ON THE DISTANCE FUNCTION IN DOMAINS WITH BOUNDARY

SULLA FUNZIONE DISTANZA IN DOMINI CON FRONTIERA

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ABSTRACT. We present some results on the regularity and the singularities of the distance function in domains with boundary.

SUNTO. Presentiamo alcuni risultati sulla regolarità e sulle singolarità della funzione distanza in domini con frontiera.

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1. Introduction and statement of the results

The aim of this paper is expository: no new results are provided but rather we will present an overview of the results on the regularity and the singularities of the distance function in domains with boundary. We also give the complete proofs of some basic regularity theorems.

This introduction is divided into two parts: in the first part we consider the regularity issue while, in the second part, the singularities of the distance function are analyzed. Throughout all the first part, we make the following assumptions. Let $\Omega \subset \mathbb{R}^n$ be an open connected set, with boundary of class C^2, let X be the closure of $\mathbb{R}^n \setminus \Omega$, and let g be a Riemannian metric defined in X, i.e.

$$g_x = \sum_{i,j=1}^{n} g_{ij}(x) dx_i dx_j,$$

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with g_{ij} of class C^2. We assume that (X, g) is a C^2 connected complete Riemannian manifold with boundary. We observe that the case of X unbounded is as well admitted. Let us also remark that the regularity results we will describe are of a local nature, then, to the price of some minor technicalities, they could be extended to the apparently more general setting of X a n–dimensional C^2-Riemannian manifold with boundary.

For $x_0, x_1 \in X$, the distance function $d(x_0, x_1)$ is defined as the infimum of the length

$$|\gamma| = \int_0^1 \left(\sum_{i,j=1}^{n} g_{ij}(\gamma(t))\dot{\gamma}_i(t)\dot{\gamma}_j(t) \right)^{\frac{1}{2}} dt$$

over the set

$$\Gamma(x_0, x_1) := \{\gamma \in AC([0,1]; X) \mid \text{ with } \gamma(0) = x_0 \text{ and } \gamma(1) = x_1\}.^1$$

A curve $\gamma \in \Gamma(x_0, x_1)$ is called a length minimizer if $d(x_0, x_1) = |\gamma|$.

1.1. **Regularity.** The first natural questions concern the existence and the (global) regularity of a length minimizer. The following result provides a solution to these basic problems.

Theorem 1.1. *For every $x_0, x_1 \in X$ there exists a length minimizer $\gamma \in \Gamma(x_0, x_1)$. Furthermore, γ is of class $C^{1,1}$.*

To our knowledge Theorem 1.1 was first established in [31] (see also [18], [29] and [20]), we observe that there is a loss of regularity w.r.t. the case of X without boundary (from C^2 to $C^{1,1}$). The following example shows that the $C^{1,1}$ regularity given in Theorem 1.1 is, in general (i.e. without additional assumptions), optimal.

Example 1.1. *Let X be the complement of a ball in $\mathbb{R}^n$ equipped with the Euclidean metric. Let r be a straight line throughout the center of the ball and let $x_0, x_1 \in X \cap r$ be in different connected components of $X \cap r$. If $x_0, x_1 \notin \partial X$, a length minimizer, γ, lies in a 2-plane passing through the straight line r, and it is given by two straight line segments through x_0 and x_1 (lying in such a plane on the same side w.r.t. r, and tangent to the intersection of the ball with the plane), and an arc of circle connecting the tangency points.*

[1] AC stands for the set of all the absolutely continuous curves.

Clearly, at the tangency points, the second derivative of the minimizer is not defined, and the $C^{1,1}$ regularity is (globally) optimal.

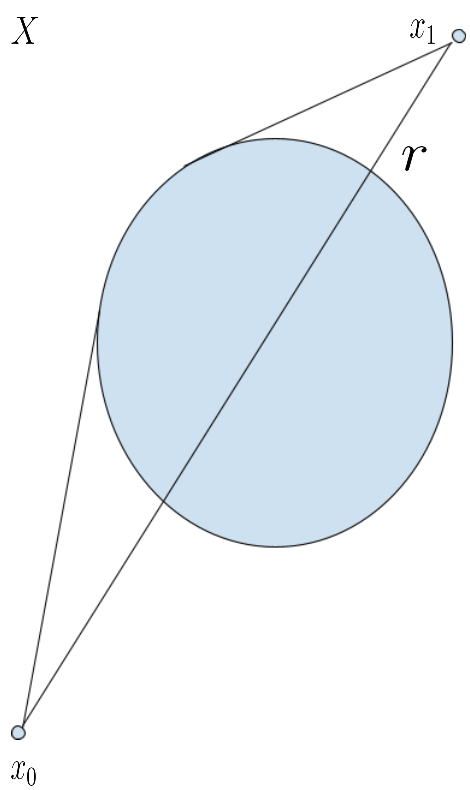

FIGURE 1. A $C^{1,1}$ length minimizer as in Example 1.1.

The previous example shows also that, even assuming metric and manifold real-analytic, in general, the best one can hope for the global regularity of a length minimizers is $C^{1,1}$. This phenomenon has the following explanation: in the normal direction to the boundary the Euler-Lagrange equation associated to the energy functional reads as a (one dimensional) differential inequality, so one can deduce only a Lipschitz regularity instead of the differentiability of the normal component of the velocity. The details of the argument are provided in Section 2, where we describe the proof given in [31].

Let us now study the regularity of the distance function d. More precisely, for $x_0 \in X$, we consider the function

$$X \ni x \mapsto d(x) := d(x_0, x). \tag{1}$$

Remark 1.1. *For the sake of brevity we omit to write the dependence on x_0, the results we will present hold for every $x_0 \in X$*

We are interested in the semiconcavity of d (which is somehow more than Lipschitz continuity and less than continuous differentiability). This kind of regularity is expected for the value function of optimal control problems with smooth data and without non-holonomic constraints (see e.g. [22] and [14]), and it can be defined as follows. We denote by $\|\cdot\|$ the Euclidean norm in $\mathbb{R}^n$.

Definition 1.1. *Let $K \subset \mathbb{R}^n$ be a closed set and let u be a real valued function defined in K. We say that u is semiconcave of exponent $\alpha \in]0,1]$ if u is Lipschitz continuous on K and there exists $C > 0$ such that*

$$tu(x) + (1-t)u(y) \leq u(tx + (1-t)y) + \frac{Ct(1-t)}{2}\|y - x\|^{1+\alpha}, \qquad \forall t \in [0,1], \tag{2}$$

for every $x, y \in K$ such that the straight line segment $[x, y]$ is fully contained in K. We denote by $SC^\alpha(K)$ the set of all the semiconcave of exponent α defined on K.

