Seminario di Analisi Matematica

Dipartimento di Matematica
dell'Università di Bologna

Anno Accademico 2009-2010

Fausto Ferrari

Sulla simmetria delle soluzioni stabili di alcune equazioni semilineari

8 aprile 2010

Sunto

Con particolare riferimento alle proprietà di simmetria, si discuterà del comportamento delle soluzioni stabili di alcune equazioni a derivate parziali semilineari ellittiche. Verranno inoltre presentate alcune disuguaglianze pesate di tipo Poincaré ottenute a partire da campi vettoriali che commutano con l'operatore.

1. Introduction

There exists a wide literature concerning solutions of semilinear problems. Below I would like to remind one of the most popular open questions.

Let $u \in C^2(\mathbb{R}^n, [-1,1])$ and $n \in \mathbb{N}$, $1 \leq n \leq 8$. If u is a solution of

$$\begin{cases} \Delta u = u^3 - u, & \mathbb{R}^n \\ \frac{\partial u}{\partial x_n} > 0, & \mathbb{R}^n, \end{cases} \tag{1}$$

then is u a one dimensional solution?

This problem, stated in an equivalent form, is known as one of the De Giorgi's conjectures, see [12]. As far as I know this conjecture has been completely proved just in dimension $n = 2, 3$, see [22], [5] for $n = 2$, and [2], [1] for $n = 3$. However, recently, several new improvements have been done in order to verify this conjecture for $4 \leq n \leq 8$. In particular the hypothesis on the dimension 8 can not be removed, see [13]. Moreover, assuming the following extra assumptions:

$$\lim_{x_n \to +\infty} u(x', x_n) = 1$$

and

$$\lim_{x_n \to -\infty} u(x', x_n) = -1,$$

Savin proved, see [26], that the solutions of (1) are indeed one dimensional. Notice indeed that this result does not prove the De Giorgi conjecture, even if when the previous limits are attained uniformly with respect to x', the 1D symmetry holds. So, the original De Giorgi's conjecture can considered still open in dimension $4 \leq n \leq 8$ even if the contribution of Savin has been a very important progress.

The attempts to solve the previous conjecture have, in a sense, forced parallel researches concerning the symmetry properties of the solutions of several other semilinear equations possibly using different methods and in different frameworks. In particular the hypothesis of monotonicity can be naturally weakened with the notion of stable solution. Indeed, following the ideas contained in [14] (see also [15]) based on some weighted Poincaré inequalities introduced in [27] and [28], the results proved in dimension $n = 2$ and $n = 3$, can be re-obtained even in a more general contexts, see the more recent survey on the subject [16] for the details.

The stability notion of a solution here arises in a very natural way. Let u be a weak solution of

$$\Delta u = f(u)$$

in $\Omega \subseteq \mathbb{R}^n$.

We know that local critical points of the functional

$$F(v) = \frac{1}{2}\int_\Omega \mid \nabla v \mid^2 + \int_\Omega (\int_0^{u(x)} f(s)ds)dx$$

are local weak solutions of the equation $\Delta u = f(u)$.

In particular, assuming that u realizes a possibly local minimum for F, then for every $\phi \in C_0^\infty(\Omega)$ we can deduce that

$$\frac{d^2F(u+\varepsilon\phi)}{ds^2}_{|\varepsilon=0} \geq 0.$$

This implies naturally what is known in this framework as a stable condition on u.

If u is a weak solution of $\Delta u = f(u)$, then u is a stable solution if for every $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega \left(\mid \nabla\phi \mid^2 + f'(u)\phi^2\right) dx \geq 0. \tag{2}$$

Notice that if u is a C^2 weak solution of $\Delta u = f(u)$, and $\frac{\partial u}{\partial x_n} > 0$, then u is a stable solution. Indeed, taking the test function $\psi = \frac{\phi^2}{\partial_n u}$, where $\phi \in C_0^\infty(\Omega)$ and $\partial_n u$ denotes $\frac{\partial u}{\partial x_n}$, then

$$\int_\Omega \langle \nabla\partial_n u, \nabla\psi\rangle dx + \int_\Omega f'(u)\partial_n u\psi dx = 0.$$

Thus developing the calculation we get

$$\int_\Omega \langle \nabla\partial_n u, \frac{\nabla(\phi)^2}{\partial_n u}\rangle dx - \int_\Omega \langle \nabla\partial_n u, \frac{\nabla\partial u_n\phi^2}{(\partial_n u)^2}\rangle dx + \int_\Omega f'(u)\phi^2 dx = 0. \tag{3}$$

Then recalling Cauchy-Schwarz inequality and Cauchy inequality we get

$$\langle \nabla\partial_n u, \frac{\nabla(\phi)^2}{\partial_n u}\rangle \leq \phi^2\frac{\mid \nabla\partial_n u \mid^2}{(\partial_n u)^2} + \mid \nabla\phi \mid^2 .$$

As a consequence, plugging previous inequality in (3) we have

$$\int_\Omega \left(\phi^2 \frac{|\nabla \partial_n u|^2}{(\partial_n u)^2} + |\nabla \phi|^2\right) - \int_\Omega \langle \nabla \partial_n u, \frac{\nabla \partial u_n \phi^2}{(\partial_n u)^2}\rangle dx + \int_\Omega f'(u)\phi^2 dx \geq 0,$$

thus

$$\int_\Omega |\nabla \phi|^2 \, dx + \int_\Omega f'(u)\phi^2 dx \geq 0,$$

and the stability follows. Thus, we could say that the problem of studying the symmetries of stable solutions, for example when $f(s) = s^3 - s$, is, in a sense, the weak version of the De Giorgi's conjecture.

