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Local Monotonicity and Isoperimetric Inequality on Hypersurfaces in Carnot groups

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Abstract

Let $\mathbb{G}$ be a k-step Carnot group of homogeneous dimension Q. Later on we shall present some of the results recently obtained in [32] and, in particular, an intrinsic isoperimetric inequality for a $\mathbf{C}^2$-smooth compact hypersurface S with boundary ∂S. We stress that S and ∂S are endowed with the homogeneous measures σ_H^{n-1} and σ_H^{n-2}, respectively, which are actually equivalent to the intrinsic $(Q-1)$-dimensional and $(Q-2)$-dimensional Hausdorff measures with respect to a given homogeneous metric ϱ on $\mathbb{G}$. This result generalizes a classical inequality, involving the mean curvature of the hypersurface, proven by Michael and Simon [29] and Allard [1], independently. One may also deduce some related Sobolev-type inequalities. The strategy of the proof is inspired by the classical one and will be discussed at the first section. After reminding some preliminary notions about Carnot groups, we shall begin by proving a linear isoperimetric inequality. The second step is a local monotonicity formula. Then we may achieve the proof by a covering argument. We stress however that there are many differences, due to our non-Euclidean setting. Some of the tools developed ad hoc are, in order, a "blow-up" theorem, which holds true also for characteristic points, and a smooth Coarea Formula for the HS-gradient. Other tools are the horizontal integration by parts formula and the 1st variation formula for the H-perimeter σ_H^{n-1} already developed in [30, 31] and then generalized to hypersurfaces having non-empty characteristic set in [32]. These results can be useful in the study of minimal and constant horizontal mean curvature hypersurfaces in Carnot groups.

1. Introduction: Euclidean case $S \subset \mathbb{R}^n$ $(n > 2)$

Theorem 1 (Isoperimetric Inequality; see [29], [1]). *Let $S \subset \mathbb{R}^n$ be a $\mathbf{C}^2$-smooth compact hypersurface with piecewise $\mathbf{C}^1$-smooth boundary. Then*

$$\{\sigma_{R}^{n-1}(S)\}^{\frac{n-2}{n-1}} \leq C_1 \Big\{ \int_S |\mathcal{H}_R|\, \sigma_R^{n-1} + \sigma_R^{n-2}(\partial S) \Big\}$$

where $C_1 > 0$ is a dimensional constant and $\mathcal{H}_R$ denotes the Riemannian mean curvature.

For a complete proof, see also [5].

Sketch of Proof. The first step is a linear isoperimetric inequality which follows from the classical integration by parts formula by making a "suitable" choice of a vector field. More precisely, if $X \in \mathfrak{X}(\mathbb{R}^n)$ we remind that

$$\int_S \{ div_{TS} X + \mathcal{H}_R \langle X, \nu \rangle \}\ \sigma_R^{n-1} = \int_{\partial S} \langle X, \eta \rangle\, \sigma_R^{n-2},$$

where ν is the unit normal vector along S and η is the unit normal vector along ∂S.

Fix $x \in \mathbb{R}^n$ and choose $X(y) = y - x$. Then, by Cauchy-Swartz, we get the next *linear isoperimetric inequality*:

$$(n-1)\, \sigma_R^{n-1}(S) \leq R \left\{ \int_S |\mathcal{H}_R|\, \sigma_R^{n-1} + \sigma_R^{n-2}(\partial S) \right\},$$

where R denotes the radius of a Euclidean ball $B(x, R)$ which contains S.

This inequality and the Coarea Formula allow to prove the *monotonicity inequality.*

Proposition 2 (Monotonicity). *At every "density-point"*[1] $x \in \mathrm{Int}(S)$, *one has*

$$-\frac{d}{dt}\frac{\sigma_R^{n-1}(S_t)}{t^{n-1}} \leq \frac{1}{t^{n-1}} \{\mathcal{A}(t) + \mathcal{B}(t)\} \tag{1}$$

for $\mathcal{L}^1$-a.e. $t > 0$, where $S_t = S \cap B(x,t)$ and

$$\mathcal{A}(t) := \int_{S_t} |\mathcal{H}_R|\, \sigma_R^{n-1}$$
$$\mathcal{B}(t) := \sigma_R^{n-2}(\partial S \cap B(x,t)).$$

Proof of (1). By Sard's Theorem we get that S_t is a $\mathbf{C}^2$-smooth manifold with boundary for $\mathcal{L}^1$-a.e. $t > 0$. Using the above linear isoperimetric inequality yields

$$(n-1)\, \sigma_R^{n-1}(S_t) \leq t \left\{ \mathcal{A}(t) + \sigma_R^{n-2}(\partial S_t) \right\}$$

for $\mathcal{L}^1$-a.e. $t > 0$. Since

$$\partial S_t = \{\partial S \cap B(x,t)\} \cup \{\partial B(x,t) \cap S\}$$

[1] By definition, $x \in \mathrm{Int}(S)$ is a *density-point* if

$$\lim_{t \searrow 0^+} \frac{\sigma_R^{n-1}(S_t)_R}{t^{n-1}} = \omega_{n-1},$$

where ω_{n-1} denotes the Lebesgue measure of the unit ball in $\mathbb{R}^{n-1}$.

we get that

$$(n-1)\sigma_{R}^{n-1}(S_t) \;\;\leq\;\; t\left\{\mathcal{A}(t)+\mathcal{B}(t)+\widetilde{\mathcal{B}}(t)\right\}$$

where we have set

$$\widetilde{\mathcal{B}}(t):=\sigma_{R}^{n-2}(\partial B(x,t)\cap S).$$

At this point we have to use the Coarea Formula (see [15]): *for all* $\phi\in\mathbf{C}^1(S)$ *one has*

$$\int_S |grad_{TS}\,\phi|\,\sigma_{R}^{n-1}=\int_{\mathbb{R}}\sigma_{R}^{n-2}(\phi^{-1}[s]\cap S)\,ds.$$