Remark 1.2. *(i) The above definition can be naturally localized, and we denote by SC^α_{loc} the relative set of functions.*
(ii) Definition 1.1 is independent of the system of coordinates (under changes of coordinates at least of class C^2).

We set

$$\rho(x) = \inf_{y \in \partial X} d(y, x), \qquad (x \in X),$$

and observe that, for $x_1 \neq x_0$, if $d(x_1) < \rho(x_1)$, then there exists a neighborhood of x_1, W (with $x_0 \notin W$), such that $d \in SC^1(W)$. Indeed, in this case the distance does not feel the presence of ∂X: d is a (viscosity) solution of the eikonal equation in W and, as a consequence of the regularity results given in [2], we have the following

Theorem 1.2. *Let $x_1 \in X$ be such that*

$$d(x_1) < \rho(x_1).$$

Then, $d \in SC^1(W)$ for a suitable W neighborhood of x_1.

One may expect that the presence of a state constraint, the boundary of X, implies a weaker (global) regularity compared with the one given by Theorem 1.2. In fact, the following result holds (see Section 3 for the proof).

Theorem 1.3. *For every $x_0 \in X$, $d \in SC^{\frac{1}{2}}_{loc}(X \setminus \{x_0\})$.*

To our knowledge Theorem 1.3 was implicitly[2] proved first in [31] (see also [8]). Comparing Theorem 1.2 with Theorem 1.3, it appears clearly a measure of the loss of regularity of the distance due to the presence of a boundary: the exponent 1/2 instead of 1. This has a geometrical meaning which we are going to describe in a special geometrical setting. We consider the distance function in the presence of a compact connected obstacle. More precisely, let $\mathcal{O} \subset \mathbb{R}^n$ be an open, bounded, and connected set with boundary of class C^2, let X be the unbounded connected component of $\mathbb{R}^n \setminus \mathcal{O}$, and let $x_0 \in X$. We take in X the Euclidean metric. In [6], it is shown that we can find two points $x_1, x_2 \in X$, $x_1 \neq x_2$, and two length minimizers $\gamma_i \in \Gamma(x_0, x_i)$, $i = 1, 2$, such that γ_1 intersects γ_2 at a point $x \in \partial\mathcal{O} \setminus \{x_0\}$ and the two minimizers do not coincide between the points x_0 and x_i, for $i = 1, 2$. From this fact it follows that

$$d \in SC^{\alpha}_{loc}(X \setminus \{x_0\}) \implies \alpha \in]0, 1/2].$$

In other words, the regularity $SC^{\frac{1}{2}}_{loc}(X \setminus \{x_0\})$ is the "fractional" semiconcavity of higher order permitting the intersection of length minimizers (on the boundary of the obstacle $\mathcal{O}$) and for C^2 obstacles and the Euclidean metric this is the optimal global regularity. So, from a geometrical point of view, it is this focusing of the length minimizers which causes the loss of regularity of the distance function on the boundary of the obstacle.

[2]More precisely, in [31] it is proved Lemma 3.1 below.

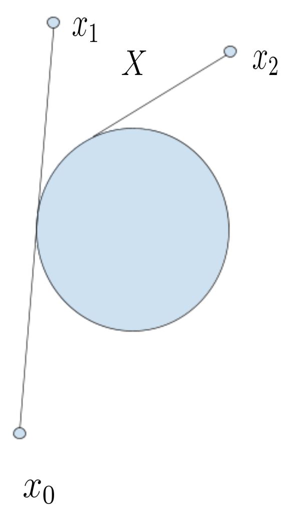

FIGURE 2. The failure of the linear semiconcavity.

1.2. **Singularities.** Let us now consider the problem of the existence of singularities and the local structure of the singular set of the distance in a domain with boundary. Even in the Euclidean case, without additional geometrical assumptions, the distance function may be smooth (away from x_0). Consider for instance X a ball and x_0 a point different from the center of the ball. In this case, $d(x) = \|x - x_0\|$ is real-analytic in $X \setminus \{x_0\}$. For this reason, we identify a class of domains with boundary as follows: we consider a compact connected obstacle, $\mathcal{O}$, such that the closure of the interior of $\mathcal{O}$ is equal to $\mathcal{O}$, and let X be the unbounded connected component of $\mathbb{R}^n \setminus \mathcal{O}$.

We recall that the distance function is the unique (viscosity) solution of a mixed boundary value problem for the eikonal equation (see e.g. [25]). Then it is expected that d is not globally smooth in $X \setminus \{x_0\}$. More precisely, C^2 regularity fails in general (this can be seen as a direct consequence of the optimality of Theorem 1.3). Since the eikonal equation is a first order nonlinear pde it is natural to study the "first" order singularities (i.e. the points where the distance function is not differentiable).

In the sequel we will refer to the Euclidean (Riemannian resp.) distance if X is equipped with the Euclidean (Riemannian resp.) metric, i.e. we will omit to write that an obstacle is present.

1.2.1. *General facts on the singularities of semiconcave functions.*
For a semiconcave function $u \in SC^{\alpha}_{loc}$, we consider two sets which are useful for the study of the singularities:

• the superdifferential of the function u at a given point x

$$D^+u(x) = \left\{ p \in \mathbb{R}^n \mid \limsup_{y \to x} \frac{u(y) - u(x) - \langle p, y - x \rangle}{|y - x|} \leq 0 \right\} ,$$

• the set of the limiting gradients

$$D^*u(x) = \left\{ \lim_{i \to \infty} Du(x_i) \mid u \text{ differentiable at } x_i \to x \right\} .$$

We observe that, since u is semiconcave, we have

(1) $D^+u(x) \neq \emptyset$ and $D^*u(x) \neq \emptyset$;
(2) $D^*u(x) \subseteq \partial D^+u(x)$, where "$\partial$" denotes the topological boundary;
(3) $\text{co}\, D^*u(x) \subseteq D^+u(x)$,

for every x in the domain of u. (Here "co" stands for the convex hull.) For the proof of the above properties, we refer the reader to [23] (see also the monograph [22]).

