Subharmonic Functions in sub-Riemannian Settings

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Abstract

In this note we present mean value characterizations of subharmonic functions related to linear second order partial differential operators with nonnegative characteristic form, possessing a well-behaved fundamental solution Γ. These characterizations are based on suitable average operators on the level sets of Γ. Asymptotic characterizations are also considered, extending classical results of Blaschke, Privaloff, Radó, Beckenbach and Reade. The results presented here generalize and carry forward former results of the authors in [6, 8].

1 $\mathcal{L}$-subharmonic functions

Let

$$\mathcal{L} := \sum_{i,j=1}^{N} \partial_{x_i}(a_{i,j}(x)\,\partial_{x_j}) = \mathrm{div}(A(x)\,\nabla) \tag{1.1}$$

be a linear second order PDO in $\mathbb{R}^N$, in divergence form, with C^2 coefficients and such that the matrix $A(x) := (a_{i,j}(x))_{i,j\le N}$ is *symmetric* and *nonnegative definite* at any point $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$. In (1.1), ∇ denotes the usual Euclidean gradient operator $\nabla = (\partial_{x_1}, \ldots, \partial_{x_N})^T$.

The operator $\mathcal{L}$ is (possibly) degenerate elliptic. However, in addition to some general hypotheses that will be fixed in the sequel, throughout the paper we always assume without further comments that $\mathcal{L}$ is *not totally degenerate* at every point. Precisely, we assume that the following condition holds:

(ND) there exists $i \le N$ such that $a_{i,i} > 0$ on $\mathbb{R}^N$.

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This condition, together with $A(x) \geq 0$, implies Picone's Maximum Principle for $\mathcal{L}$: If $V \subset \mathbb{R}^N$ is open and bounded and $u \in C^2(V, \mathbb{R})$ satisfies

$$\mathcal{L}u \geq 0 \quad \text{in } V \text{ and} \quad \limsup_{x \to y} u(x) \leq 0 \quad \text{for every } y \in \partial V,$$

then $u \leq 0$ in V. (See [20, Corollary 1.3].)

A function h will be said $\mathcal{L}$*-harmonic* in an open set $\Omega \subseteq \mathbb{R}^N$ if $h \in C^2(\Omega, \mathbb{R})$ and $\mathcal{L}h = 0$ in Ω. An upper semicontinuous function (u.s.c. function, for short) $u : \Omega \to [-\infty, \infty)$ will be called $\mathcal{L}$*-subharmonic* in Ω if

(i) the set $\Omega(u) := \{x \in \Omega \,|\, u(x) > -\infty\}$ is dense in Ω, and

(ii) for every bounded open set $V \subset \overline{V} \subset \Omega$ and for every $\mathcal{L}$-harmonic function $h \in C^2(V, \mathbb{R}) \cap C(\overline{V}, \mathbb{R})$ such that $u \leq h$ on ∂V, one has $u \leq h$ in V.

We shall denote by $\underline{\mathcal{S}}_{\mathcal{L}}(\Omega)$, or simply by $\underline{\mathcal{S}}(\Omega)$, the family (actually, the cone) of the $\mathcal{L}$-subharmonic functions in Ω.

It is well known that the subharmonic functions play crucial rôles in Potential Theory of linear second order PDE's (just think about Perron's method for the Dirichlet problem) as well as in studying the notion of convexity in Euclidean and non-Euclidean settings. (See the Bibliographical Notes at the end of the Introduction for some related references.)

When $\mathcal{L}$ is the classical Laplace operator Δ, several characterizations of the Δ-subharmonicity, involving surface and solid average operators on Euclidean balls, are given in literature. Some of them are quite well known, others are less so. If H_α is the α-dimensional Hausdorff measure, and if we denote by

$$\begin{aligned} \mathbf{S}_r(u)(x) &:= \frac{1}{H_{N-1}(\partial B(x,r))} \int_{\partial B(x,r)} u(y)\, \mathrm{d}H_{N-1}(y) \quad \text{and} \\ \mathbf{B}_r(u)(x) &:= \frac{1}{H_N(B(x,r))} \int_{B(x,r)} u(y)\, \mathrm{d}H_N(y), \end{aligned} \tag{1.2}$$

respectively, the mean value operator on the Euclidean sphere of center x and radius r, and on the corresponding solid ball $B(x, r)$, we can list the previously mentioned characterizations as follows.