Choosing $\phi(y):=|y-x|$ into this formula yields

$$\begin{aligned}\sigma_{R}^{n-1}(S_{t+h})-\sigma_{R}^{n-1}(S_t) \;&\geq\; \int_{S_{t+h}\setminus S_t}|grad_{TS}\,\phi|\,\sigma_{R}^{n-1}\\ &=\;\int_t^{t+h}\sigma_{R}^{n-2}(\phi^{-1}[s]\cap S)\,ds\\ &=\;\int_t^{t+h}\widetilde{\mathcal{B}}(s)\,ds\end{aligned}$$

for all (small enough) $h>0$. From the last inequality we infer that

$$\widetilde{\mathcal{B}}(t)\leq\frac{d}{dt}\sigma_{R}^{n-1}(S_t)$$

for $\mathcal{L}^1$-a.e. $t>0$. Hence

$$(n-1)\,\sigma_{R}^{n-1}(S_t)\leq t\left(\mathcal{A}(t)+\mathcal{B}(t)+\frac{d}{dt}\sigma_{R}^{n-1}(S_t)\right)$$

which is equivalent to the monotonicity inequality (1). □

We now remind the following well-known:

Lemma 3 (Vitali-type Covering Lemma)**.** *Let* (X,ϱ) *be a compact metric space and* $A\subset X$*. Further, let* $\mathcal{C}$ *be a covering of* A *by closed* ϱ*-balls with centers in* A*. We also assume that each point* $x\in A$ *is the center of at least one closed* ϱ*-ball belonging to* $\mathcal{C}$ *and that the radii of the balls of the covering are uniformly bounded by some positive constant. Then, for every* $\lambda>2$ *there exists a no more than countable subset* $\mathcal{C}'\subset\mathcal{C}$ *of pairwise non-intersecting closed* ϱ*-balls* $\overline{B}_\varrho(x_k,R_k)$ *such that*

$$A\subset\bigcup_{k=1}^{\infty}B_\varrho(x_k,\lambda R_k).$$

By using the monotonicity inequality one proves, by contradiction, an estimate modelled on the previous covering lemma.

Lemma 4 (Calculus Lemma)**.** *Let* $x\in\mathrm{Int}(S)$ *and set* $R_0=2\left\{\frac{\sigma_{R}^{n-1}(S)}{\omega_{n-1}}\right\}^{1/n-1}$*. Then, for every* $\lambda\geq 2$ *there exists* $R\in]0,R_0[$ *such that*

$$\sigma_{R}^{n-1}(S_{\lambda R})\leq\lambda^{Q-1}R_0\left\{\mathcal{A}(R)+\mathcal{B}(R)\right\}.$$

Putting all together, the proof of the Isoperimetric Inequality easily follows. □

1.1. **Applications.** The monotonicity inequality is actually equivalent to an asymptotic estimate of exponential-type. More precisely one proves the following:

Corollary 5 (Asymptotic of $\sigma_{\scriptscriptstyle R}^{n-1}(S_t)$.). *For every $x \in \mathrm{Int}(S)$ one has*

$$\sigma_{\scriptscriptstyle R}^{n-1}(S_t) \geq \omega_{n-1}\, t^{n-1} e^{-\mathcal{H}_0 t} \qquad (t \to 0^+)$$

where $\mathcal{H}_0$ is a positive constant satisfying $|\mathcal{H}_{\scriptscriptstyle R}| \leq \mathcal{H}_0$. In particular, if $\mathcal{H}_{\scriptscriptstyle R} = 0$, i.e. S is a minimal hypersurface, one gets that

$$\sigma_{\scriptscriptstyle R}^{n-1}(S_t) \geq \omega_{n-1}\, t^{n-1} \qquad (t \to 0^+).$$

Finally, by using a well-known argument due to Federer-Fleming and Maz'ya, one can show the following (see [16], [28]):

Corollary 6 (Sobolev-type Inequality). *Let $S \subset \mathbb{R}^n$ be a $\mathbf{C}^2$-smooth closed hypersurface. Then there exists a dimensional constant $C_2 > 0$ such that*

$$\left\{\int_S |\psi|^{\frac{n-1}{n-2}} \sigma_{\scriptscriptstyle R}^{n-1}\right\}^{\frac{n-2}{n-1}} \leq C_2 \int_S \{|\mathcal{H}_{\scriptscriptstyle R}||\psi| + |grad_{\scriptscriptstyle TS}\psi|\}\, \sigma_{\scriptscriptstyle R}^{n-1}.$$

for every $\psi \in \mathbf{C}_0^1(S)$.

2. Carnot groups, Hypersurfaces and measures

Over the last years *Sub-Riemannian* or *Carnot-Carathéodory* geometries have become a subject of great interest because of their connections with many areas of Mathematics and Physics, such as PDE's, Calculus of Variations, Control Theory, Mechanics, Theoretical Computer Science. As an introduction to this subject we refer the reader to Montgomery's book [33] and to the surveys by Gromov [22] and Vershik and Gershkovich [39]. See also the works by Cheeger, Kleiner and Naor [9, 10] for some new perspectives.

Very recently, the so-called Visual Geometry has also received new impulses from this field; see for example [12] and references therein.

Furthermore, Carnot groups constitute a wide class of examples of sub-Riemannian geometries. Indeed, by a well-know result due to Mitchell (see Montgomery's book [33]), the *Gromov-Hausdorff tangent cone* at the regular points of a sub-Riemannian manifold is a suitable Carnot group. Thus, in a sense, Carnot groups play for sub-Riemannian geometries an analogous role to that of Euclidean spaces in Riemannian geometry.

The beginning of Geometric Measure Theory in this setting was perhaps an intrinsic isoperimetric inequality proven by Pansu in his Thesis [34], for the case of the *Heisenberg group* $\mathbb{H}^1$. For what concerns isoperimetric inequalities on Lie groups and C-C spaces, see also [6], [22], [36], [17], [38].

For an introduction to Analysis and GMT in this setting we shall refer the reader to [2], [3], [4], [6], [8], [13], [18, 19, 20, 21], [23, 24], [25, 26, 27], [31, 32], [35, 36].

We also refer to [7], [11], [14], [37], for recent results concerning minimal hypersurfaces in the Heisenberg group.