In particular this different point of view is useful to consider the parallel problem associated with different semilinear partial differential equations. For instance, I am thinking about the case of the real part of the Kohn-Laplace operator, that preserves a lot of properties that the classical Laplace operator has or to Grushin operator or the Laplace operator in some Riemannian manifold.

Among this type of problems I recall first the contributions of [8], [9], [10] and following the general notion of stable solution also the researches contained in [20] ,[21], [6] and [7].

The notion of stability is the key to exploit the particular structure of the semilinear equation

$$\Delta u = f(u)$$

in Ω, where usually Ω is all the space $\mathbb{R}^n$.

Indeed, assuming that f is C^1 and u is smooth enough, we can differentiate both terms of the equation. As a consequence for every $j = 1, \ldots, n$, $\partial_j u$ is still solution of the equation

$$\Delta(\partial_j u) = f'(u)\partial_j u$$

in Ω.

How to connect this information with the notion of stability? Just considering the weak meaning of solution for $\partial_j u$. Namely, for every $j = 1, \ldots, n$, and for every $\phi \in C_0^\infty(\Omega)$ we get, integrating by parts,

$$\int_\Omega \langle \nabla \partial_j u, \nabla \psi\rangle + \int_\Omega f'(u)\partial_j u \psi = 0. \tag{4}$$

So, by choosing in the previous equation a nice test function for every j and a parallel test function in the stability inequality (2), we are able to generate a weighted Poincaré inequality.

More precisely, putting in (4), respectively for every $j = 1, \dots, n$, $\psi = \partial_j u \phi^2$ we get

$$\int_\Omega \langle \nabla \partial_j u, \nabla(\partial_j u \phi^2)\rangle + \int_\Omega f'(u)(\partial_j u)^2 \phi^2 = 0,$$

and summing from $j = 0, \dots, n$ we get

$$\int_\Omega \sum_{j=1}^n \langle \nabla \partial_j u, \nabla(\partial_j u \phi^2)\rangle + \int_\Omega f'(u) \mid \nabla u \mid^2 \phi^2 = 0. \tag{5}$$

We remark that if $\partial_n u > 0$, then $\nabla u \neq 0$, thus $\mid \nabla u \mid \phi \in \mathrm{Lip}_0$. However in general, we have to relax the definition of stability we introduced before (as follows) in order to consider more general cases.

A weak solution u of $\Delta u = f(u)$ is stable if for every $\phi \in C_0^\infty(\Omega)$

$$\int_\Omega \left(\mid \nabla \eta \mid^2 + f'(u)\eta^2\right) dx \geq 0, \tag{6}$$

for every $\eta = \phi \mid \nabla u \mid$.

Notice that $\nabla(\mid \nabla u \mid \phi) = 0$ almost everywhere in $\{\mid \nabla u \mid = 0\} \cap \Omega$, see Stampacchia's Theorem (Theorem 6.19 in [23]) .

Thus putting $\eta = \mid \nabla u \mid \phi$, where $\phi \in C_0^\infty(\Omega)$, in the stability inequality (6), we get

$$\int_{\Omega \cap \{\nabla u \neq 0\}} \mid \nabla(\mid \nabla u \mid \phi) \mid^2 + \int_\Omega f'(u) \mid \nabla u \mid^2 \phi^2 \geq 0. \tag{7}$$

Comparing (5) with (7), immediatedely follows

$$\int_{\Omega \cap \{\nabla u \neq 0\}} \mid \nabla(\mid \nabla u \mid \phi) \mid^2 \geq \int_\Omega \sum_{j=1}^n \langle \nabla \partial_j u, \nabla(\partial_j u \phi^2)\rangle.$$

Developing the calculation we get

$$
\begin{aligned}
&\int_{\Omega\cap\{\nabla u\neq 0\}} |\nabla|\nabla u||^2\,\phi^2 + \int_{\Omega\cap\{\nabla u\neq 0\}} |\nabla\phi^2||\nabla u|^2 \\
&+2\int_{\Omega\cap\{\nabla u\neq 0\}} \langle\nabla|\nabla u|,\nabla\phi\rangle\phi|\nabla u| \geq \int_\Omega \sum_{j=1}^n \langle\nabla\partial_j u,\nabla\partial_j u\rangle\phi^2 \\
&+\int_\Omega \sum_{j=1}^n \langle\nabla\partial_j u,\nabla(\phi^2)\rangle\partial_j u \\
&\geq \int_{\Omega\cap\{\nabla u\neq 0\}} \sum_{j=1}^n \langle\nabla\partial_j u,\nabla\partial_j u\rangle\phi^2 + \int_{\Omega\cap\{\nabla u\neq 0\}} \sum_{j=1}^n \langle\nabla\partial_j u,\nabla(\phi^2)\rangle\partial_j u.
\end{aligned}
\tag{8}
$$

Thus

$$
\int_{\Omega\cap\{\nabla u\neq 0\}} |\nabla|\nabla u||^2\,\phi^2 + \int_{\Omega\cap\{\nabla u\neq 0\}} |\nabla\phi^2||\nabla u|^2 \geq \int_{\Omega\cap\{\nabla u\neq 0\}} \sum_{j=1}^n \langle\nabla\partial_j u,\nabla\partial_j u\rangle\phi^2,
$$

and

$$
\int_\Omega |\nabla\phi^2||\nabla u|^2 \geq \int_{\Omega\cap\{\nabla u\neq 0\}} \left(\sum_{j=1}^n \langle\nabla\partial_j u,\nabla\partial_j u\rangle - |\nabla|\nabla u||^2\right)\phi^2.
$$

Eventually we can rearrange the previous inequality getting the key relation

$$
\int_\Omega |\nabla\phi^2||\nabla u|^2 \geq \int_{\Omega\cap\{\nabla u\neq 0\}} \left(|D^2u|^2 - |\nabla|\nabla u||^2\right)\phi^2,
\tag{9}
$$

where D^2u denotes the Hessian matrix of u and $|D^2u|^2=\sum_{i,j=1}^n(\partial_{ij}u)^2$.