Theorem A. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let $u : \Omega \to [-\infty, \infty)$ be an u.s.c. function with $\Omega(u)$ dense in Ω. Given $x \in \Omega$, we set $R(x) := \sup\{r > 0 : \overline{B(x,r)} \subset \Omega\}$.*

Then, the following statements are equivalent:

(i) $u \in \underline{\mathcal{S}}_\Delta(\Omega)$.

(ii) *For every $x \in \Omega$ and $0 < r < R(x)$, it holds that $u(x) \leq \mathbf{S}_r(u)(x)$.*

(iii) *For every $x \in \Omega$ and $0 < r < R(x)$, it holds that $u(x) \leq \mathbf{B}_r(u)(x)$.*

(iv) *(Blaschke) For every* $x \in \Omega(u)$*, it holds that* $\limsup_{r\to 0} \dfrac{\mathbf{S}_r(u)(x) - u(x)}{r^2} \geq 0.$

(v) *(Privaloff) For every* $x \in \Omega(u)$*, it holds that* $\limsup_{r\to 0} \dfrac{\mathbf{B}_r(u)(x) - u(x)}{r^2} \geq 0.$

(vi) *For every* $x \in \Omega$*, the function* $r \mapsto \mathbf{S}_r(u)(x)$ *is monotone increasing on* $(0, R(x))$ *and* $\lim_{r\to 0} \mathbf{S}_r(u)(x) = u(x)$*.*

(vii) *For every* $x \in \Omega$*, the function* $r \mapsto \mathbf{B}_r(u)(x)$ *is monotone increasing on* $(0, R(x))$ *and* $\lim_{r\to 0} \mathbf{B}_r(u)(x) = u(x)$*.*

(viii) *(Beckenbach-Radó) For every* $x \in \Omega$ *and* $r \in (0, R(x))$*, it holds that* $\mathbf{B}_r(u)(x) \leq \mathbf{S}_r(u)(x)$ *and* $\lim_{r\to 0} \mathbf{S}_r(u)(x) = u(x)$*.*

(ix) *(Reade) For every* $x \in \Omega(u)$*, it holds that* $\liminf_{r\to 0} \dfrac{\mathbf{S}_r(u)(x) - \mathbf{B}_r(u)(x)}{r^2} \geq 0$ *and* $\lim_{r\to 0} \mathbf{S}_r(u)(x) = u(x)$*.*

Sharp versions of Blaschke and Privaloff conditions were proved by Saks.
If u is Δ-subharmonic in Ω then, by Riesz's Representation Theorem, there exists a Radon measure μ_u (called the *Riesz measure* of u) such that $\Delta u = \mu_u$ in $\mathcal{D}'(\Omega)$. On the other hand, from the Lebesgue Differentiation Theorem of a measure, the symmetric derivative of μ_u, say

$$D_{\mathrm{s}}\mu_u(x) := \lim_{r\to 0} \frac{\mu_u(B(x,r))}{H_N(B(x,r))},$$

exists H_N-almost everywhere in Ω. The following result holds.

Theorem B (Saks)**.** *Let* u *be a* Δ*-subharmonic function in* $\Omega \subseteq \mathbb{R}^N$ *and let* μ_u *be its Riesz measure. Then, at every point* $x \in \Omega$ *where* $D_{\mathrm{s}}\mu_u(x)$ *exists, one has:*

(i) $\lim_{r\to 0} \dfrac{\mathbf{S}_r(u)(x) - u(x)}{r^2} = \dfrac{1}{2N} D_{\mathrm{s}}\mu_u(x),$

(ii) $\lim_{r\to 0} \dfrac{\mathbf{B}_r(u)(x) - u(x)}{r^2} = \dfrac{1}{2(N+2)} D_{\mathrm{s}}\mu_u(x),$

(iii) $\lim_{r\to 0} \dfrac{\mathbf{S}_r(u)(x) - \mathbf{B}_r(u)(x)}{r^2} = \dfrac{1}{N(N+2)} D_{\mathrm{s}}\mu_u(x).$

The goals of our work is to recast Theorems A and B above in more general settings, today usually called of *sub-Riemannian* type.

