, 00185 Roma (Italy)

Email address: leoni@mat.uniroma1.it