Inequality (9) is the second important ingredient to consider in this approach.

The analysis of such weighted inequality is worth to be considered. Notice that the weight on the right hand side is positive, i.e.

$$
|D^2u|^2 - |\nabla|\nabla u||^2 \geq 0.
$$

Indeed, in $\Omega\setminus\{\nabla u\neq 0\}$

$$
\nabla|\nabla u| = \frac{D^2u\nabla u}{|\nabla u|}
$$

and so

$$\mid D^2u\mid^2-\mid\nabla\mid\nabla u\mid\mid^2=\mid D^2u\mid^2-\langle D^2u\frac{\nabla u}{\mid\nabla u\mid},D^2u\frac{\nabla u}{\mid\nabla u\mid}\rangle$$
$$=\mid D^2u\mid^2-\langle(D^2u)^2\frac{\nabla u}{\mid\nabla u\mid},\frac{\nabla u}{\mid\nabla u\mid}\rangle=\mathrm{Tr}(D^2u)^2-\langle(D^2u)^2\frac{\nabla u}{\mid\nabla u\mid},\frac{\nabla u}{\mid\nabla u\mid}\rangle.$$

Actually, in order to convince the reader that $\mid D^2u\mid^2-\mid\nabla\mid\nabla u\mid\mid^2\geq 0$, we argue as follows, see [17] for further details. For any $x\in\Omega\setminus\{\nabla u\neq 0\}$ we choose an orthonormal basis of eigenvectors of $D^2u(x)$ of $\mathbb{R}^n$. Let us denote such basis $\{v_1,\dots,v_n\}$ and λ_i denotes, for $i=1,\dots,n$ the eigenvalue associated with the eigenvector v_i. Let Q be the unit matrix obtained considering the matrix

$$[v_1,\dots,v_n]$$

where v_i are considered as columns. Then

$$\mathrm{Tr}(D^2u)^2=\mathrm{Tr}(D^2uQ^TQD^2uQ^TQ)=\mathrm{Tr}(Q^TD^2uQQ^TD^2uQ^T)=\sum_{i=1}^n\lambda_i^2.$$

Thus assuming also that $D^2u(x)\neq 0$, otherwise the result is true, we get

$$\mid D^2u\mid^2-\mid\nabla\mid\nabla u\mid\mid^2=\mathrm{Tr}(D^2u)^2-\langle(D^2u)^2\frac{\nabla u}{\mid\nabla u\mid},\frac{\nabla u}{\mid\nabla u\mid}\rangle$$
$$=\sum_{i=1}^n\lambda_i^2-\langle D^2u\frac{\nabla u}{\mid\nabla u\mid},D^2u\frac{\nabla u}{\mid\nabla u\mid}\rangle\geq\sum_{i=1,i\neq\bar{i}}^n\lambda_i^2>0,$$

where $\bar{i}\in\{1,\dots,n\}$ is defined as follows

$$\lambda_{\bar{i}}^2(x)=\max\{\lambda_i(x)^2:\ \ i=1,\dots,n,\ \ \lambda_i\ \text{ is an eigenvalue of }\ D^2u(x)\}.$$

Let us define, see [6],

$$\mathcal{D}(u)=\mid D^2u\mid^2-\mid\nabla\mid\nabla u\mid\mid^2$$

the defect of u. Thus, summarizing, we got

$$\int_{\Omega\cap\{|\nabla u|\neq 0\}}\phi^2\mathcal{D}(u)dx\leq\int_\Omega\mid\nabla\phi^2\mid\mid\nabla u\mid^2dx, \tag{10}$$

where $\mathcal{D}(u)$ and $\mid\nabla u\mid^2$ are two weights. The defect $\mathcal{D}(u)$ can be characterized more precisely. Indeed it can be proved, se [27] ans [28] that

$$\mathcal{D}(u)=\mid\nabla u\mid^2\mathcal{C}_2(u)+\mid\nabla_T\mid\nabla u\mid\mid^2,$$

where $\mathcal{C}_2(u)$ is the sum of the squares of the principal curvatures of the level set $\{u = c\}$ and ∇_T is the tangential gradient along the level set of u. We remind here that $\{\nabla u \neq 0\}$ so that in our hypothesis $\{u = c\}$ is a smooth manifold on $\{\nabla u \neq 0\}$.

The last but not the least important point concerns with the possibility that a sequence of functions $\{\phi_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\Omega)$ such that $\phi_j \to 1$, as $j \to \infty$ and the right hand size of (10) goes to zero exists. Namely we would like that

$$\int_\Omega \mid \nabla \phi_j^2 \mid\mid \nabla u \mid^2 dx \to 0,$$

as $j \to \infty$.