2.1. **Carnot groups.** A *k-step Carnot group* $(\mathbb{G}, \bullet)$ is a connected, simply connected, nilpotent and stratified Lie group (with respect to the group law $\bullet$). Its Lie algebra $\mathfrak{g} \cong \mathbb{R}^n$ satisfies:

$$\mathfrak{g} = H_1 \oplus ... \oplus H_k$$

$$[H_1, H_{i-1}] = H_i \quad (i = 2, ..., k), \;\; H_{k+1} = \{0\}.$$

Let 0 be the identity of $\mathbb{G}$ and $\mathfrak{g} \cong T_0\mathbb{G}$. Let $h_i := \mathrm{dim} H_i$ for $i = 1, ..., k$ and $h_1 := h$. Moreover set $H := H_1$ and

$$V := H_2 \oplus ... \oplus H_k.$$

H and V are smooth subbundles of $T\mathbb{G}$ called *horizontal* and *vertical*, respectively. The horizontal bundle H is generated by a frame $\underline{X_H} := \{X_1, ..., X_h\}$ of left-invariant vector fields. The horizontal frame can be completed to a global graded, left-invariant frame $\underline{X} := \{X_1, ..., X_n\}$ for $T\mathbb{G}$. We stress that the standard basis $\{\mathrm{e}_i : i = 1, ..., n\}$ of $\mathbb{R}^n$ can be relabelled to be *graded* or *adapted to the stratification.* Any left-invariant vector field of $\underline{X}$ is given by $X_i(x) = L_{x*}\mathrm{e}_i \, (i = 1, ..., n)$, where L_{x*} denotes the differential of the left-translation at $x \in \mathbb{G}$. We fix a Euclidean metric on $\mathfrak{g} = T_0\mathbb{G}$ which makes $\{\mathrm{e}_i : i = 1, ..., n\}$ an orthonormal basis; this metric extends to the whole tangent bundle by left-translations and makes $\underline{X}$ an orthonormal left-invariant frame. We shall denote by $g = \langle \cdot, \cdot \rangle$ this metric. Note that $(\mathbb{G}, g)$ is a Riemannian manifold.

We shall use the so-called *exponential coordinates of 1st kind* and so $\mathbb{G}$ will be identified with its Lie algebra $\mathfrak{g}$, via the (Lie group) exponential map $exp : \mathfrak{g} \longrightarrow \mathbb{G}$.

A *sub-Riemannian metric* g_H is a symmetric positive bilinear form on the horizontal bundle H. The *CC-distance* $d_{\mathbf{cc}}(x, y)$ between $x, y \in \mathbb{G}$ is given by

$$d_{\mathbf{cc}}(x, y) := \inf \int \sqrt{g_H(\dot\gamma, \dot\gamma)}\, dt,$$

where the infimum is taken over all piecewise-smooth horizontal paths γ joining x to y. From now on we choose $g_H := g_{|H}$.

Carnot groups are *homogeneous groups*, i.e. they admit a 1-parameter group of automorphisms $\delta_t : \mathbb{G} \longrightarrow \mathbb{G}$ $(t \geq 0)$. By definition

$$\delta_t x := \exp\left(\sum_{j, i_j} t^j \, x_{i_j}\mathrm{e}_{i_j}\right),$$

where $x = exp\left(\sum_{j,i_j} x_{i_j}\mathrm{e}_{i_j}\right) \in \mathbb{G}$. The *homogeneous dimension* of $\mathbb{G}$ is the integer

$$Q := \sum_{i=1}^{k} i \, \mathrm{h}_i$$

coinciding with the *Hausdorff dimension* of $(\mathbb{G}, d_{\mathbf{cc}})$ as a metric space.

The *structural constants* of $\mathfrak{g}$ associated with $\underline{X}$ are defined by

$$C^r_{ij} := \langle [X_i, X_j], X_r \rangle \quad (i, j, r = 1, ..., n).$$

They are skew-symmetric and satisfy Jacobi's identity.

The stratification hypothesis on $\mathfrak{g}$ implies that

$$X_i \in H_l,\, X_j \in H_m \Longrightarrow [X_i, X_j] \in H_{l+m}.$$

The last condition can be rephrased in terms of (vanishing of) structural constants.

Henceforth, we shall set

- $C^\alpha_H := [C^\alpha_{ij}]_{i,j=1,...,h} \in \mathcal{M}_{h\times h}(\mathbb{R}) \qquad (\alpha = h+1, ..., h+h_2)$;
- $C^\alpha := [C^\alpha_{ij}]_{i,j=1,...,n} \in \mathcal{M}_{n\times n}(\mathbb{R}) \qquad (\alpha = h+1, ..., n)$.

We introduce the left-invariant co-frame

$$\underline{\omega} := \{\omega_i : i = 1, ..., n\}$$

dual to $\underline{X}$, i.e. $\omega_i = X_i^*$ for every $i = 1, ..., n$. In particular, the *left-invariant 1-forms* ω_i are uniquely determined by

$$\omega_i(X_j) = \langle X_i, X_j\rangle = \delta_i^j \qquad (i, j = 1, ..., n)$$

where δ_i^j denotes the Kronecker delta.

Let ∇ denote the unique left-invariant Levi-Civita connection on $\mathbb{G}$ associated with the left-invariant metric $g = \langle\cdot,\cdot\rangle$. For every $i, j = 1, ..., n$, it turns out that

$$\nabla_{X_i} X_j = \frac{1}{2}\sum_{r=1}^{n}(C^r_{ij} - C^i_{jr} + C^j_{ri})X_r.$$

If $X, Y \in \mathfrak{X}(H)$, we set $\nabla^H_X Y := \mathcal{P}_H(\nabla_X Y)$. ∇^H is called *H-connection* and it is *flat, compatible with the metric g_H* and *torsion-free.*

Horizontal gradient and horizontal divergence operators are denoted, respectively, by $grad_H$ and div_H.