To be more specific, we have extended Theorem A to every operator $\mathcal{L}$ endowing $\mathbb{R}^N$ with the structure of a $\mathfrak{S}^*$-harmonic space, and having a *nonnegative global fundamental solution*

$$\mathbb{R}^N \times \mathbb{R}^N \setminus \{x = y\} \ni (x,y) \mapsto \Gamma(x,y) \in \mathbb{R},$$

with pole at any point of the diagonal $\{x = y\}$ of $\mathbb{R}^N$ (For the notion of $\mathfrak{S}^*$-harmonic space, see [8, Section 6.10]).

In our version of Theorem A the classical mean value operators $\mathbf{S}_r$ and $\mathbf{B}_r$ are replaced by suitable average operators on the *level sets* of Γ,

$$\partial\Omega_r(x) = \big\{y \in \mathbb{R}^N \,:\, \Gamma(x,y) = 1/r\big\},$$

and on their solid counterpart

$$\Omega_r(x) := \big\{y \in \mathbb{R}^N \,:\, \Gamma(x,y) > 1/r\big\}.$$

We explicitly remark that study of the average operators related to the general PDO's considered in this paper is complicated by the presence of non-trivial kernels. For instance, when $\mathcal{L}$ in (1.1) is a sub-Laplacian on a stratified Lie group $\mathbb{G}$, the kernels appearing in the relevant mean-integrals cannot be identically 1, *unless* $\mathbb{G}$ *is the usual Euclidean group* $(\mathbb{R}^N, +)$, as it is proved in [7].

A crucial tool for our extension of Theorem A is a Lemma providing a unifying approach to several characterizations of $\mathcal{L}$-subharmonicity. This result traces back to a 1933 theorem by W. Kozakiewicz [19] related to the case of the ordinary Laplace operator.

For our versions of Theorems B, we impose a further restriction. Indeed, our approach to this last extension exploits *Poisson-Jensen Formulas* for $\mathcal{L}$-subharmonic functions, together with a *homogeneity* property for the measure of the level sets of Γ. Therefore, Theorem B can be conveniently extended to the sub-Laplacians on stratified Lie groups, which naturally satisfy these requirements.

We close this Section with the following bibliographical notes.

Bibliographical Notes

Gauss Theorem on mean value properties for classical harmonic functions has been generalized in countless directions. The historical development of the problems related to this property, both for harmonic and caloric functions, is presented in the survey paper [22] by Netuka and Veselý.

A mean value theorem for solutions to $\mathcal{L}u = 0$, when $\mathcal{L}$ is a general operator as in (1.1) of "elliptic-type", has been proved by Hoh and Jacob [17]. Citti, Garofalo and Lanconelli [10] proved some representation formulas for smooth solutions to $\mathcal{L}u = f$, for operators $\mathcal{L}$ which are sum of squares of vector fields satisfying the Hörmander rank condition.

Later on, these formulas were used in [6] to derive representation formulas for smooth solutions to $\Delta_{\mathbb{G}} u = f$, where $\Delta_{\mathbb{G}}$ denotes a sub-Laplacian on a Carnot group $\mathbb{G}$. When $f = 0$ and $\Delta_{\mathbb{G}}$ is the Kohn Laplacian on the Heisenberg group, the formulas in [6], as well as those in [10] and [17], give back a mean value property first proved by Gaveau [15].

The use of *asymptotic average operators* in the characterization of classical subharmonic functions has a long history, starting with the papers [3] and [23] by Blaschke and Privaloff, respectively. Beckenbach and Radó [2] characterized the Δ-subharmonic functions in terms of the inequality "*solid average* $\leq$ *surface average*". [The original Beckenbach-Radó condition was stated and proved for *continuous* functions in $\mathbb{R}^2$. An extension to any dimension, still for continuous functions, is contained in the very recent paper [13] by Freitas and Matos.]