Remark 1.3. *We notice that, in* (3) *above, the equality holds if x is an interior point (at a boundary point the inclusion is strict)*

Now, denoting by

$$\Sigma(u) = \{x \ : \ u \text{ is not differentiable at } x\},$$

we have that

$$x \in \Sigma(u) \iff D^*u(x) \text{ contains at least two elements.}$$

Remark 1.4. *We recall that, in a control theoretical perspective, the singular points of the distance function can be understood as the "final" point of two length minimizers with different "final" velocities. Furthermore, for every* $x_1 \in X \setminus \{x_0\}$ *and for every* $-v \in D^*d(x_1)$ *there is a length minimizer, parametrized by the arc-length, whose velocity at* x_1 *is* v *(and vice versa: every final velocity of a length minimizer, parametrized by the arc-length, is an element of* $-D^*d(x_1)$*). All these facts are proved in* [8] *in the Euclidean case and in* [17] *in the Riemannian case.*

One may wonder if some kind of information on the geometry of $\Sigma(u)$ is encoded in the sets D^+u and D^*u. In particular, one may look for sufficient conditions guaranteeing that $x \in \Sigma(u)$ is not an isolated point. This is done in the papers [9] and [10] for functions in the class SC^1_{loc} (see also [19] for some related results), and extended in [1] and [7] to functions in the class SC^α_{loc}, for $\alpha \in]0,1]$. These results can be summarized in the naive statement: if u is a semiconcave function, and $\partial D^+u(x) \setminus D^*u(x) \neq \emptyset$ (in particular $x \in \Sigma(u)$), then x belongs to a continuum of singular points (as shown in [10], some additional regularity and lower estimates on the Hausdorff dimension of such a continuum can be given in the case of a function in SC^1_{loc}). More precisely, if $u \in SC^1_{loc}$ is not differentiable at x, then for every $p \in \partial D^+u(x) \setminus D^*u(x)$, and every vector $v \neq 0$ in the normal cone to $D^+u(x)$ at p, one can find a Lipschitz curve made of singular points with initial velocity $-v$.

1.2.2. *Singularities of solutions of Hamilton-Jacobi equations.* Assume now that $u \in SC^1_{loc}$ is a weak solution (for instance in the viscosity sense) of an Hamilton-Jacobi equation of the form

$$F(x,u,Du) = 0, \tag{3}$$

with $F = F(x,u,p)$ convex w.r.t. p and sufficiently regular (for the precise assumptions we refer the reader to [12]). Furthermore, we suppose that $x \in \Sigma(u)$, x is in the interior of the domain of u, and

$$0 \notin \operatorname{co} D_pF(x,u(x),D^+u(x)). \tag{4}$$

Then, there exist $\sigma > 0$ and a Lipschitz continuous (non-constant) curve γ defined on $[0, \sigma[$ and made of singular points for u, with $\gamma(0) = x$, such that

$$\dot{\gamma}(t) \in \operatorname{co} D_p F(\gamma(t), u(\gamma(t)), D^+ u(\gamma(t))) \qquad \text{for a.e. } t \in [0, \sigma[. \tag{5}$$

We call a solution of the differential inclusion (5) a generalized characteristic[3]. We can rephrase the result above by saying that the singularities propagate[4] (local in time) along the generalized characteristics provided that (4) holds.

Unfortunately, in general, $\Sigma(u)$ isn't a closed set. For this reason, one can consider its closure. We observe that if all the solutions of (3) are semiconcave and if $F(x, u, p) = F(x, u, -p)$, for every (x, u, p) in the domain of F, then

$$\overline{\Sigma(u)} = \ \operatorname{singsupp}_{C^{1,1}} u,$$

for every solution u of (3), i.e. if $x \notin \overline{\Sigma(u)}$ then u is of class $C^{1,1}$ in a suitable neighborhood of x. This regularity property can be proved arguing as follows: if $x \notin \overline{\Sigma(u)}$ the there exists a neighborhood of x such that, in such a neighborhood,

- u is differentiable;
- $-u$ is a solution of (3).

Then, one conclude that $u \in C^{1,1}$ in such a neighborhood (see [22] for this last deduction). Unfortunately, $\overline{\Sigma(u)}$ may be a large set (e.g. of positive measure, see [28]). Furthermore, more surprisingly, the interior of $\overline{\Sigma(u)}$ may be non-empty (see [33]).

In the special case of a Hamiltonian F related to an optimal control problem it is natural consider one more set, somehow intermediate between the singular set and the singular support, the cut locus (see e.g. [30], [27], [3], [4]).

1.2.3. *Existence of singularities.*

[3]The differential inclusion (5) was introduced in [12] (see also [23] for some related results). In [12] a solution of (5) was constructed by iteration using the construction recalled in Subsection 1.2.1. Then, the construction was simplified in [34] (see also [24]).

[4]The terminology is due to the fact that (3) can be an equation of evolution. In the stationary case, it could be more appropriate to say that the singular set is invariant w.r.t. the semiflow (5).

While in the case of Hamilton-Jacobi equation of evolution the generation of singularities can be understood in terms of geometrical property of the initial datum (see [16]) in the case of stationary equations the situation is less clear (there is no a natural direction of propagation since there is no a natural "time" variable).

Let us begin with a result on the existence of singularities of the Euclidean distance on the boundary of the obstacle (see [8]).

Theorem 1.4. *Let d be the Euclidean distance, and let $\mathcal{O}$ be a convex set. Then, there exists $x \in \partial\mathcal{O}$, with $x \neq x_0$, such that d is not differentiable at x.*

In other words, the convexity of the obstacle is a sufficient condition for $\Sigma(d) \cap \partial\mathcal{O} \neq \emptyset$.

For the Riemannian distance function in the presence of an obstacle (without assumptions of convexity on $\mathcal{O}$), in [17], it is proved the following

Theorem 1.5. *There exists x in the interior of $X \setminus \{x_0\}$ such that d is not differentiable at x.*

Remark 1.5. *To the author's knowledge the following problem is completely open: let d be the Riemannian distance, find a sufficient condition, possibly in terms of g and $\mathcal{O}$, guaranteeing that $\Sigma(d) \cap \partial\mathcal{O} \neq \emptyset$.*

Once the existence of a singularity (different from x_0) is established a natural question is the following:

can there be an isolated singular point in $\Sigma(d) \setminus \{x_0\}$?

1.2.4. *Propagation of singularities: local results.*

• *Local propagation from an interior point.* As already recalled there are mainly two approaches to the study of the singularities of d:

(A) the regularity results show that d is a semiconcave function, then one can try to apply to d the general results on the structure of the singular sets (which follow from the only knowledge that d is a semiconcave function);

(B) d is a solution (in a suitable weak sense) of an Hamilton-Jacobi equation, then one can try to apply to d the results on the propagation of singularities along generalized characteristics.

We point out that strategy (A) is well suited for a local analysis while (B) is the starting point for global results[5].