A continuous distance $\varrho : \mathbb{G}\times\mathbb{G} \longrightarrow \mathbb{R}_+ \cup \{0\}$ is called *homogenous* if one has

$$\begin{aligned}\varrho(x, y) &= \varrho(z \bullet x, z \bullet y) \quad \forall\, x,\, y,\, z \in \mathbb{G};\\ \varrho(\delta_t x, \delta_t y) &= t\varrho(x, y) \quad \forall t \geq 0.\end{aligned}$$

Remark 7 (Assumptions on ϱ)**.** *Let $\varrho(x) = \varrho(0, x) = \|x\|_\varrho$. We shall assume that:*

- *ϱ is piecewise $\mathbf{C}^1$-smooth;*
- *$|grad_H\, \varrho| \leq 1$ at each regular point of ϱ;*
- *$|x_H| \leq \varrho(x)$;*
- *there exist constants $c_i > 0$ such that*

$$|x_{H_i}| \leq c_i\varrho^i(x) \qquad \textit{for every} \quad i = 2, ..., k.$$

Example 8. *CC-distance $d_{\mathbf{cc}}$ and Korany distance on the Heisenberg group $\mathbb{H}^n$ satisfy all the assumptions of Remark 7.*

2.2. **Hypersurfaces and measures.** The Riemannian left-invariant volume form on $\mathbb{G}$ is defined as $\sigma^n_R := \bigwedge_{i=1}^n \omega_i \in \bigwedge^n(T^*\mathbb{G})$.

The measure σ^n_R is the Haar measure of $\mathbb{G}$ and equals the push-forward of the usual n-dimensional Lebesgue measure $\mathcal{L}^n$ on $\mathfrak{g} \cong \mathbb{R}^n$.

Let $S \subset \mathbb{G}$ be a $\mathbf{C}^1$-smooth hypersurface. Then $x \in S$ is a *characteristic point* if

$$\dim H_x = \dim(H_x \cap T_x S).$$

The *characteristic set* of S is given by

$$C_S := \{x \in S : \dim H_x = \dim(H_x \cap T_x S)\}.$$

Note that $x \in S$ is non-characteristic if, and only if, H is transversal to S at x. Moreover the $(Q-1)$-dimensional CC Hausdorff measure of the characteristic set C_S vanishes, i.e.

$$\mathcal{H}^{Q-1}_{\mathbf{cc}}(C_S) = 0.$$

Let ν denote the unit normal vector along S. The $(n-1)$-dimensional Riemannian measure along S is defined by

$$\sigma^{n-1}_{R} \llcorner S := (\nu \lrcorner \sigma^n_{R})|_S,$$

where $\lrcorner$ denotes the "contraction" operator on differential forms. Remind that

$$\lrcorner : \bigwedge^k (T^*\mathbb{G}) \to \bigwedge^{k-1}(T^*\mathbb{G})$$

is defined, for $X \in T\mathbb{G}$ and $\alpha \in \bigwedge^k(T^*\mathbb{G})$, by

$$(X \lrcorner \alpha)(Y_1, ..., Y_{k-1}) := \alpha(X, Y_1, ..., Y_{k-1}).$$

If S is non-characteristic the *unit H-normal* along S is the normalized projection of ν onto H, i.e.

$$\nu_{H} := \frac{\mathcal{P}_H \nu}{|\mathcal{P}_H \nu|}.$$

We define the $(n-1)$-dimensional homogeneous measure $\sigma^{n-1}_{H} \in \bigwedge^{n-1}(T^*S)$ by

$$\sigma^{n-1}_{H} \llcorner S := (\nu_{H} \lrcorner \sigma^n_{R})|_S.$$

If $C_S \neq \emptyset$ we extend σ^{n-1}_{H} to the whole of S by setting $\sigma^{n-1}_{H} \llcorner C_S = 0$. Note that

$$\sigma^{n-1}_{H} \llcorner S = |\mathcal{P}_H \nu|\, \sigma^{n-1}_{R} \llcorner S.$$

It follows that $C_S = \{x \in S \,|\, |\mathcal{P}_H \nu| = 0\}$. Since σ^{n-1}_{H} coincides on smooth hypersurfaces with the H-perimeter measure, σ^{n-1}_{H} is also called *H-perimeter form*.

Let $\mathcal{S}^{Q-1}_{\mathbf{cc}}$ denote the $(Q-1)$-dimensional spherical Hausdorff measure associated with the CC-distance $d_{\mathbf{cc}}$. Then we have

$$\sigma^{n-1}_{H}(S \cap B) = k(\nu_{H})\, \mathcal{S}^{Q-1}_{\mathbf{cc}} \llcorner (S \cap B),$$

for all $B \in \mathcal{B}or(\mathbb{G})$, where the density $k(\nu_{H})$, called *metric factor*, depends on ν_{H}; see below. The *horizontal tangent bundle $HS \subset TS$* and the *horizontal normal bundle $\nu_{H} S$* split the horizontal bundle H into an orthogonal direct sum, i.e.

$$H = \nu_{H} \oplus HS.$$

We also remind that the stratification of $\mathfrak{g}$ induces a stratification of $TS := \oplus_{i=1}^k H_i S$, where $HS := H_1 S$; see [22].

Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface and let ∇^{TS} be the induced connection on S from ∇. The tangential connection ∇^{TS} induces a partial connection on HS defined by

$$\nabla^{HS}_X Y := \mathcal{P}_{HS}(\nabla^{TS}_X Y) \quad (X, Y \in HS).$$

It turns out that $\nabla^{HS}_X Y = \nabla^{H}_X Y - \langle \nabla^{H}_X Y, \nu_{H}\rangle \nu_{H}$. The HS-gradient and the HS-divergence are denoted by $grad_{HS}$ and div_{HS}, respectively.

The horizontal 2nd fundamental form of S is the $\mathbf{C}^\infty(S)$-bilinear map given by

$$B_H(X,Y) := \langle \nabla^H_X Y, \nu_H \rangle$$

for every $X,\, Y \in \mathfrak{X}(HS)$. The horizontal mean curvature $\mathcal{H}_H$ is, by definition, the trace of B_H, i.e.

$$\mathcal{H}_H := \mathrm{Tr} B_H = -div_H \nu_H.$$

The torsion T_{HS} of the HS-connection ∇^{HS} is defined by

$$\mathrm{T}_{HS}(X,Y) := \nabla^{HS}_X Y - \nabla^{HS}_Y X - \mathcal{P}_H[X,Y]$$

for every $X, Y \in HS$. We have a non-zero torsion because, in general, B_H is *not symmetric.*

Finally, we define some important geometric objects:

- $\varpi_\alpha := \frac{\nu_\alpha}{|\mathcal{P}_H \nu|} \quad (\alpha = h+1, ..., n)$;
- $\varpi := \frac{\mathcal{P}_V \nu}{|\mathcal{P}_H \nu|} = \sum_{\alpha=h+1}^n \varpi_\alpha X_\alpha$;
- $C_H := \sum_\alpha \varpi_\alpha C^\alpha_H \quad (\alpha = h+1, ..., h+h_2)$.