It was Saks [26] who proved, in 1941, Theorem B. Two years later, Reade [24] introduced his asymptotic version of the Beckenbach-Radó condition. [We call *Reade condition* the one contained in number (ix) of Theorem A. Actually, Reade stated in [24], but without any proof, that a *continuous* function u is Δ-subharmonic in an open set $\Omega \subseteq \mathbb{R}^2$ iff $\limsup_{r\to 0}(\mathbf{S}_r(u)(x) - \mathbf{B}_r(u)(x))/r^2 \geq 0$ for every $x \in \Omega$. We have not been able to find any proof of this statement in literature. Instead, number (ix) of Theorem A follows from our Theorem 4.2 applied to Δ.]

A modern reference for some asymptotic-mean characterizations of Δ-subharmonicity is the monograph [1] by Armitage and Gardiner (see [1, Section 3.2]). This monograph, which mainly deals with classical Potential Theory, also contains some applications of subharmonicity to the usual convexity (a systematic subharmonic approach to convexity in the Euclidean setting can also be found in Hörmander's monograph [18]).

Mean value characterizations of subharmonic functions in Carnot groups are contained in [6], see also the monograph [8], Chapter 8. We directly refer to this chapter for some applications, and a list of references, about convexity in the stratified Lie group setting. Furthermore, [6] and [8] also deal with the problem of the smooth approximation of subharmonic functions in Carnot groups. The results proved therein use a version of the Friedrichs's mollifiers, resting on the homogeneous Lie group structure of $\mathbb{G}$. This approach does not work in absence of such an algebraic structure underlying the operator $\mathcal{L}$. For our approximation theorem in the present paper we exploit an idea used in [14] by Garofalo and one of us, for classical parabolic operators with variable coefficients.

2 Assumptions on the operator $\mathcal{L}$. The $\mathcal{L}$-harmonic space

We assume that the operator $\mathcal{L}$ in (1.1) is equipped with a *global fundamental solution* Γ, that is, there exists a function $\Gamma : D = \{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N \ : \ x \neq y\} \longrightarrow \mathbb{R}$ with the following properties:

(G.1) $\Gamma \in L^1_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^N) \cap C^2(D, \mathbb{R})$, $\Gamma(x,y) > 0$ for every $(x,y) \in D$;

(G.2) for every fixed $x \in \mathbb{R}^N$, we have $\lim_{y\to x} \Gamma(x,y) = \infty$ and $\lim_{y\to\infty} \Gamma(x,y) = 0$;

(G.3) for any $\varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R})$,

$$\int_{\mathbb{R}^N} \Gamma(x,y)\, \mathcal{L}\varphi(y)\, \mathrm{d}y = -\varphi(x), \quad \text{for every } x \in \mathbb{R}^N. \tag{2.1}$$

This last property, since $\mathcal{L}^* = \mathcal{L}$, can be restated as follows: $-\mathcal{L}\Gamma(x,\cdot)$ equals the Dirac measure at $\{x\}$, in the sense of distributions. It in particular implies that $y \mapsto \Gamma(x,y)$ is $\mathcal{L}$-harmonic in $\mathbb{R}^N \setminus \{x\}$. As a consequence, since $\Gamma(x,y) \to \infty$ as $y \to x$, an easy application of Picone's Maximum Principle shows that $-\Gamma(x,\cdot)$ is $\mathcal{L}$-subharmonic in $\mathbb{R}^N$.

Our second general assumption is that $\mathcal{L}$ endows $\mathbb{R}^N$ with the structure of $\mathfrak{S}^*$-*harmonic space*, in the sense of [8, Definition 6.10.1]. Due to (G.1) and (G.2), this amounts to make the following extra hypotheses on $\mathcal{L}$:

(D) *Doob convergence property:* If $\{u_n\}_n$ is a monotone increasing sequence of $\mathcal{L}$-harmonic functions on an open set $\Omega \subseteq \mathbb{R}^N$, then $u := \sup_n u_n$ is $\mathcal{L}$-harmonic in Ω, provided that u is finite in a dense subset of Ω.

(R) *Regularity axiom:* The $\mathcal{L}$-regular open sets form a basis of the Euclidean topology.

Here, we agree to call *$\mathcal{L}$-regular* any bounded open set $V \subset \mathbb{R}^N$ such that: for every $f \in C(\partial V, \mathbb{R})$, there exists a (unique) $\mathcal{L}$-harmonic function in V, denoted by H_f^V, satisfying $\lim_{y\to x} H_f^V(y) = f(x)$, for every $x \in \partial V$.