Thus, assuming that such sequence of functions exists, we deduce that $\mathcal{C}_2(u) = 0$ and $\mid \nabla_T \mid \nabla u \mid\mid = 0$. In particular all the principal curvatures of the level surfaces $\{u = c\}$ are zero and $\nabla_T \mid \nabla u \mid = 0$. In particular the level surfaces of u are hyperplanes and the projection of the gradient of $\mid \nabla u \mid$ to the tangent plane $\{u = c\}$ is constant. In dimension 2, it can be proved the existence of this type of sequence for any stable solution just considering for every $j \in \mathbb{N} \setminus \{0\}$

$$\phi_j(x) = \begin{cases} 1, & \text{if } \mid x \mid \leq \sqrt{j} \\ 2^{\frac{\log j - \log|x|}{\log j}}, & \text{if } \sqrt{j} \leq \mid x \mid \leq j \\ 0, & \text{if } \mid x \mid \geq j. \end{cases} \tag{11}$$

In particular, following the steps previously described, it can be proved a more general statement of the De Giorgi's conjecture for stable solutions. Even in dimension 3 the De Giorgi's conjecture can be proved (i.e. assuming that $\partial u_3 > 0$ and taking $f(s) = s^3 - s$) following this idea.

Hence, the generality of this approach is quite evident and it can be adapted to several types of equations. Indeed the problem was studied, applying this strategy, in the Heisenberg group, see [20] and for Grushin operator, see [21]. I would like to stress some interesting aspects that in these cases appear and that have been studied in [6]

2. The role of the right invariant vector fields

We can assume to study our problem in a more general framework. Say, from an abstract point of view. Roughly speaking, let $\mathcal{M}$ be a set endowed with a structure such that the integration by parts make sense. We can think, for instance, to a Riemannian manifold or a sub-Riemannian manifold.

Thus, suppose that in $\mathcal{M}$ a measure dV is given and for every $x \in \mathcal{M}$ there exists a vector space $\mathcal{M}_x$. Moreover we can assume that there exist also a divergence operator div, a measure dS a vector field n on ∂Q, for any open set $Q \subset \mathcal{M}$, such that for each vector fields b on Q makes sense the following divergence theorem:

$$\int_Q \mathrm{div} b dV = \int_{\partial Q} \langle b, n\rangle dS.$$

In particular, if for every function u on $\mathcal{M}$ there exists a vector field

$$\nabla u(x) = (X_1 u(x), \ldots, X_m u(x))$$

where $X_1, \ldots, X_m$ are some vector fields, we can assume that the following operator $\mathcal{L}$ can be written in divergence form and $\mathcal{L} = \mathrm{div}\nabla u = \sum_{i=1}^n X_i^2 u$. Hence,

$$\int_Q \mathrm{div}\nabla u dV = \int_{\partial Q} \langle \nabla u, n\rangle dS.$$

In this case, denoting $\mathcal{L} = \sum_{i=1}^n X_i^2$ we shall consider the solution u defined in M of

$$\mathcal{L}u = f(u).$$

In this case we can define the weak solution, as usual, considering as solutions those functions u such that for every $\phi \in C_0^\infty(\mathcal{M})$

$$\int_{\mathcal{M}} \sum_{i=1}^n X_i u X_i \phi dV + \int_{\mathcal{M}} f(u)\phi dV = 0.$$

Notice that taking in account the functional

$$A(v) = \frac{1}{2}\int_{\mathcal{M}} \sum_{i=1}^n (X_i v)^2 + \int_{\mathcal{M}} (\int_0^u f(s)ds)dV$$

the critical points of A are weak solutions and the assumption

$$\frac{d^2 A(u+\varepsilon\phi)}{d\varepsilon^2}_{|\varepsilon=0} \geq 0,$$

reads as follows, for every $\phi \in C_0^\infty(M)$

$$\int_{\mathcal{M}} \sum_{i=1}^n (X_i\phi)^2 dV + \int_{\mathcal{M}} f'(u)\phi^2 dV \geq 0$$

i.e. the definition of stable solution in this abstract framework.

Now the problem arising in the next step depends on the commutativity properties of the vector fields X_i, $i = 1, \dots, m$. Indeed, in general it is not true that $X_iX_ju = X_jX_iu$, whenever $i \neq j$. This fact is measured, in a sense, by the commutator, i.e. the vector field defined as

$$[X_i, X_j]u = X_iX_ju - X_jX_iu.$$

For example, in the first approach described in [20], in that paper the authors dealt with in the Heisenberg group, the problem was traited considering the equation that naturally arises differentiating with respect to X_i. It came out that X_iu was still solution of the equation

$$\mathcal{L}v = f'(u)v - H_i([\nabla, \nabla]X_mu, \dots, [\nabla, \nabla]X_mu),$$

where H_i was a function of the remainders due to the noncommutativity of the vector fields, see also [18].

However the extra terms H_i introduce some distortion effects in the procedure and more complicate calculations that in principle it should be better to avoid. In the case of the Heisenberg group, for example, we get see [20]

$$\begin{aligned} \Delta_{\mathbb{H}} Xu &= f'(u)Xu - 2TYu \\ \Delta_{\mathbb{H}} Yu &= f'(u)Yu + 2TXu. \end{aligned} \tag{12}$$

Here $H_X([X, Y]Yu) = -2[X, Y]Yu$ and $H_Y([X, Y]Yu) = 2[X, Y]Xu$.

Thus in [6] the idea to avoid the extra terms H_i was investigated. Indeed if there are vectors that commute with the operator the extra terms H_i desappear. This suggests that some nice path, and the related vector fields that we simply denote $\tilde{X}$ and $\tilde{Y}$, could exist such that $\tilde{X}u$ and $\tilde{Y}$ are solutions of

$$\mathcal{L}v = f'(u)v.$$

For instance in the simplest Heisenberg group $\mathbb{H}^1$ the vector fields $\tilde{X} = \partial_x - 2y\partial_t$ and $\tilde{Y} = \partial_y + 2x\partial_t$ enjoy this property. Let me recall that here $\mathbb{H}^1$ is $\mathbb{R}^3$ endowed with the non-commutative law defined as follows. For every $(x_1, y_1, t_1), (x_2, y_2, t_2) \in \mathbb{R}^3$,

$$(x_1, y_1, t_1) \circ (x_2, y_2, t_2) = (x_1 + x_2, y_1 + y_2, t_1, t_2 + 2(x_2y_1 - x_1y_2)).$$

Usually $X = (1, 0, 2y)$ and $Y = (0, 1, -2x)$ also denote the vector fields $X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}$ and $Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}$. Notice that X and Y does not commute

$$[X, Y] = T = -4\frac{\partial}{\partial t}$$

and all the further vector fields obtained by subsequent commutations vanish. In particular, the algebra spanned by X and Y has dimension 3.