3. Preliminary tools

We begin by recalling a horizontal integration by parts formula and the 1st variation formula of the H-perimeter σ^{n-1}_H already developed in [30, 31] and then generalized in [32] to hypersurfaces having non-empty characteristic set. Then we will state a smooth Coarea Formula for the HS-gradient and a "Blow-up" procedure (see also [3], [18, 19], [25, 26]), which can also applied to characteristic points. Finally, we shall discuss a linear inequality which is the key ingredient in the proof of the Isoperimetric Inequality.

3.1. **Integration by parts and 1st variation.** Let assume that ∂S be a $\mathbf{C}^1$-smooth $(n-2)$-dimensional manifold, oriented by its unit normal vector $\eta \in TS \cap \mathrm{Nor}(\partial S)$. We shall denote by σ^{n-2}_R the Riemannian measure on ∂S, which can be defined as follows:

$$\sigma^{n-2}_R \llcorner \partial S = (\eta \lrcorner \sigma^{n-1}_R)|_{\partial S}.$$

We stress that from the previous definitions it follows that for every $X \in \mathfrak{X}(S)$, one has

$$(X \lrcorner \sigma^{n-1}_H)|_{\partial S} = \langle X, \eta \rangle |\mathcal{P}_H \nu|\, \sigma^{n-2}_R \llcorner \partial S.$$

The *characteristic set* of ∂S is defined by

$$C_{\partial S} := \{p \in \partial S : |\mathcal{P}_{HS} \eta| = 0\}.$$

Moreover, the *unit HS-normal* along ∂S is given by

$$\eta_{HS} := \frac{\mathcal{P}_{HS}\eta}{|\mathcal{P}_{HS}\eta|}.$$

We may define the $(n-2)$-dimensional homogeneous measure $\sigma^{n-2}_H \in \bigwedge^{n-2}(T^*\partial S)$ by

$$\sigma^{n-2}_H \llcorner \partial S := \left(\eta_{HS} \lrcorner \sigma^{n-1}_H\right)\big|_{\partial S}.$$

Equivalently, one may set

$$\sigma^{n-2}_H \llcorner \partial S = |\mathcal{P}_H \nu|\, |\mathcal{P}_{HS}\eta| \sigma^{n-2}_R \llcorner \partial S.$$

It turns out that

$$(X \lrcorner\, \sigma_{H}^{n-1})|_{\partial S} = \langle X, \eta_{HS}\rangle \sigma_{H}^{n-2} \llcorner\, \partial S \qquad \text{for every } \ X \in \mathfrak{X}(HS).$$

Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth compact hypersurface. Then, for every $X \in \mathfrak{X}^1(H)$ the following *horizontal integration by parts* formula holds up to the characteristic set C_S:

$$\text{(2)} \quad \int_S \left\{ div_{HS} X + \langle C_H \nu_H, X\rangle \right\} \sigma_H^{n-1} = -\int_S \mathcal{H}_H \langle X, \nu_H\rangle\, \sigma_H^{n-1} + \int_{\partial S} \langle X, \eta_{HS}\rangle\, \sigma_H^{n-2}.$$

The horizontal matrix C_H is a key object in this setting and it is connected with the skew-symmetric part of the horizontal 2nd fundamental form B_H.

We may therefore prove a horizontal linear isoperimetric inequality. More precisely, one may show that

$$(h-1)\, \sigma_H^{n-1}(S) \le R \left\{ \int_S \left\{ |\mathcal{H}_H| + |C_H \nu_H| \right\} \sigma_H^{n-1} + \sigma_H^{n-2}(\partial S) \right\},$$

where R is the radius of a ϱ-ball circumscribed about S. This formula can be proved by using the horizontal position vector $x_H \equiv (x_1, ..., x_h)$ and (2). However, this inequality is not sufficient in order to get the "right" monotonicity formula. To this end we will use in the sequel a more general vector field; see Section 3.4.

Let us state the *1st variation* formula of σ_H^{n-1} for a $\mathbf{C}^2$-smooth compact hypersurface S with piecewise $\mathbf{C}^1$-smooth boundary ∂S. For every $X \in \mathfrak{X}(\mathbb{G})$ one has

$$I_S(X, \sigma_H^{n-1}) := \frac{d}{ds}\left(\int_S \vartheta_s^* \sigma_H^{n-1} \right)\bigg|_{s=0} = -\int_S \mathcal{H}_H \, \langle X, (\nu_H + \varpi)\rangle \, \sigma_H^{n-1} + \int_{\partial S} \langle X, \eta\rangle \, |\mathcal{P}_H \nu| \, \sigma_R^{n-2},$$

where $\vartheta_s^* \sigma_H^{n-1}$ denotes the push-forward of σ_H^{n-1} by the flows ϑ_s related to X. The formula holds true even if $C_S \neq \emptyset$.

3.2. **Coarea formula for the *HS*-gradient.** Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface and let $\phi \in \mathbf{C}^1(S)$. Then

$$\int_S |grad_{HS}\, \phi(x)| \sigma_H^{n-1}(x) = \int_{\mathbb{R}} \sigma_H^{n-2}(\phi^{-1}[s] \cap S) ds.$$

3.3. **Blow-up of σ_H^{n-1}.** Let $S \subset \mathbb{G}$ be a smooth hypersurface and let us study the density of σ_H^{n-1} at $x \in S$, i.e.

$$\lim_{R \to 0^+} \frac{\sigma_H^{n-1}(S \cap B_\varrho(x, R))}{R^{Q-1}},$$

where $B_\varrho(x, R)$ is the ϱ-ball of center $x \in S$ and radius R. We remind that ϱ is a fixed smooth homogeneous distance on $\mathbb{G}$ as in Remark 7.