Under the previous assumptions, the map

$$\Omega \mapsto \mathcal{H}(\Omega) = \{u \in C^2(\Omega, \mathbb{R}) \mid u \text{ is } \mathcal{L}\text{-harmonic in } \Omega\} \tag{2.2}$$

is a *harmonic sheaf* and $(\mathbb{R}^N, \mathcal{H})$ is a $\mathfrak{S}^*$-harmonic space, which we call the *$\mathcal{L}$-harmonic space*. Indeed, the functions of the type $\max\{-\Gamma(x,\cdot), -k\}$ (with $k \in \mathbb{N}$) provide non-positive continuous $\mathcal{L}$-subharmonic functions separating points of $\mathbb{R}^N$.

Remark 2.1. Conditions (D) and (R) are satisfied if $\mathcal{L}$ is hypoelliptic (see [8], Chapter 7, Exercise 7). In particular, this holds true if $\mathcal{L} = \sum_{j=1}^m X_j^2$, where the X_j's are smooth vector fields in $\mathbb{R}^N$ satisfying the Hörmander rank condition

$$\dim\big(\mathrm{Lie}\{X_1, \ldots, X_m\}(x)\big) = N, \quad \forall\, x \in \mathbb{R}^N.$$

The hypoellipticity of $\mathcal{L}$, together with its *homogeneity with respect to a group of dilations in* $\mathbb{R}^N$ (in the sense of Folland, Stein [12]), is a sufficient condition for the existence of a function Γ satisfying (G.1)–(G.3). This follows by arguing as in [8, Section 5.3]. Indeed, from the hypoellipticity of $\mathcal{L}$, we infer the existence of a "local" fundamental solution (see Trèves [28, Theorems 52.1, 52.2]); by the homogeneity of $\mathcal{L}$, a "local-to-global" argument can be performed (see Folland [11, Theorem 2.1]). Then one argues as in [8, Section 5.3] to obtain the requested properties of Γ.

Particular examples of hypoelliptic homogenous operators are the sub-Laplacians on Carnot groups, If V is any $\mathcal{L}$-regular open set and $x \in V$, the map $C(\partial V, \mathbb{R}) \ni f \mapsto H_f^V(x) \in \mathbb{R}$ is linear and it is nonnegative on nonnegative f's. Hence, there exists a unique Radon measure μ_x^V on ∂V such that

$$H_f^V(x) = \int_{\partial V} f(y)\,\mathrm{d}\mu_x^V(y), \quad \text{for every } f \in C(\partial V, \mathbb{R}).$$

One says that μ_x^V is the *$\mathcal{L}$-harmonic measure* related to V and x.

In the sequel, we write $u \in \mathrm{USC}(\Omega)$ if $u : \Omega \to [-\infty, \infty)$ is upper semicontinuous in Ω. We explicitly remark that a function u is subharmonic in terms of the $\mathcal{L}$-harmonic space $(\mathbb{R}^N, \mathcal{L})$ if and only if it is $\mathcal{L}$-subharmonic according to the definition given in Section 1 above (see [8, Definition 6.5.1 and Theorem 6.5.2]).

Without any further comment, throughout the note we assume that the operator $\mathcal{L}$ in (1.1) satisfies assumptions (ND), (G.1), (G.2), (G.3), (D) and (R) introduced above.

3 Level sets of Γ. Average operators, representation formulas

We introduce a family of sets which will play a central rôle in the sequel. For every given pair of $x \in \mathbb{R}^N$ and $r > 0$, we set

$$\Omega_r(x) := \big\{ y \in \mathbb{R}^N \setminus \{0\} \;:\; \Gamma(x, y) > 1/r \big\} \cup \{x\}. \tag{3.1}$$

For the sake of simplicity, we assume that, for every $x \in \mathbb{R}^N$ and $r > 0$, $\nabla(\Gamma(x, \cdot)) \neq 0$ on $\partial\Omega_r(x)$, whence $\partial\Omega_r(x)$ is a smooth-manifold of class C^2. Thanks to Sard's Theorem, this hypothesis could be easily removed (in this case, all the following results will hold for almost every r). We also have

$$\partial\Omega_r(x) = \big\{ y \in \mathbb{R}^N \;:\; \Gamma(x, y) = 1/r \big\}.$$