On each fiber $\mathcal{H}_P$ given by the vector space generated by $X(P)$ and $Y(P)$, $\mathcal{H}_P = \mathrm{span}\{X(P), Y(P)\}$, a metric can be defined as follows. Let $U, V \in \mathcal{H}_P$, and $U = \alpha_1 X + \beta_1 Y$ e $V = \alpha_2 X + \beta_2 Y$. We set

$$\langle U, V\rangle_{\mathbb{H}^1} = \alpha_1\alpha_2 + \beta_1\beta_2.$$

This internal dot makes orthonormal, in $\mathcal{H}_P$, the vectors X e Y. Moreover the norm of a vector in $U \in \mathcal{H}_P$ will be defined as

$$\mid U \mid_{\mathbb{H}^1} = \sqrt{\langle U, X\rangle_{\mathbb{H}^1}^2 + \langle U, Y\rangle_{\mathbb{H}^1}^2}.$$

If u is a C^1 function then $\nabla_{\mathbb{H}^1} u(P) = (Xu(P), Yu(P)) = Xu(P)X(P) + Yu(P)Y(P)$. The vector $\nabla_{\mathbb{H}^1} u$ is the intrinsic gradient of u. A point $P \in \Sigma$ is characteristic for $\Sigma = \{u = c\}$, $u \in C^1$, if $\mathcal{H}_P = T_P\Sigma$ in P, where $T_P\Sigma$ is the tangent space to Σ at P. In particular if $\nabla_{\mathbb{H}^1} u(P) \neq 0$, then P is not characteristic.

If u is a C^1 function, $P \in \{u = k\} \cap \{\nabla_{\mathbb{H}^1} u \neq 0\}$, and $\{u = k\}$ then

$$\nu = \frac{\nabla_{\mathbb{H}^1} u(P)}{\mid \nabla_{\mathbb{H}^1} u(P) \mid} \tag{13}$$

is the intrinsic normal to $\{u = k\}$ at P.

If ν is the intrinsic normal at $P \in \{u = k\}$ (noncharacteristic), then

$$v = \frac{(Yu(P), -Xu(P))}{\mid \nabla_{\mathbb{H}^1} u \mid} \tag{14}$$

is the intrinsic tangent unit vector to $\{u = k\}$at P. In particular $\langle \nu, v\rangle_{\mathbb{H}^1} = 0$. For further details see e.g. [3] and [4]. The real part of the Kohn-Laplace operator is

$$\Delta_{\mathbb{H}^1} u = X^2 u + Y^2 u,$$

moreover

$$\Delta_{\mathbb{H}} u = \mathrm{div}_{\mathbb{H}^1}(\nabla_{\mathbb{H}^1} u) = X(Xu) + Y(Yu)$$

and for every bounded smooth set $\Omega \subset \mathbb{H}^1$

$$\int_{\Omega} \mathrm{div}_{\mathbb{H}^1}(\nabla_{\mathbb{H}^1} u) dV = \int_{\partial\Omega} \langle \nabla_{\mathbb{H}^1} u, \nu \rangle_{\mathbb{H}^1} dS,$$

where dV is the Lebesgue measure in $\mathbb{R}^3$, $dS = \sqrt{\langle X, n\rangle^2 + \langle Y, n\rangle^2} d\mathcal{H}^2$, n is the Euclidean normal to $\partial\Omega$ at P and $\mathcal{H}^2$ is the Hausdorff measure.

The Hessian horizontal matrix of u is

$$Hu = \begin{bmatrix} XXu, & YXu \\ XYu, & YYu \end{bmatrix}.$$

This matrix is not symmetric and its norm is:

$$|Hu| = \sqrt{(XXu)^2 + (YXu)^2 + (XYu)^2 + (YYu)^2} \tag{15}$$

and

$$(Hu)^2 = (Hu)(Hu)^T. \tag{16}$$

The vector fields $\tilde{X}$ and $\tilde{Y}$ have the nice property of commuting with X and Y. So that

$$\Delta_{\mathbb{H}^1}(\tilde{X}u) = f'(u)\tilde{X}u$$

and

$$\Delta_{\mathbb{H}^1}(\tilde{Y}u) = f'(u)\tilde{Y}u.$$

Then we can repeat the procedure described in the introduction in order to obtain the following inequality

$$\int_{\Omega\cap\{\tilde{\nabla}u\neq 0\}} \phi^2 \tilde{\mathcal{D}}(u) dx \leq \int_{\Omega} \mid \nabla_{\mathbb{H}^1}\phi \mid^2 \mid \tilde{\nabla}_{\mathbb{H}^1} u \mid^2, \tag{17}$$

where the *defect* of u is, compare it with the Euclidean case of formula (10),

$$\tilde{\mathcal{D}}(u) = \mid \tilde{\nabla}_{\mathbb{H}^1} Xu \mid^2 + \mid \tilde{\nabla}_{\mathbb{H}^1} Yu \mid^2 - \left(\langle \tilde{\nu}, \tilde{\nabla}_{\mathbb{H}^1} Xu \rangle^2_{\tilde{\mathbb{H}}^1} + \langle \tilde{\nu}, \tilde{\nabla}_{\mathbb{H}^1} Yu \rangle^2_{\tilde{\mathbb{H}}^1} \right),$$

$$\tilde{\nabla}_{\mathbb{H}^1} u(P) = (\tilde{X}u, \tilde{Y}u) = \tilde{X}u(P)\tilde{X}(P) + \tilde{Y}(P)\tilde{Y}(P),$$