Case (a). Let S be a $\mathbf{C}^1$-smooth hypersurface and let $x \in S \setminus C_S$; then it turns out that

$$\sigma_H^{n-1}(S \cap B_\varrho(x, R)) \sim \kappa_\varrho(\nu_H) R^{Q-1}$$

for $R \to 0^+$, where the constant $\kappa_\varrho(\nu_H)$ is explicitly given by

$$\kappa_\varrho(\nu_H) = \sigma_H^{n-1}(\mathcal{I}(\nu_H(x)) \cap B_\varrho(x, 1)).$$

Here $\mathcal{I}(\nu_H(x))$ denotes the vertical hyperplane orthogonal to ν_H at x; see also [25, 26].

Case (b). Assume that, locally around $x \in C_S$, there exists $\alpha = h+1, .., n$, $\mathrm{ord}(\alpha) = i$, such that S is the image of a $\mathbf{C}^i$-smooth X_α-graph. With no loss of generality, let us fix $x = 0 \in \mathbb{G}$. In such a case one has

$$S \cap B_\varrho(x, r) \subset exp\left\{\left(\zeta_1, ..., \zeta_{\alpha-1}, \psi(\zeta), \zeta_{\alpha+1}, ..., \zeta_n\right) \,\middle|\, \zeta := (\zeta_1, ..., \zeta_{\alpha-1}, 0, \zeta_{\alpha+1}, ..., \zeta_n) \in \mathrm{e}_\alpha^\perp\right\},$$

where $\psi : \mathrm{e}_\alpha^\perp \cong \mathbb{R}^{n-1} \to \mathbb{R}$ is a function of class $\mathbf{C}^i$. If ψ satisfies

$$\frac{\partial^{(l)}\psi}{\partial\zeta_{j_1}...\partial\zeta_{j_l}}(0) = 0$$

whenever $\mathrm{ord}(j_1) + ... + \mathrm{ord}(j_l) < i$ for every $l = 1, ..., i$, then it follows that there exists a positive constant $\kappa_\varrho(C_S)$ such that

$$\sigma_H^{n-1}(S \cap B_\varrho(0, R)) \sim \kappa_\varrho(C_S)\, R^{Q-1}$$

for $R \to 0^+$. The constant $\kappa_\varrho(C_S)$ can be computed by integrating the H-perimeter σ_H^{n-1} along a homogeneous polynomial hypersurface of order $i = \mathrm{ord}(\alpha)$ only depending on the Taylor's expansion up to order $j \leq i$ of ψ at $0 \in \mathbb{R}^{n-1}$; see [32].

Example 9. *Consider the case of the Heisenberg group* $(\mathbb{H}^n, \varrho)$ *where*

$$\varrho(x) = \sqrt[4]{|x_H|^4 + 16t^2}$$

denotes the Korany distance. Let

$$S = \{x \equiv \exp(x_H, t) \in \mathbb{H}^n : t = 0\}.$$

Then it turns out that $C_S = \{0 \in \mathbb{H}^n\}$ *and that*

$$\kappa_\varrho(C_S) = \frac{O_{2n-1}}{4n},$$

where O_{2n-1} *denotes the surface measure of the unit sphere* $\mathbb{S}^{2n-1}$ *of* $H \cong \mathbb{R}^{2n}$*.*

3.4. **Another linear isoperimetric inequality.** Fix $x \in \mathbb{G}$ and consider the *Carnot homothety* centered at x, i.e. $\vartheta^x(t, y) := x \bullet \delta_t(x^{-1} \bullet y)$. With no loss of generality, let $x = 0 \in \mathbb{G}$. Then

$$\vartheta^0(t, y) := \exp\left(ty_H, t^2 y_{H_2}, ..., t^i y_{H_i}, ..., t^k y_{H_k}\right)$$

for $t \geq 0$, where $y_{H_i} = \sum_{j_i} y_{j_i}\mathrm{e}_{j_i}$. The variational vector field of ϑ^0_t is defined as

$$Z_0 := \frac{\partial\vartheta^0_t}{\partial t}\Big|_{t=1} = \frac{\partial\delta_t}{\partial t}\Big|_{t=1} = y_H + 2y_{H_2} + ... + ky_{H_k}.$$

Analogously, we shall denote by Z_x the variational vector field of ϑ^x_t. By invariance of σ_H^{n-1} under Carnot dilations, one gets

$$I_S(Z_0, \sigma_H^{n-1}) = (Q-1)\,\sigma_H^{n-1}(S).$$

By applying the 1st variation formula of σ_H^{n-1} we get

$$(Q-1)\,\sigma_H^{n-1}(S) = -\int_S \mathcal{H}_H\,\langle Z_0, (\nu_H + \varpi)\rangle\,\sigma_H^{n-1} + \int_{\partial S}\left\langle Z_0, \frac{\eta}{|\mathcal{P}_{HS}\,\eta|}\right\rangle \underbrace{|\mathcal{P}_H\,\nu_H|\,|\mathcal{P}_{HS}\,\eta|\sigma_R^{n-2}}_{=\sigma_H^{n-2}}.$$

From now on, we shall set

$$\eta_{HS} + \chi := \frac{\eta}{|\mathcal{P}_{HS}\eta|}, \qquad \chi := \frac{\mathcal{P}_V\eta}{|\mathcal{P}_{HS}\eta|} = \sum_{i=2}^{k} \chi_{H_iS}.$$

By Cauchy-Schwartz inequality it follows that

$$(Q-1)\,\sigma_H^{n-1}(S) \leq R\left\{\int_S |\mathcal{H}_H|\left(1+\sum_{i=2}^{k} i\,c_i\varrho^{i-1}|\varpi_{H_i}|\right)\sigma_H^{n-1} + \int_{\partial S}\left(1+\sum_{i=2}^{k} i\,c_i\varrho^{i-1}|\chi_{H_iS}|\right)\sigma_H^{n-2}\right\},$$

where $\varrho(y)=\varrho(y,0)$ and R is the radius of a ϱ-ball centered at $0\in\mathbb{G}$ and circumscribed about S. Notice that the constants $c_i\,(i=2,...,k)$ have been defined at Remark 7. By invariance under left translations, this formula holds true even for an arbitrary choice of $x\in\mathbb{G}$. In this case $\varrho(y)=\varrho(y,x)$ is the ϱ-distance from the center of a ϱ-ball $B_\varrho(x,R)$ containing S. The last inequality is the key tool in the proof of the local monotonicity.