Note that any $\Omega_r(x)$ is a bounded open neighborhood of x and

$$\textstyle\bigcap_{r>0} \Omega_r(x) = \{x\}, \qquad \bigcup_{r>0} \Omega_r(x) = \mathbb{R}^N. \tag{3.2}$$

Here and in the sequel, if E is any (Lebesgue-)measurable subset of $\mathbb{R}^N$, we denote by $|E|$ its Lebesgue measure. Moreover, $\mathrm{d}y$ and $\mathrm{d}\sigma(y)$ will respectively denote, without possibility of ambiguity, the Lebesgue measure and the surface measure in $\mathbb{R}^N$, the latter being the Hausdorff $(N-1)$-dimensional measure.

By the Bouligand regularity theorem holding true in any $\mathfrak{S}^*$-harmonic space (see [8, Theorem 6.10.4]), the set $\Omega_r(x)$ is $\mathcal{L}$-regular, for every $r > 0$ and every $x \in \mathbb{R}^N$. Indeed, the function $y \mapsto \Gamma(x_0, y) - 1/r$ is an $\mathcal{H}$-barrier function (in the sense of [8, Definition 6.10.3]) at any point x_0 of $\partial\Omega_r(x)$.

To state our main theorem, we need the following notation and definitions about some distinguished average operators related to $\mathcal{L}$.

Definition 3.1 (Mean-Integral Operators)**.** Let $x \in \mathbb{R}^N$ and let us consider the functions, defined for $y \neq x$,

$$\Gamma_x(y) := \Gamma(x, y), \qquad \mathcal{K}_x(y) := \frac{\big\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\big\rangle}{|\nabla\Gamma_x(y)|}.$$

Let $\Omega \subseteq \mathbb{R}^N$ be an open set and suppose $u \in \mathrm{USC}(\Omega)$. For every fixed $\alpha > 0$, and every $x \in \mathbb{R}^N$ and $r > 0$ such that $\overline{\Omega_r(x)} \subset \Omega$, we introduce the following integrals:

$$m_r(u)(x) = \int_{\partial\Omega_r(x)} u(y)\, \mathcal{K}_x(y)\, \mathrm{d}\sigma(y), \quad M_r(u)(x) = \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha\, m_\rho(u)(x)\, \mathrm{d}\rho,$$

Furthermore, for every $x \in \mathbb{R}^N$ and every $r > 0$, we set

$$\begin{aligned} q_r(x) &= \int_{\Omega_r(x)} \Big(\Gamma_x(y) - \frac{1}{r}\Big)\, \mathrm{d}y, \qquad Q_r(x) = \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha\, q_\rho(x)\, \mathrm{d}\rho, \\ \omega_r(x) &= \frac{1}{\alpha\, r^{\alpha+1}} \int_{\Omega_r(x)} \big(r^\alpha - \Gamma_x^{-\alpha}(y)\big)\, \mathrm{d}y. \end{aligned} \tag{3.3}$$

The notation $M_r^{(\alpha)}, Q_r^{(\alpha)}, \omega_r^{(\alpha)}$ will also apply. An alternative representation for $M_r(u)(x)$ is given by the following formula

$$M_r(u)(x) = \frac{\alpha+1}{r^{\alpha+1}} \int_{\Omega_r(x)} u(y)\, K^{(\alpha)}(x, y)\, \mathrm{d}y, \tag{3.4}$$

where we have set

$$K^{(\alpha)}(x, y) := \frac{\big\langle A(y)\nabla\Gamma_x(y), \nabla\Gamma_x(y)\big\rangle}{\Gamma_x^{2+\alpha}(y)}. \tag{3.5}$$