$\tilde{X}(P) = (1, 0, -2y)$, $\tilde{Y}(P) = (0, 1, 2x)$. A parallel metric associated with span$\{\tilde{X}, \tilde{Y}\}$, denoted as $\langle \cdot, \cdot \rangle_{\tilde{\mathbb{H}}^1}$, can be defined in such a way that $\tilde{X}, \tilde{Y}$ are orthonormal and

$$\tilde{\nu}(P) = \frac{\tilde{\nabla}_{\mathbb{H}^1} u(P)}{\sqrt{(\tilde{X}u(P))^2 + (\tilde{Y}u(P))^2}},$$

here we omit all the details nevertheless a parallel geometric structure can be defined around the the vector fields $\tilde{X}$ and $\tilde{Y}$. Moreover it can be proved that

$$\tilde{D}(u) = \frac{(X\tilde{X}u\tilde{Y}u - X\tilde{Y}u\tilde{X}u)^2 + (Y\tilde{X}u\tilde{Y}u - Y\tilde{Y}u\tilde{X}u)^2}{(\tilde{X}u)^2 + (\tilde{Y}u)^2}.$$

Thus, if a sequence of functions $\{\phi_j\} \subset C_0^\infty(\Omega)$ converging to 1 and such that as $j \to \infty$

$$\int_\Omega \mid \nabla_{\mathbb{H}^1} \phi_j \mid^2 \mid \tilde{\nabla}_{\mathbb{H}^1} u \mid^2 \to 0,$$

exists, then $\tilde{D}(u) = 0$. In particular we deduce that $X\tilde{X}u\tilde{Y}u - X\tilde{Y}u\tilde{X}u = 0$ and $Y\tilde{X}u\tilde{Y}u - Y\tilde{Y}u\tilde{X}u = 0$. As a consequence $X(\frac{\tilde{X}}{\tilde{Y}}) = 0$ and $Y(\frac{\tilde{X}}{\tilde{Y}}) = 0$, thus $\frac{\tilde{X}}{\tilde{Y}}$ has to be constant.

This approach can be generalized and the details are contained in [6]. I shall come back to some specific results in the next section.

3. Applications

3.1. **Some Poincaré Inequalities.** If we consider the inequality (17) some tools useful in the study of the regularity theory of PDE's can be proved. Notice that, in particular, it is enough to choose in the right way the function u, see [19].

(i) For instance, in two dimension, just considering the usual Laplace operator, we can take $\tilde{X} = X = \partial_x$ and $\tilde{Y} = Y = \partial_y$. Then

$$\int_{\mathbb{R}^2 \cap \{\nabla u \neq 0\}} \phi^2 \tilde{D}(u) \leq \int_{\mathbb{R}^2} \mid \nabla \phi \mid^2 \mid \nabla u \mid^2 .$$

The function $u = x^2 + y^2$ is stable. In particular, with this substitution, we get:

$$\int_{\mathbb{R}^2 \cap \{\nabla u \neq 0\}} \phi^2 \leq \int_{\mathbb{R}^2} \mid \nabla \phi \mid^2 (x^2 + y^2).$$

(ii) In the Heisenberg group $\mathbb{H}^1$, let $u = t$. Then $Xt = 2y$, $Yt = -2x$, $XXt = 0$, $YYt = 0$ so that $\Delta_{\mathbb{H}^1} t = 0$, thus t is also stable. Moreover $\tilde{X}t = -2y$, $\tilde{Y}t = 2x$

$X\tilde{X}t = 0$, $Y\tilde{X}t = -2$, $Y\tilde{Y} = 0$, $X\tilde{Y}t = 2$ and $\tilde{D}t = 4$. Thus for every $\phi \in C_0^\infty(\mathbb{H}^1)$ we get

$$\int_{\mathbb{H}^1} \phi^2 \leq \int_{\mathbb{H}^1} \mid \nabla_{\mathbb{H}^1}\phi \mid^2 (x^2+y^2).$$

In particular, for every $\phi \in C_0^\infty(B_R)$

$$\int_{B_R} \phi^2 \leq \int_{B_R} \mid \nabla_{\mathbb{H}^1}\phi \mid^2 \sqrt{(x^2+y^2)^2+t^2} \leq R^2 \int_{B_R} \mid \nabla_{\mathbb{H}^1}\phi \mid^2,$$

where

$$B_R = \{(x,y,t) \in \mathbb{H}^1 :\ (x^2+y^2)^2+t^2 < R^4\}$$

is the Korányi ball.