4. Main results

Throughout this section we shall present some of the results obtained in [32] and, more precisely, an isoperimetric inequality for the case of a $\mathbf{C}^2$-smooth compact hypersurface with boundary. The hypersurface and its boundary are endowed with the homogeneous measures σ_H^{n-1} and σ_H^{n-2}, respectively. These measures are actually equivalent to the intrinsic $(Q-1)$-dimensional and $(Q-2)$-dimensional Hausdorff measures associated with some given homogeneous metric ϱ on $\mathbb{G}$. As already said, this generalizes a classical inequality proved by Michael and Simon, [29], and Allard, [1]. We shall also deduce some related Sobolev-type inequalities. The strategy of the proof is inspired by the Euclidean one but there are many differences, due to the different geometric setting.

We remark that a monotonicity estimate for the H-perimeter has been recently proved by Danielli, Garofalo and Nhieu in [14] for graphical strips in the Heisenberg group $\mathbb{H}^1$.

Let $S\subset\mathbb{G}$ be a $\mathbf{C}^2$-smooth hypersurface and set $S_t := S\cap B_\varrho(x,t)$, $t>0$. Further set

$$\begin{aligned}\mathcal{A}_\infty(t) &:= \int_{S_t}|\mathcal{H}_H|\Big(1+\sum_{i=2}^{k} i\,c_i\varrho^{i-1}|\varpi_{H_i}|\Big)\sigma_H^{n-1},\\ \mathcal{B}_\infty(t) &:= \int_{\partial S\cap B_\varrho(x,t)}\Big(1+\sum_{i=2}^{k} i\,c_i\varrho^{i-1}|\chi_{H_iS}|\Big)\sigma_H^{n-2},\end{aligned}$$

where $\varrho(y)=\varrho(x,y)$ denotes the ϱ-distance from the (fixed) point $x\in S$ and the constants $c_i\,(i=2,...,k)$ have been defined at Remark 7.

Theorem 10 (Local monotonicity of σ_H^{n-1})**.** *For any $x\in\mathrm{Int}(S\setminus C_S)$ there exists $R(x)>0$ such that*

$$-\frac{d}{dt}\frac{\sigma_H^{n-1}(S_t)}{t^{Q-1}} \leq \frac{1}{t^{Q-1}}\left\{\mathcal{A}_\infty(t)+\mathcal{B}_\infty(t)\right\}$$

for $\mathcal{L}^1$-a.e. $t\in]0,R(x)[$.

The monotonicity inequality could be proved, following a classical pattern, by assuming that *there exists a smooth homogeneous norm* ϱ *satisfying:*

$$\frac{1}{\varrho}|\langle Z_x, grad_{TS}\,\varrho\rangle| \leq 1$$

for every $x, y \in S$; see [32]. This would be "natural" in the Riemannian case and trivially true in the Euclidean one, where

$$Z_x(y) = y - x, \quad grad\,\varrho(y) = \frac{y-x}{|y-x|}.$$

In our case, the above assumption could be too strong and we have to replaced it by a weaker one. More precisely, one may use a "local" integral estimate.

Lemma 11. *For every* $x \in \mathrm{Int}(S \setminus C_S)$, *there exists* $R(x) > 0$ *such that*

$$\int_{S_R} \frac{1}{\varrho}|\langle Z_x, grad_{TS}\,\varrho\rangle|\,\sigma_{H}^{n-1} \leq \sigma_{H}^{n-1}(S_R)$$

for every $R \leq R(x)$, *where* $\varrho(y) = \varrho(x, y)$.

At this point we may state our main results. To this aim, set

$$\begin{aligned}
\mathcal{A}_\infty(S) &:= \int_S |\mathcal{H}_H| \left(1 + \sum_{i=2}^{k} i\, c_i \varrho_S^{i-1} |\varpi_{H_i}|\right) \sigma_H^{n-1},\\
\mathcal{B}_\infty(S) &:= \int_{\partial S} \left(1 + \sum_{i=2}^{k} i\, c_i \varrho_S^{i-1} |\chi_{H_i S}|\right) \sigma_H^{n-2},
\end{aligned}$$

where $\varrho_S := \frac{\mathrm{diam}_\varrho(\mathrm{S})}{2}$.

Theorem 12 (Isoperimetric inequality)**.** *Let* $S \subset \mathbb{G}$ *be a compact* $\mathbf{C}^2$*-smooth hypersurface with* $\mathbf{C}^1$*-smooth boundary. Then there exists a dimensional constant* C_1 *such that*

$$\{\sigma_H^{n-1}(S)\}^{\frac{Q-2}{Q-1}} \leq C_1 \{\mathcal{A}_\infty(S) + \mathcal{B}_\infty(S)\}.$$

Remark 13 (Estimate of $\mathcal{A}_\infty(S)$)**.** *Let* S *be a* $\mathbf{C}^1$*-smooth hypersurface. It is well-known that* $\dim C_S < n-1$. *Note that* $\varpi \in [L^1_{\mathrm{loc}}(\sigma_H^{n-1})]^{n-h}$ *and that* $\varpi \in \mathbf{C}(S \setminus C_S)$. *So let* $\mathcal{U}_\epsilon \subset S$ *be a family of open neighborhoods of* C_S *such that* $\sigma_R^{n-1}(\mathcal{U}_\epsilon) \longrightarrow 0$ *for* $\epsilon \to 0^+$. *For instance, take* $\mathcal{U}_\epsilon = \{|\mathcal{P}_H \nu| < \epsilon \leq 1\}$. *We also stress that* $|\varpi| = O\left(\frac{1}{|\mathcal{P}_H \nu|}\right)$ *as* $\epsilon \to 0^+$. *Therefore*

$$\int_{\mathcal{U}_\epsilon} |\varpi_{H_i}| \sigma_H^{n-1} \leq \sigma_H^{n-1}(S) \qquad (i = 2, ..., k)$$