There holds alternative formulas for $q_r(x), Q_r(x), \omega_r(x)$ too (all proved in the Appendix), having interest in their own: for every $x \in \mathbb{R}^N$ and every $\alpha, r > 0$, one has

$$\begin{aligned} q_r(x) &= \int_0^r \frac{|\Omega_s(x)|}{s^2}\, \mathrm{d}s, \\ Q_r(x) &= \int_0^r \frac{|\Omega_s(x)|}{s^2} \Big(1 - \Big(\frac{s}{r}\Big)^{\alpha+1}\Big)\, \mathrm{d}s, \\ \omega_r(x) &= \frac{1}{r^{\alpha+1}} \int_0^r s^{\alpha-1}\, \big|\Omega_s(x)\big|\, \mathrm{d}s. \end{aligned} \tag{3.6}$$

Remark 3.2. *The above definitions are well-posed*. Indeed, note that $m_r(u)(x)$ is well-posed because $\partial\Omega_r(x)$ is a compact subset of Ω (see also hypothesis (G.2) on the fundamental solution), and u is bounded from above on the compact sets (since it is upper semicontinuous). We also remark that $q_r(x), Q_r(x), \omega_r(x)$ are strictly positive, for any $x \in \mathbb{R}^N$ and $r > 0$. Also, they are finite since $\Gamma(x, \cdot)$ is locally summable.

Moreover, in the hypotheses of the above definition, we claim that the map $r \mapsto m_r(u)(x)$ is upper semicontinuous, so that $M_r(u)(x)$ is well posed too. The claim follows from the following argument. Being $u \in \mathrm{USC}(\Omega)$ and being $\partial\Omega_r(x)$ compact, there exists a decreasing sequence of continuous functions $\{u_j\}_j$ on $\partial\Omega_r(x)$ converging pointwise to u in a compact neighborhood od any fixed r; it is easily seen that $r \mapsto m_r(u_j)(x)$ is continuous (for every $j \in \mathbb{N}$) and that $m_r(u)(x) = \lim_{j\to\infty} m_r(u_j)(x)$. Hence $r \mapsto m_r(u)(x)$ is upper semicontinuous.

We shall prove in the Appendix that the above average operators do intervene, when u is C^2, in remarkable mean-value formulas generalizing the classical Gauss-Green formulas for Laplace's operator. These are recalled, for later use, in the following theorem.

Theorem 3.3 (Mean-Value Formulas for $\mathcal{L}$)**.** *Let m_r, M_r be the average operators in Definition 3.1. Let also $x \in \mathbb{R}^N$ and $r > 0$.*

Then, for every function u of class C^2 on an open set containing $\overline{\Omega_r(x)}$, we have the following $\mathcal{L}$-representation formulas:

$$u(x) = m_r(u)(x) - \int_{\Omega_r(x)} \Big(\Gamma(x,y) - \frac{1}{r}\Big)\, \mathcal{L}u(y)\,\mathrm{d}y, \tag{3.7}$$

$$u(x) = M_r(u)(x) - \frac{\alpha+1}{r^{\alpha+1}} \int_0^r \rho^\alpha \bigg(\int_{\Omega_\rho(x)} \Big(\Gamma(x,y) - \frac{1}{\rho}\Big)\, \mathcal{L}u(y)\,\mathrm{d}y \bigg) \mathrm{d}\rho, \tag{3.8}$$

We shall refer to (3.7) *as the* Surface Mean-Value Formula for $\mathcal{L}$*, whereas* (3.8) *will be called the* Solid Mean-Value Formula for $\mathcal{L}$.

For our main results (Theorem 4.2 in the next section), we need a representation formula also *for the difference of the mean-integral operators* $m_r - M_r$ (involved in what we shall call the 'Reade-type Condition'). This formula seems to be new in many contexts of interest (e.g., for sub-Laplacian operators). For instance, we shall prove the following result.

Theorem 3.4. *Let $x \in \mathbb{R}^N$, $r > 0$ and $\alpha > 0$. Then, we have*

$$m_r(u)(x) - M_r(u)(x) = \frac{1}{\alpha\, r^{\alpha+1}} \int_{\Omega_r(x)} \big(r^\alpha - \Gamma^{-\alpha}(x,y)\big)\, \mathcal{L}u(y)\,\mathrm{d}y, \tag{3.9}$$

for every $u \in C^2(\overline{\Omega_r(x)}, \mathbb{R})$.