(iii) In the case of the Beltrami operator in $\mathbb{R}\times\mathbb{R}^+$ we have $\mathcal{L} = y^2\Delta$. Here $\nabla_h = (y\partial_x, y\partial_y)$, the vector fields are $\tilde{X} = \partial_x$ and $\tilde{Y} = x\partial_x + y\partial_y$. They commute with $\mathcal{L}$, moreover we can integrate by parts considering the area measure $dV = \frac{dxdy}{y^2}$. This is an example of a Riemannian manifold given by the hyperbolic metric. In particular we get

$$\int_{\mathbb{R}\times\mathbb{R}^+\cap\{\tilde{\nabla}u\neq 0\}} \phi^2\tilde{D}(u)dV \leq \int_{\mathbb{R}\times\mathbb{R}^+} \mid \nabla_h\phi \mid^2 \mid \tilde{\nabla}u \mid^2 dV.$$

If we take $u = \frac{x}{y^m}$ in $\mathbb{R}\times\mathbb{R}^+$ we get $\mathcal{L}u = m(m+1)u$. Thus u is stable. Moreover $\tilde{X}u = y^{-m}$, $\tilde{Y}u = (1-m)xy^{-m}$, $X\tilde{X}u = 0$, $Y\tilde{X}u = -my^{-m}$, $Y\tilde{Y}u = -m(1-m)xy^{-m}$, and $X\tilde{Y}u = (1-m)y^{-m+1}$

(18)
$$\tilde{D}(u) = \frac{(X\tilde{X}u\tilde{Y}u - X\tilde{Y}u\tilde{X}u)^2 + (Y\tilde{X}u\tilde{Y}u - Y\tilde{Y}u\tilde{X}u)^2}{(\tilde{X}u)^2+(\tilde{Y}u)^2} = \frac{(1-m)^2y^{-4m+2}}{y^{-2m}+(1-m)^2x^2y^{-2m}}$$
$$= \frac{(1-m)^2y^{-2m+2}}{1+(1-m)^2x^2}$$

Thus

$$\int_{\mathbb{R}\times\mathbb{R}^+} \phi^2 \frac{(1-m)^2y^{-2m}}{1+(1-m)^2x^2}dxdy \leq \int_{\mathbb{R}\times\mathbb{R}^+} \mid \nabla\phi \mid^2 (1+(1-m)^2x^2)y^{-2m}dxdy.$$

3.2. **Some results about symmetries of solutions of semilinear equations.** I recall below few cases concerning some symmetric properties of stable solutions.

(i) In the case of the Laplace operator if we can prove that the second member of the inequality is zero, then in $\{\nabla u \neq 0\}$ there exists a constant c such that $\partial_x = c\partial_y$. Thus the level surfaces are always orthogonal to the vector $(1,-c)$. Namely, the stable solutions of $\Delta u = f(u)$ have the level surfaces given by the straight lines $x - cy = 0$. Thus, up to a rotation, stable solutions depend only on one variable, i.e. there exists a smooth function $g : \mathbb{R} \to \mathbb{R}$ and a constant vector $a \in \mathbb{R}^2$ such that $u(x,y) = g(\langle a, (x,y)\rangle)$. In particular this means that it is enough to study the one dimensional equation $g'' = f(g)$.

(ii) In the case of the Heisenberg group it can be proved the following no existence result in $\mathbb{H}^1$, see Theorem 1 in [6] for the general version in $\mathbb{H}^n$.

There is no solution u of $\Delta_{\mathbb{H}^1} u = f(u)$ in $\mathbb{H}^1$ satisfying the following two conditions: $Xu > 0$ in $\mathbb{H}^1$ or $Yu > 0$ in $\mathbb{H}^1$ and

$$\limsup_{R\to+\infty} R^{-4} \int_{B_R} (x^2+y^2) \mid \tilde{\nabla}_{\mathbb{H}^1} u(x,y,t) \mid^2 dxdydt < \infty.$$

Indeed, the result is based on the fact that if $\tilde{D}(u) = 0$ in $\Omega \subseteq \{\tilde{\nabla}_{\mathbb{H}^1} u \neq 0\}$, then $\frac{\tilde{\nabla}_{\mathbb{H}^1} u}{|\tilde{\nabla}_{\mathbb{H}^1} u|}$ is constant in Ω, and moreover, see Lemma 8 in [6] for the general version in $\mathbb{H}^n$, if $\{\tilde{\nabla}_{\mathbb{H}^1} u = 0\} = \emptyset$, then there exists a suitable function $g : \mathbb{R} \to \mathbb{R}$ and a vector $a \in \mathbb{R}^2$ such that for every $(x,y,t) \in \mathbb{H}^1$, $u(x,t) = g(\langle a, (x,y)\rangle)$.

(iii) As far as the Beltrami operator in dimension 2 concerns, if the second term of the main inequality goes to zero, possibly by choosing an opportune sequence of functions, it would mean that there exists a constant c such that in $\{\tilde{\nabla} u \neq 0\}$,

$$\tilde{X}u + c\tilde{Y}u = 0.$$

In this case it comes out that $\partial_x + c(x\partial_x u + y\partial_y u) = 0$. Thus $(1+cx)\partial_x u + cy\partial_y u = 0$. In particular this would imply that the level sets are given for every $k \in \mathbb{R}^+$ by

$$y = k \mid 1 + cx \mid$$

in $\mathbb{R} \times \mathbb{R}^+$. Otherwise it could be $\partial_x u = 0$, that in any case would imply that the level lines are straight line parallel to the x axis of the type $y = k$. Eventually it could happen that $x\partial_x u + y\partial_y u = 0$. This last case would imply that the level lines are given by $y = k \mid x \mid$. Notice that this theoretical discussion would be coherent with the results concerning the symmetries of the solutions of semilinear equations in the hyperbolic space obtained in [7].

Unfortunately, it is not clear if it is possible to prove that there exists a sequence $\{\phi_j\}_{j\in\mathbb{N}} \subset C_0^\infty(\mathbb{R} \times \mathbb{R}^+)$ such that $\phi_j \to 1$ and

$$\int_{\mathbb{R}\times\mathbb{R}^+} \mid \nabla_h \phi_j \mid^2 \mid \tilde{\nabla} u \mid^2 dV \to 0,$$

without assume too restrictive conditions on u.