In the special case of the eikonal equation the generalized characteristics are a (natural) generalization of the gradient flow. In particular, for the case of the Euclidean distance a singularity of d propagates along the generalized gradient flow:

$$\dot{\gamma}(t) \in D^+d(\gamma(t)), \qquad \text{for a.e. } t \in [0, \sigma]. \tag{6}$$

In other words, if d is not differentiable at a point $\gamma(0)$ in the interior of X, and γ is a solution of the differential inclusion (6), then there exists $\sigma > 0$ (depending on the initial point $\gamma(0)$) such that d is not differentiable in the set $\gamma([0, \sigma])$. We observe that, at a first look, also this appears only as a local propagation result with a specific characteristic: a special singular curve is selected while, in principle, there could be several different curves that carry a singularity. The drawback of this approach is that, if $\gamma(0)$ is a critical point (i.e. $0 \in D^+d(\gamma(0))$), then the solution of (6) is the trivial curve $\gamma(t) \equiv \gamma(0)$, while the approach (A) would give at least a singular curve provided that $\partial D^+d(\gamma(0)) \setminus D^*d(\gamma(0)) \neq \emptyset$. We observe that, in the present context, (4) reads as

$$0 \notin D^+d(\gamma(0)), \tag{7}$$

i.e. $\gamma(0)$ should be not a critical point for d. Hence, in order to apply these results to the Euclidean distance at $x \in \Sigma(d) \cap \mathrm{int}(X)$, we need to verify that or $\partial D^+d(x) \setminus D^*d(x) \neq \emptyset$ or that (7) holds at $\gamma(0) = x$.

We need to distinguish two cases: if $x \in \mathrm{co}\,\mathcal{O}$ or $x \notin \mathrm{co}\,\mathcal{O}$ (both cases were analyzed in [8]). We have that

- if $x \in \mathrm{co}\,\mathcal{O} \cap \Sigma(d)$, then $\partial D^+d(x) \setminus D^*d(x) \neq \emptyset$. In this case, strategy (A) yields the existence of a non-constant Lipschitz singular curve.

We observe that this local propagation result is optimal, i.e. singularities may disappear. This phenomenon can be understood considering the case of $\mathcal{O} \subset \mathbb{R}^3$ equal to the 2 torus.

[5]It is useful for the analysis of some phenomena of global nature, e.g. the study of the topological structure of the singular set (see e.g. [13])

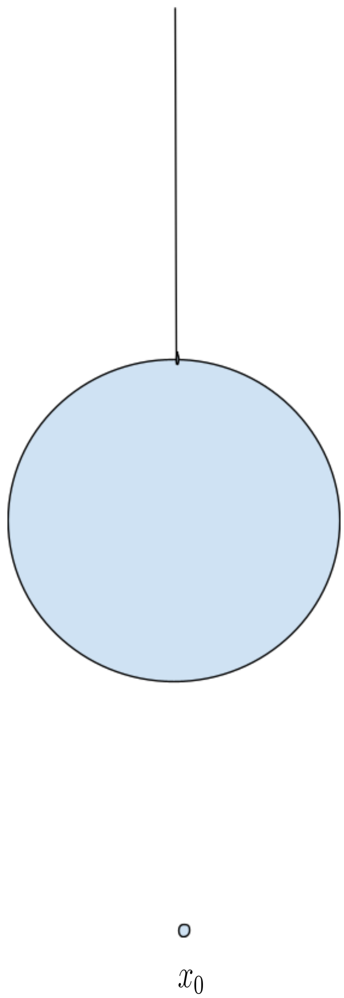

FIGURE 3. Propagation of singularities.

- If $x \in \Sigma(d) \setminus \mathrm{co}\,\mathcal{O}$, then $0 \notin D^+d(x)$ and one may apply strategy (B).

For the Riemannian distance function, in [17], it is shown that if $x \in \Sigma(d) \cap \mathrm{int}(X)$, then $\partial D^+d(x) \setminus D^*d(x) \neq \emptyset$ and strategy (A) yields the existence of a non-constant Lipschitz singular curve.

• *Local propagation from a boundary point.* Let d be the Euclidean distance, and let $x \in \Sigma(d) \cap \partial\mathcal{O}$. Arguing as in [7], first one can extend d to a function in $SC^{\frac{1}{2}}_{loc}$ on an open neighborhood of x and then try to apply strategy (A) (this approach was followed in [8]). We point out that while in the case of a convex obstacle the proof is rather simple the case of a general obstacle is much more tricky. Let us also remark that, for $\mathcal{O}$ non-convex a local propagation result is the best one can hope for (the already recalled example of a 2-torus shows that a singularity may disappear).

Remark 1.6. *Whether the previous result extends to the Riemannian distance function is still an open problem. In the case of a positive answer one would conclude that the Riemannian distance function (in the presence of an obstacle) has not isolated singularities.*

1.2.5. *Propagation of singularities: global results.*

To our knowledge, the first results on global propagation of singularities were given in [11] and [21] (for Hamilton-Jacobi equations of evolution with constant coefficients) and in [5] (for the variable coefficients). These results can be rephrased saying that singularities propagate for all the times provided that they do not reach a boundary and the coefficients are sufficiently smooth. In the case of the Euclidean distance, in [8], it is shown that if either

- $\mathcal{O}$ is convex and $x \in \Sigma(d) \cap \partial\mathcal{O}$, or
- $x \in \Sigma(d) \setminus \mathrm{co}\,\mathcal{O}$,

then the solution of (6) is defined for all $t \geq 0$, $d(\gamma(t))$ is strictly increasing and the singularity goes to infinity, i.e. there is a global propagation of singularities.

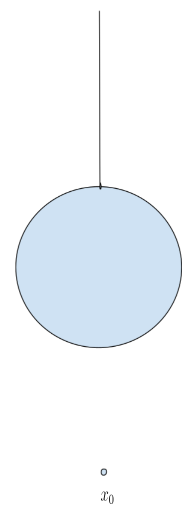

FIGURE 4. Global propagation of singularities in the case of a convex obstacle.

Remark 1.7. *We point out that the key ingredients for the proof of the global propagation results are the property that* $0 \notin D^+ \mathrm{d}(x)$ *for every* $x \notin \mathrm{co}\,\mathcal{O}$ *and the fact that the gradient flow selects a singular curve, i.e. we follow strategy* (B) *as stated in Subsection 1.2.4.*

2. Proof of Theorem 1.1

• Existence of a length minimizer:

Since X is connected, there exists $\gamma \in \Gamma(x_0, x_1)$ such that $|\gamma| < \infty$. Let $\gamma_h \in \Gamma(x_0, x_1)$ be a minimizing sequence, i.e.

$$\lim_{h\to\infty} |\gamma_h| = \inf_{\gamma\in\Gamma(x_0,x_1)} |\gamma| =: \ell.$$

In particular, we have that

$$\ell_h := |\gamma_h| = \ell + a_h, \text{ with } a_h \downarrow 0.$$

We denote by $\hat\gamma_h$, γ_h parametrized by the arc-length, i.e. for a.e. $t \in [0, \ell_h]$,

$$g_{\hat\gamma_h(t)}(\dot{\hat\gamma}_h(t), \dot{\hat\gamma}_h(t)) := \sum g_{ij}(\hat\gamma_h(t))\dot{\hat\gamma}_{hi}(t)\dot{\hat\gamma}_{hj}(t) = 1. \tag{8}$$