for all small enough $\epsilon > 0$. *Furthermore*

$$\int_{S\setminus\mathcal{U}_\epsilon} |\varpi_{H_i}| \sigma_H^{n-1} \leq \|\varpi\|_{L^\infty(S\setminus\mathcal{U}_\epsilon)}\, \sigma_H^{n-1}(S) \leq \frac{n-h}{\epsilon}\, \sigma_H^{n-1}(S)$$

for all small enough $\epsilon \in]0, 1]$. *It follows that there exists* $K_1 > 0$ *such that*

$$\mathcal{A}_\infty(S) \leq K_1 \int_S |\mathcal{H}_H| \sigma_H^{n-1}.$$

Remark 14 (Estimate of $\mathcal{B}_\infty(S)$)**.** *First note that if* $\dim C_{\partial S} < n-2$*, then* $\sigma_{R}^{n-2}(C_{\partial S}) = 0$*. Let* $\mathcal{V}_\epsilon \subset \partial S$ *be a family of neighborhoods of* $C_{\partial S}$ *such that* $\sigma_{R}^{n-2}(\mathcal{V}_\epsilon) \longrightarrow 0$ *as* $\epsilon \to 0^+$*. For instance, take* $\mathcal{V}_\epsilon = \{|\mathcal{P}_{HS}\eta| < \epsilon \leq 1\}$*. Since*

$$\chi = \sum_{i=2}^{k} \chi_{H_i S} \in [L^1(\partial S, \sigma_H^{n-2})]^{n-h},$$

one gets that

$$\int_{\mathcal{V}_\epsilon} |\chi_{H_i S}|\sigma_H^{n-2} \leq \sigma_H^{n-2}(\partial S) \qquad (i = 2, ..., k)$$

for all small enough $\epsilon > 0$*. Furthermore*

$$\int_{\partial S \setminus \mathcal{V}_\epsilon} |\chi_{H_i S}|\sigma_H^{n-2} \leq \|\chi\|_{L^\infty(\partial S\setminus\mathcal{V}_\epsilon)}\, \sigma_H^{n-2}(\partial S) \leq \frac{n-h}{\epsilon}\, \sigma_H^{n-2}(\partial S)$$

for all small enough $\epsilon \in]0, 1]$*. It follows that there exists* $K_2 > 0$ *such that*

$$\mathcal{B}_\infty(S) \leq K_2\, \sigma_H^{n-2}(\partial S).$$

By applying the previous remarks we may prove the following:

Corollary 15 (Michael-Simon-type Isoperimetric Inequality)**.** *Let* $S \subset \mathbb{G}$ *be a compact* $\mathbf{C}^2$*-smooth hypersurface with piecewise* $\mathbf{C}^1$*-smooth boundary* ∂S*. Furthermore, let assume that* $\dim C_{\partial S} < n-2$*. Then there exists* $C_2 > 0$ *such that*

$$\{\sigma_H^{n-1}(S)\}^{\frac{Q-2}{Q-1}} \leq C_2 \left\{ \int_S |\mathcal{H}_H|\sigma_H^{n-1} + \sigma_H^{n-2}(\partial S) \right\}.$$

In particular, if $\mathcal{H}_H = 0$*, one has*

$$\{\sigma_H^{n-1}(S)\}^{\frac{Q-2}{Q-1}} \leq C_2 \left\{\sigma_H^{n-2}(\partial S)\right\}.$$

5. Applications

Corollary 16 (Asymptotic of σ_H^{n-1} for $x \in \mathrm{Int}(S \setminus C_S)$)**.** *Let* $S \subset \mathbb{G}$ *be a* $\mathbf{C}^2$*-smooth hypersurface and assume that* $\partial S \cap B_\varrho(x, t) = \emptyset$*. Furthermore, let* $\mathcal{H}_H^0$ *be a positive constant such that* $|\mathcal{H}_H| \leq \mathcal{H}_H^0 < +\infty$*. Then for every* $x \in \mathrm{Int}(S \setminus C_S)$ *one has*

$$\sigma_H^{n-1}(S_t) \geq \kappa_\varrho(\nu_H)\, t^{Q-1} e^{-t\{\mathcal{H}_H^0\,(1+O(t))\}}$$

for $t \to 0^+$*, where* $\kappa_\varrho(\nu_H)$ *is the "density" of* σ_H^{n-1} *at the point* x*, also called* metric factor*; see Section 3.3.*

We may also tract the case where S is immersed in $\mathbb{H}^n$ and $x \in C_S$.

Corollary 17 (Asymptotic of σ_H^{n-1} for $x \in C_S$)**.** *Let* $S \subset \mathbb{H}^n$ *be a* $\mathbf{C}^2$*-smooth hypersurface. Let* $\partial S \cap B_\varrho(x, t) = \emptyset$ *and* $|\mathcal{H}_H| \leq \mathcal{H}_H^0 < +\infty$*. Then for every* $x \in C_S$*, there exists* $\varepsilon_0 > 0$ *such that*

$$\sigma_H^{2n}(S_t) \geq \kappa_\varrho(C_S(x))\, t^{Q-1} e^{-t\mathcal{H}_H^0\,\varepsilon_0}$$

for $t \to 0^+$*, where* $\kappa_\varrho(C_S)$ *is the "density" of* σ_H^{n-1} *at* $x \in C_S$*; see Section 3.3.*

Finally we may state another useful consequence, that is a Sobolev-type inequality.

Corollary 18. *Let $S \subset \mathbb{G}$ be a $\mathbf{C}^2$-smooth closed hypersurface and assume that for every smooth $(n-2)$-dimensional submanifold $N \subset S$ one has $\dim C_N < n-2$. Then there exists $C_3 > 0$ such that*

$$\left\{ \int_S |\psi|^{\frac{Q-1}{Q-2}} \sigma_H^{n-1} \right\}^{\frac{Q-2}{Q-1}} \leq C_3 \int_S \{ |\mathcal{H}_H| |\psi| + |grad_{HS}\psi| \} \sigma_H^{n-1}$$

for all $\psi \in \mathbf{C}^1_0(S)$.

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