References

[1] G. Alberti, L. Ambrosio, X. Cabré, *On a long-standing conjecture of E. De Giorgi: symmetry in 3D for general nonlinearities and a local minimality property.* Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday, Acta Appl. Math.,**65** (2001) 9-33.

[2] L. Ambrosio, X. Cabré, *Entire solutions of semilinear el- liptic equations in R^3 and a conjecture of De Giorgi* J. Amer. Math. Soc., 13(4):725739 (electronic), 2000.

[3] N. Arcozzi, F. Ferrari, *Metric normal and distance function in the Heisenberg group*, Math. Z. 256(3), 661–684 (2007).

[4] N. Arcozzi, F. Ferrari, *The Hessian of the distance from a surface in the Heisenberg group*, Ann. Acad. Sci. Fenn. Math. 33(1), 35–63 (2008). (http://mathstat.helsinki.fi/Annales/Vol33/vol33.html)

[5] H. Berestycki, L. Caffarelli, L. Nirenberg, *Further qualitative properties for elliptic equations in unbounded domains.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4), 25(1-2):6994 (1998), 1997. Dedicated to Ennio De Giorgi.

[6] I. Birindelli, F. Ferrari, E. Valdinoci, *Semilinear PDEs in the Heisenberg group: the role of the right invariant vector fields.* Nonlinear Anal. **72** (2010) 987–997.

[7] I. Birindelli, R. Mazzeo, *Symmetry for solutions of two-phase semilinear elliptic equations on hyperbolic space.* Indiana Univ. Math. J. **58** (2009), 2347–2368.

[8] I. Birindelli, E. Lanconelli, *A note on one dimensional symmetry in Carnot groups.* Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., **13** (2002) 17-22.

[9] I. Birindelli, E. Lanconelli, *A negative answer to a one-dimensional symmetry problem in the Heisenberg group.* Calc. Var. Partial Differential Equations, **18** (2003) 357-372.

[10] I. Birindelli, J. Prajapat, *Monotonicity and symmetry results for degenerate elliptic equations on nilpotent Lie groups.* Pacific J. Math., **204** (2002) 1-17.

[11] X. Cabré, A. Capella, *Regularity of radial minimizers and extremal solutions of semilinear elliptic equations.* J. Funct. Anal., **238** (2006) 709-733.

[12] E. De Giorgi, *Convergence problems for functionals and operators.* Proceedings of the International Meeting on Recent Methods in Nonlinear Analysis (Rome, 1978),Pitagora, Bologna, (1979) 131-188.

[13] M. Del Pino, M. Kowalczyk, and J. Wei, *A counterexample to a conjecture by De Giorgi in large dimensions.* C. R. Math. Acad. Sci. Paris 346 (2008), no. 23-24, 1261–1266.

[14] A. Farina, *Propriétés qualitatives de solutions d'équations et systèmes d'équations non-linéaires.* Habilitation à diriger des recherches, Paris VI (2002).

[15] A. Farina, B. Sciunzi, E. Valdinoci. *Bernstein and De Giorgi type problems: new results via a geometric approach.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **7** (2008) 741-791.

[16] A. Farina, E. Valdinoci, *The state of the art for a conjecture of De Giorgi and related problems.*, to appear in Ser. Adv. Math. Appl. Sci., World Sci. Publ., Hackensack, NJ (2008).

[17] F. Ferrari, *Some inequalities associated with semilinear elliptic equations with variable coefficients and applications*, in press, to appear in Contemp. Math.

[18] F. Ferrari, *On some geometric inequalities in Carnot groups*, preprint (2009).

[19] F. Ferrari, E. Valdinoci. *Some weighted Poincaré inequalities.* Indiana Univ. Math. J. 58 (2009), 1619–1637.

[20] F. Ferrari, E. Valdinoci. *A geometric inequality in the Heisenberg group and its applications to stable solutions of semilinear problems*, Math. Annalen, **343** (2009) 351-370.

[21] F. Ferrari, E. Valdinoci. *Geometric PDEs in the Grushin plane: weighted inequalities and flatness of level sets.* , Int. Math. Res. Not. **22**, (2009) 4232–4270.

[22] N. Ghoussoub, C. Gui, *On a conjecture of De Giorgi and some related problems.* Math. Ann., **311** (1998) 481–491 .

[23] E.H. Lieb and M. Loss, Analysis, Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 1997.

[24] L. Modica, *The gradient theory of phase transitions and the minimal interface criterion*, Arch. Rational. Mech. Anal., **98** (1987) 123-142 .

[25] V. O. Savin, *Phase transitions: Regularity of Flat Level sets.* PhD thesis, University of Texas at Austin, 2003. https://dspace.lib.utexas.edu/bitstream/2152/993/1/savinov032.pdf

[26] V. O. Savin. *Phase transitions: Regularity of Flat Level sets.* Ann. of Math., **169** (2008) 41-78.

[27] P. Sternberg, K. Zumbrun, *A Poincaré inequality with applications to volume-constrained area-minimizing surfaces.* J. Reine Angew. Math., **503** (1998) 63-85 .

[28] P. Sternberg, K. Zumbrun, *Connectivity of phase boundaries in strictly convex domains.* Arch. Rational Mech. Anal. **141** (1998) 375-400.

Fausto Ferrari
Dipartimento di Matematica dell'Università
Piazza di Porta S. Donato, 5, 40126 Bologna, Italy
and
C.I.R.A.M.
Via Saragozza, 8, 40123 Bologna, Italy.
E-mail: `ferrari@dm.unibo.it`