Hence, the sequence $\hat\gamma_h$ is equi-Lipschitz continuous on the interval $[0, \ell]$ and, possibly taking a subsequence, we find that, by the Ascoli-Arzelà Theorem, there exists $\hat\gamma_* : [0, \ell] \longrightarrow X$, which is the uniform limit of $\hat\gamma_h$ and that it is Lipschitz continuous. We claim that

$$\ell = \int_0^\ell g_{\hat\gamma_*(t)}(\dot{\hat\gamma}_*(t), \dot{\hat\gamma}_*(t))^{\frac{1}{2}}\, dt. \tag{9}$$

In order to prove (9), we need the following bounds

$$g_{\hat\gamma_*(t)}(\dot{\hat\gamma}_*(t), \dot{\hat\gamma}_*(t)) \le 1, \qquad \text{for a.e. } t \in [0, \ell]. \tag{10}$$

Basically, the proof of (10) is a convexity argument we are going to describe (see [33]). Let t be a point of differentiability for $\hat\gamma_*$, and let V be a neighborhood of $\hat\gamma_*(t)$ in X. Due to the regularity of the metric, there exists $c_0 > 0$ such that

$$g_x(\xi, \xi) \ge c_0^2 \|\xi\|^2, \tag{11}$$

for every $x \in V$ and $\xi \in \mathbb{R}^n$. Let $\epsilon \in]0, 1[$ be such that

$$\bar B_\epsilon(\hat\gamma_*(t)) := \{x \in \mathbb{R}^n \ : \ \|x - \hat\gamma_*(t)\| \le \epsilon\} \subset V,$$

and define

$$\Xi(\epsilon) = \mathrm{co}\{\xi \in \mathbb{R}^n \ : \ g_x(\xi, \xi) \le 1, \quad x \in \bar B_\epsilon(\hat\gamma_*(t))\}.$$

Let $\delta \in]0, \epsilon c_0/2[$, and let $\bar{h}$ large enough such that

$$\|\hat{\gamma}_h(t) - \hat{\gamma}_*(t)\| \le \frac{\epsilon}{2}, \tag{12}$$

for every $h \ge \bar{h}$. Then, for $s \in]t-\delta, t+\delta[$ and $h \ge \bar{h}$, we have that

$$\hat{\gamma}_h(t) - \hat{\gamma}_h(s) = \int_s^t \dot{\hat{\gamma}}_h(\sigma)\, d\sigma.$$

Hence, in view of (11), we find that

$$\|\hat{\gamma}_h(t) - \hat{\gamma}_h(s)\| \le |s-t|/c_0 \le \epsilon/2, \tag{13}$$

provided that[6] $\hat{\gamma}_h(\sigma) \in V$ for every σ between s and t. Then, in view of (12) and (13), we deduce that, for every $s \in]t-\delta, t+\delta[$,

$$\hat{\gamma}_h(s) \in \bar{B}_\epsilon(\hat{\gamma}_*(t)) \subset V.$$

Therefore, due to the definition of the set $\Xi(\epsilon)$, we have that

$$\dot{\hat{\gamma}}_h(s) \in \Xi(\epsilon),$$

for every $s \in]t-\delta, t+\delta[$ and $h \ge \bar{h}$. Now, from the fact that $\Xi(\epsilon)$ is a convex set, it follows that

$$\frac{\hat{\gamma}_h(t+\delta) - \hat{\gamma}_h(t)}{\delta} = \frac{1}{\delta}\int_t^{t+\delta} \dot{\hat{\gamma}}_h(s)\, ds \in \Xi(\epsilon),$$

and, passing to the limit as $h \to +\infty$, we deduce that

$$\frac{\hat{\gamma}_*(t+\delta) - \hat{\gamma}_*(t)}{\delta} \in \Xi(\epsilon).$$

Hence, taking the limit as $\delta \to 0^+$, we find that

$$\dot{\hat{\gamma}}_*(t) \in \Xi(\epsilon).$$

Since the above inclusion holds for every $\epsilon > 0$, we obtain that (10) holds, i.e.

$$g_{\hat{\gamma}_*(t)}(\dot{\hat{\gamma}}_*(t), \dot{\hat{\gamma}}_*(t)) \le 1.$$

[6]This property is a consequence of the uniform convergence of the sequence $\hat{\gamma}_h$ to $\hat{\gamma}_*$.

Finally, in order to prove claim (9), we observe that, assuming the above inequality strict on a subset of $[0,\ell]$ of positive measure, it would follow that

$$|\hat{\gamma}_*| < \ell$$

in contrast with the minimality of ℓ. This complete the proof of the existence of a length-minimizer. We denote by $\gamma_* : [0,1] \longrightarrow X$ a parametrization of the curve $\hat{\gamma}_*$.

• Regularity of length minimizers:

We claim that γ_* is a $C^{1,1}$ curve. We need to distinguish points in the interior of X from those on the boundary of X. For this purpose, we set

$$I = \{t \in [0,1] \mid \gamma_*(t) \in int(X)\}, \quad J = \{t \in [0,1] \mid \gamma_*(t) \in \partial X\}.$$

Instead of working with the length functional, it more convenient to use the energy:

$$E(\gamma_*) = \int_0^1 g_{\gamma_*(t)}(\dot{\gamma}_*(t), \dot{\gamma}_*(t))\, dt.$$

The Cauchy-Schwarz inequality and the fact that γ_* is a length minimizer imply that

$$c_0^2 = E(\gamma_*) \leq |\gamma|^2 \leq E(\gamma), \qquad \forall \gamma \in \Gamma(x_0, x_1).$$

Hence, a length minimizer is also a minimizer of the energy. Let $t_0 \in]0,1[$, then either $t_0 \in I$ or $t_0 \in J$.

We begin with the case of $t_0 \in I$. Then, the (local) regularity of the length minimizer under exam is a direct consequence of the Euler-Lagrange equations. Indeed, we can find $\delta > 0$ such that $\gamma_*(t) \in int(X)$, for every $t \in [t_0 - \delta, t_0 + \delta]$. Furthermore, since γ_* minimizes the energy over $[0,1]$ (with fixed endpoints x_0, x_1), it minimizes the energy over $[t_0 - \delta, t_0 + \delta]$, with fixed endpoints $\gamma_*(t_0 \pm \delta)$. Then, γ_* is a solution (in the sense of the distributions) of the Euler-Lagrange equations, i.e.

$$\frac{d}{dt}\sum_{j=1}^n g_{ij}(\gamma_*(t))\dot{\gamma}_{*j}(t) = \frac{1}{2}\sum_{k,j=1}^n \frac{\partial g_{kj}}{\partial x_i}(\gamma_*(t))\dot{\gamma}_{*k}(t)\dot{\gamma}_{*j}(t), \tag{14}$$

for every $i = 1, \ldots, n$. More precisely, the measurable function

$$p_i(t) = \sum_{j=1}^n g_{ij}(\gamma_*(t))\dot{\gamma}_{*j}(t) + \int_t^1 \frac{1}{2}\sum_{k,j=1}^n \frac{\partial g_{kj}}{\partial x_i}(\gamma_*(s))\dot{\gamma}_{*k}(s)\dot{\gamma}_{*j}(s)\, ds,$$

satisfies the equation

$$\int_0^1 p_i(t)\phi'(t)\,dt = 0$$

for $i = 1, \dots, n$ and for every $\phi \in C_0^\infty([0,1])$. Then, we conclude that $p_i(\cdot)$ is a constant function and

$$p_i(t) \equiv p_i(1), \text{ for a.e. } t \in [0,1],$$

i.e., for every $i = 1, \dots, n$, we have that

$$\sum_{j=1}^n g_{ij}(\gamma_*(t))\dot\gamma_{*j}(t) = p_i(1) - \int_t^1 \frac{1}{2}\sum_{k,j=1}^n \frac{\partial g_{kj}}{\partial x_i}(\gamma_*(s))\dot\gamma_{*k}(s)\dot\gamma_{*j}(s)\,ds. \tag{15}$$

Now, the right hand side in (15) is a continuous function, and it follows that also the left hand side in (15) is a continuous function. Multiplying both the sides of (15) by the inverse of the matrix g_{ij}, we deduce that

$$\dot\gamma_{*j}(t) = \sum_{i=1}^n g_{ij}^{-1}(\gamma_*(t))\left(p_i(1) - \int_t^1 \frac{1}{2}\sum_{k,j=1}^n \frac{\partial g_{kj}}{\partial x_i}(\gamma_*(s))\dot\gamma_{*k}(s)\dot\gamma_{*j}(s)\,ds\right), \tag{16}$$

for every $j = 1, \dots, n$ and for every $t \in [0,1]$. Then, (16) implies that $y_j(\cdot)$ is of class C^1 and, using once more (16), we conclude that it is a C^2 function.

Remark 2.1. *We observe that, more in general, on I the length minimizer has the same regularity of the metric. Let us also point out that, due to* (16)*, a bound for the Lipschitz constant of $\dot\gamma_*$ can be given in terms of the metric (and its first derivatives) and the Lipschitz constant of γ_*.*

It remains to consider the case of $t_0 \in J$ where, in general, it is expected a loss of regularity w.r.t. the previous case. To handle this case it is useful to use the adapted system of coordinates given in the following

Lemma 2.1. *There exist local coordinates $y_1, \dots, y_n$ near $\gamma_*(t_0)$ such that*

(1) *X is given by $y_n \geq 0$;*
(2) *$g_{jn} \equiv 0$, for $j = 1, \dots, n-1$, and $g_{nn} \equiv 1$.*

(We postpone the proof of Lemma 2.1 after the end of the proof of the theorem.)

Let $a < b$ be such that $0 < a < t_0 < b < 1$, and let $\gamma_*([a,b])$ be in a neighbourood of $\gamma_*(t_0)$, W, with coordinates $y_1, \dots, y_n$ given as in Lemma 2.1.

Then, for every $\phi = (\phi_1, \dots, \phi_n) \in [C_0^\infty(]a,b[)]^n$, with $\phi_n \geq 0$, and for $\epsilon > 0$ small enough,we have that

$$(\gamma_* + \epsilon\phi)([a,b]) \subset W.$$

Using the fact that γ_* minimizes the energy, we find that

$$\int_a^b g_{\gamma_*(t)}(\dot{\gamma}_*(t), \dot{\gamma}_*(t))\, dt \leq \int_a^b g_{\gamma_*(t)+\epsilon\phi(t)}(\dot{\gamma}_*(t) + \epsilon\dot{\phi}(t), \dot{\gamma}_*(t) + \epsilon\dot{\phi}(t))\, dt.$$

Then, we deduce that

$$0 \leq \int_a^b g_{\gamma_*(t)}(\dot{\gamma}_*(t), \dot{\phi}(t))\, dt + \frac{1}{2}\int_a^b \langle (\nabla_y g)_{\gamma_*(t)}(\dot{\gamma}_*(t), \dot{\gamma}_*(t)), \phi(t)\rangle\, dt. \tag{17}$$

Now, taking $j \in \{1, \dots, n-1\}$ and ϕ zero except the j-component which is $\phi_j \in C_0^\infty(]a,b[)$, we find the Euler-Lagrange "tangential" equations:

$$\frac{d}{dt}\sum_{i=1}^n g_{ij}(\gamma_*(t))[\dot{\gamma}_*(t)]_i = \frac{1}{2}\frac{\partial g_{\gamma_*(t)}}{\partial y_j}(\dot{\gamma}_*(t), \dot{\gamma}_*(t)) \tag{18}$$

in the sense of the distributions in $]a,b[$, for every $j = 1, \dots, n-1$. Recalling that, in view of Lemma 2.1, $g_{nj}(\gamma_*(t)) \equiv 0$ and, due to the Lipschitz continuity of γ_*,

$$\frac{\partial g_{ik}(\gamma_*)}{\partial y_j}[\dot{\gamma}_*]_i[\dot{\gamma}_*]_k \in L^\infty(]a,b[).$$

The equation above and (18) yield that $\sum_{i=1}^{n-1} g_{ij}(\gamma_*(t))[\dot{\gamma}_*(t)]_i$ is Lipschitz continuous, for $j = 1, \dots, n-1$. Finally, using the fact that the inverse of a matrix with Lipschitz continuous entries is Lipschitz too, we deduce that $[\dot{\gamma}_*(t)]_i$ is Lipschitz continuous, for $i = 1, \dots, n-1$. In order to complete the proof let us consider the normal component. Using once more Lemma 2.1, (17) yields

$$0 \leq \int_a^b \left([\dot{\gamma}_*(t)]_n \dot{\phi}_n(t) + \frac{1}{2}\phi_n(t)\frac{\partial g(\gamma_*(t))}{\partial y_n}(\dot{\gamma}_*(t), \dot{\gamma}_*(t)) \right) dt,$$

for every $\phi_n \in C_0^\infty(]a,b[)$. Hence, we find that

$$\mu := -\frac{d}{dt}[\dot{\gamma}_*(\cdot)]_n + \frac{1}{2}\frac{\partial g(\gamma_*(\cdot))}{\partial y_n}(\dot{\gamma}_*(\cdot), \dot{\gamma}_*(\cdot)) \tag{19}$$

is a positive distribution (in dimension 1).

We claim that μ is (locally) a measure of bounded variation.

Indeed, we may localize μ using a cut-off function $\psi \in C_0^\infty(]a,b[)$, i.e. we consider the distribution $\psi\mu$. Furthermore let $\eta \in C_0^\infty(]a,b[)$ with $\eta \equiv 1$ in the support of ψ, and set

$$M := \psi\mu(\eta).$$

For every $\phi \in C_0^\infty(]a,b[)$, we have that $0 \leq \eta\|\phi\|_{L^\infty} \pm \phi \in C_0^\infty(]a,b[)$ and

$$\psi\mu(\eta\|\phi\|_{L^\infty} \pm \phi) \geq 0,$$

i.e.

$$|\psi\mu(\phi)| \leq M\|\phi\|_{L^\infty}, \qquad \forall \phi \in C_0^\infty(]a,b[).$$

Then, the estimate above and the Riesz Lemma yield that $\psi\mu$ is a (positive) measure of bounded variation.

We denote by $f(s_\pm) = \lim_{t\to s^\pm} f(t)$. It follows that $[\dot\gamma_*]_n$ is locally of bounded variation and, due to the continuity of the second term in the left hand side of (19), we find that

$$[\dot\gamma_*]_n(s_+) \leq [\dot\gamma_*]_n(s_-),$$

at every discontinuity point s. We observe that, at such a point s, $[\gamma_*(s)]_n = 0$ and, by construction $[\gamma_*(\cdot)]_n \geq 0$, then

$$[\dot\gamma_*]_n(s_-) \leq 0 \leq [\dot\gamma_*]_n(s_+).$$

It follows that

$$[\dot\gamma_*]_n(s_-) = [\dot\gamma_*]_n(s_+) = 0,$$

i.e. $[\dot\gamma_*]_n(\cdot)$ is continuous on $]a,b[$. So, recalling the Euler-Lagrange equation (18), we deduce that $[\gamma_*]_i \in C^2(]a,b[)$, for $i = 1,\dots,n-1$.

In order to complete the proof, it remains to show that $[\dot\gamma_*]_n$ is uniformly Lipschitz continuous on $]a,b[$, i.e. there exists $L > 0$ such that

$$|[\dot\gamma_*]_n(s_2) - [\dot\gamma_*]_n(s_1)| \leq L|s_2 - s_1| \tag{20}$$

for every $a < s_1 < s_2 < b$.

If $\gamma_*(]s_1,s_2[) \subset int(X)$ then γ_* is a length minimizer in $int(X)$ and

$$|[\dot\gamma_*]_n(s_2) - [\dot\gamma_*]_n(s_1)| \leq L|s_2 - s_1|, \tag{21}$$

(with, in light of Remark 2.1, L independent of s_1 and s_2). Let us suppose that

$$\gamma_*(]s_1, s_2[) \cap \partial X \neq \emptyset.$$

We consider

$$\tau_i = \inf\{t \in]s_1, s_2[\mid \gamma_*(t) \in \partial X\}, \qquad \text{and} \qquad \tau_s = \sup\{t \in]s_1, s_2[\mid \gamma_*(t) \in \partial X\}.$$

If $\tau_i = s_1$ and $\tau_s = s_2$, then $\gamma_*(s_j) \in \partial X$ (i.e. $[\gamma_*(s_j)]_n = 0$ and $[\dot\gamma_*(s_j)]_n = 0$) and the estimate (20) is trivial. If $\tau_i = s_1$ and $\tau_s < s_2$ (the symmetric situation, $\tau_i > s_1$ and $\tau_s = s_2$, can be handled similarly), then, $\gamma_*(]\tau_s, s_2[) \cap \partial X = \emptyset$, and we have that

$$|[\dot\gamma_*]_n(s_2) - [\dot\gamma_*]_n(s_1)| = |[\dot\gamma_*]_n(s_2) - [\dot\gamma_*]_n(\tau_s)| \leq L|s_2 - \tau_s| \leq L|s_2 - s_1|$$

(We point out that in the first inequality above we used (21).)

It remains to consider the case of $s_1 < \tau_i < \tau_s < s_2$. In particular, we have that

$$\gamma_*(\tau_i), \gamma_*(\tau_s) \in \partial X, \qquad \gamma_*(]s_1, \tau_i[), \gamma_*(]\tau_s, s_2[) \subset int(X).$$

Recalling that $[\dot\gamma_*(\tau_i)]_n = [\dot\gamma_*(\tau_s)]_n = 0$, we deduce that

$$\begin{aligned}|[\gamma_*]_n(s_2) - [\gamma_*]_n(s_1)| &\leq |[\gamma_*]_n(s_2) - [\gamma_*]_n(\tau_s)| + |[\gamma_*]_n(\tau_i) - [\gamma_*]_n(s_1)| \\ &\leq L(s_2 - \tau_s + \tau_i - s_1) \leq L|s_2 - s_1|.\end{aligned}$$

Then, the uniform Lipschitz continuity of $[\dot\gamma_*]_n$ follows varying $]a, b[\subset [0, 1]$.

In order to complete the proof, it remains the last step:

Proof of Lemma 2.1. Let $\bar d(x)$ be the signed distance function from ∂X. Since ∂X is a $(n-1)$-manifold of class C^2, we have that $\nabla \bar d(\gamma_*(t_0))$ is different from the vector zero. Without loss of generality, we may suppose that $\partial \bar d(\gamma_*(t_0))/\partial x_n \neq 0$. Then, in a suitable neighborhood of $\gamma_*(t_0)$, we may introduce the coordinates

$$z_1 = x_1, \ldots, z_{n-1} = x_{n-1}, z_n = \bar d(x).$$

We denote by $A(x)$ the $n \times n$ family of matrices that represent the metric g in the coordinates x, i.e.

$$g_x(\xi, \xi) = \langle A(x)\xi, \xi\rangle, \qquad x \in X\,, \xi \in \mathbb{R}^n,$$

where $\langle\cdot,\cdot\rangle$ stands for the standard Euclidean scalar product. We define

$$y_n := z_n.$$

For $j = 1, \dots, n-1$, we consider the system

$$\begin{cases} \langle A(x)^{-1}\nabla \bar{d}(x), \nabla y_j(x)\rangle = 0, \\ \\ y_j(x)|_{z_n(x)=0} = z_j(x). \end{cases}$$

We notice that, for every $j = 1, \dots, n-1$, the equation above is a non-characteristic Cauchy problem for a first order linear pde, and it admits a unique local classical solution near $\gamma_*(t_0)$. Finally, the fact that, in the new coordinates, the metric has the desired form follows from the equation above, the identity

$$A(y) = \left(\left(\frac{\partial y}{\partial x}\right)^{-1}\right)^t A(x) \left(\frac{\partial y}{\partial x}\right)^{-1} = \left(\left(\frac{\partial y}{\partial x}\right) A(x)^{-1} \left(\frac{\partial y}{\partial x}\right)^t\right)^{-1},$$

and the fact that $\langle A(x)^{-1}\nabla \bar{d}(x), \nabla \bar{d}(x)\rangle = 1$.

□

3. Proof of Theorem 1.3

Lemma 3.1. *Let $x_1 \in X \setminus \{x_0\}$, set $\ell = d(x_1) > 0$, and let $\gamma : [0,\ell] \longrightarrow X$ be a length minimizer parametrized by the arc-length, with $\gamma(0) = x_0$ and $\gamma(\ell) = x_1$. Then, for every y near x_1 we have that*

$$d(y)^2 \le d(x_1)^2 + 2\ell g_{x_1}(\dot\gamma(\ell), y - x_1) + O\left(\|y - x_1\|^{\frac{3}{2}}\right).$$

Proof. Let us fix a neighborhood of x_1 (if $x_1 \in \partial X$ we use the coordinates given by Lemma 2.1). For y in such a neighborhood of x_1, set

$$\sigma = \|y - x_1\| \quad \text{and} \quad v = \frac{y - x_1}{\sigma}.$$

Let

$$\delta\gamma(t) = \begin{cases} 0, & t \in [0, \ell - \sigma^\alpha] \\ \sigma^{1-\alpha}(t - \ell + \sigma^\alpha)v & t \in [\ell - \sigma^\alpha, \ell]. \end{cases}$$

We observe that

- $\tilde{\gamma} := \gamma + \delta\gamma$ is a piecewise C^1 curve such that $\tilde{\gamma}(0) = x_0$ and $\tilde{\gamma}(\ell) = y$;
- $\|\delta\gamma(t)\| \leq \sigma$, for every $t \in [0, \ell]$;
- $\|\delta\dot{\gamma}(t)\| \leq \sigma^{1-\alpha}$, for every $t \in [0, \ell]$.

Then, we have that

$$\begin{aligned} E(\tilde{\gamma}) - E(\gamma) &= \int_{\ell-\sigma^\alpha}^{\ell} [g_{\tilde{\gamma}(t)}(\dot{\tilde{\gamma}}(t), \dot{\tilde{\gamma}}(t)) - g_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))]\, dt \\ &= \int_{\ell-\sigma^\alpha}^{\ell} \left(O(\sigma) + 2g_{\gamma(t)}(\dot{\gamma}(t), \delta\dot{\gamma}(t)) + g_{\gamma(t)}(\delta\dot{\gamma}(t), \delta\dot{\gamma}(t))\right)\, dt \\ &= O(\sigma^{1+\alpha}) + 2g_{\gamma(t)}(\dot{\gamma}(t), \delta\gamma(t))\Big|_{t=\ell-\sigma^\alpha}^{t=\ell} \\ &- 2\sum_{i,j=1}^{n} \int_{\ell-\sigma^\alpha}^{\ell} \frac{d}{dt}\left(g_{ij}(\gamma(t))\dot{\gamma}_i(t)\right)\delta\gamma_j(t)\, dt + O(\sigma^{2-\alpha}) \\ &\underbrace{=}_{\text{using the Euler-Lagrange eq.}} O(\sigma^{1+\alpha}) + 2g_{x_1}(\dot{\gamma}(\ell), y - x_1) \\ &- 2\sum_{i=1}^{n} \int_{\ell-\sigma^\alpha}^{\ell} \frac{\partial g_{\gamma(t)}}{\partial x_i}(\dot{\gamma}(t), \dot{\gamma}(t))\delta\gamma_i(t)\, dt + O(\sigma^{2-\alpha}) \\ &= 2g_{x_1}(\dot{\gamma}(\ell), y - x_1) + O(\sigma^{1+\alpha}) + O(\sigma^{2-\alpha}). \end{aligned}$$

Now, we take α such that $1+\alpha = 2-\alpha$ and, in view of $d(x_1)^2 = \ell E(\gamma)$ and $d(y)^2 \leq \ell E(\tilde{\gamma})$, we deduce that

$$d(y)^2 \leq d(x_1)^2 + 2\ell g_{x_1}(\dot{\gamma}(\ell), y - x_1) + O\left(\|y - x_1\|^{\frac{3}{2}}\right).$$

This completes the proof. □

We need also the following

Lemma 3.2. *Let $x_1 \in X \setminus \{x_0\}$, set $\ell = d(x_1) > 0$, and let $\gamma : [0, \ell] \longrightarrow X$ be a length minimizer parametrized by the arc-length, with $\gamma(0) = x_0$ and $\gamma(\ell) = x_1$. Then, for every y near x_1 we have that*

$$d(y) \leq d(x_1) + g_{x_1}(\dot{\gamma}(\ell), y - x_1) + O\left(\|y - x_1\|^{\frac{3}{2}}\right).$$

Proof. As a consequence of Lemma 3.1, we have that

$$d(y)^2 \leq d(x_1)^2 + 2\ell g_{x_1}(\dot{\gamma}(\ell), y - x_1) + O\left(\|y - x_1\|^{\frac{3}{2}}\right).$$

Then,

$$
\begin{aligned}
d(y) \leq d(x_1) + & \frac{2\ell g_{x_1}(\dot\gamma(\ell), y - x_1) + O\left(\|y - x_1\|^{\frac{3}{2}}\right)}{d(y) + d(x_1)} \\
& \leq d(x_1) + \frac{2\ell g_{x_1}(\dot\gamma(\ell), y - x_1)}{2d(x_1)} + O\left(\|y - x_1\|^2\right) + O\left(\|y - x_1\|^{\frac{3}{2}}\right) \\
& \qquad = d(x_1) + g_{x_1}(\dot\gamma(\ell), y - x_1) + O\left(\|y - x_1\|^{\frac{3}{2}}\right).
\end{aligned}
$$

This completes the proof of the lemma. □

We observe that, in the above result, one can take as a neighborhood of x_1 a small Euclidean ball with center at x_1, B. Then for every $x, y \in B$ and for every $t \in [0, 1]$ taking $z = (1 - t)x + ty$, with γ a length minimizer (parametrized by the arc length) joining x_0 with z, we find that

$$
d(x) \leq d(z) + g_z(\dot\gamma(\ell), x - z) + O\left(\|x - z\|^{\frac{3}{2}}\right), \tag{22}
$$

$$
d(y) \leq d(z) + g_z(\dot\gamma(\ell), y - z) + O\left(\|y - z\|^{\frac{3}{2}}\right). \tag{23}
$$

Then, multiplying both the sides of (22) by $1 - t$, multiplying both the sides of (23) by t, and taking the sum of the obtained inequalities, we find that

$$
(1 - t)d(x) + td(y) \leq d(z) + O\left(\|x - y\|^{\frac{3}{2}}\right),
$$

which is the desired semiconcavity estimate.

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