We also have

Proposition 3.5. *Following the notation in Definition 3.1, for every* $x \in \mathbb{R}^N$, $R > 0$ *and for every* $u \in C^2(\overline{\Omega_R(x)}, \mathbb{R})$ *we have*

$$\lim_{r\to 0} \frac{m_r(u)(x) - u(x)}{q_r(x)} = \lim_{r\to 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)} = \lim_{r\to 0} \frac{m_r(u)(x) - M_r(u)(x)}{\omega_r(x)} = \mathcal{L}u(x). \tag{3.10}$$

Furthermore, the functions $(0, \infty) \ni r \mapsto q_r(x), Q_r(x)$ *are monotone increasing and*

$$\lim_{r\to 0} q_r(x) = \lim_{r\to 0} Q_r(x) = \lim_{r\to 0} \omega_r(x) = 0. \tag{3.11}$$

Finally, if the function $s \mapsto |\Omega_s(x)|/s$ *is monotone increasing, then* $r \mapsto \omega_r(x)$ *is monotone increasing too.*

We next make explicit the values of our average operators in the classical case of the Laplacian in $\mathbb{R}^N$.

Remark 3.6 (*The average operators for* $\mathcal{L} = \Delta$)**.** Let $N \geq 3$ be fixed. In the classical case $A =$ identity matrix, i.e., $\mathcal{L} = \Delta$ is Laplace operator in $\mathbb{R}^N$, it holds that $\Gamma(x, y) = c_N\, |y - x|^{2-N}$, for a suitable dimensional constant $c_N > 0$. Actually, $c_N = (N(N-2)\omega_N)^{-1}$, where $\omega_N = |B(0, 1)|$.

A direct computation gives $\mathcal{K}(x, y) = c_N(N-2)|y - x|^{1-N}$, *which is constant on* $\partial\Omega_r(x)$, so that (if $B(x, \rho)$ denotes the usual Euclidean ball of center x and radius ρ) a simple computation produces

$$m_r(u)(x) = \mathbf{S}_\rho(u)(x), \quad \text{where } \rho = (c_N\, r)^{\frac{1}{N-2}}.$$

Here, $\mathbf{S}_\rho$ is the classical surface average operator which we introduced in (1.2). Note that $\sigma(\partial B(x, \rho)) = \sigma_N\, \rho^{N-1}$, where $\sigma_N = \sigma(\partial B(0, 1)) = N\, \omega_N$, whence $\sigma_N c_N (N-2) = 1$.

With reference to (3.5), it holds that $K^{(\alpha)}(x, y) = c_N^{-\alpha}(N-2)^2\, |y - x|^{\alpha(N-2)-2}$. *This is constant iff we choose* $\alpha = \frac{2}{N-2}$. With this choice of α, a simple computation gives

$$M_r(u)(x) = \mathbf{B}_\rho(u)(x), \quad \text{where } \rho = (c_N\, r)^{\frac{1}{N-2}},$$

where $\mathbf{B}_\rho$ is the classical solid average operator in (1.2). We next compute q_r and Q_r:

$$\begin{cases} q_r(x) &= \frac{1}{2\,N}\, \rho^2, \\ Q_r(x) &= \frac{\alpha+1}{2\,N}\, \frac{N-2}{2+(\alpha+1)(N-2)}\, \rho^2 \end{cases} \qquad \text{where } \rho = (c_N\, r)^{\frac{1}{N-2}}.$$

The above choice $\alpha = \frac{2}{N-2}$ produces $Q_r(x) = \frac{1}{2\,(N+2)}\, \rho^2$.

Finally, $(c_N\, s)^{\frac{1}{N-2}}$, formula (**??**) in the Appendix gives

$$\omega_r(x) = \frac{1}{N}\, \frac{1}{\alpha(N-2) + N}\, \rho^2, \qquad \text{where } \rho = (c_N\, r)^{\frac{1}{N-2}}.$$

and the normalizing choice $\alpha = \frac{2}{N-2}$ produces $\omega_r(x) = \frac{1}{N\,(N+2)}\, \rho^2$.

4 Mean value characterizations of $\mathcal{L}$-subharmonic functions: Main Theorem

Before stating our main theorem, we need a last definition.

Definition 4.1 (m, M-Continuity)**.** A function $u \in \mathrm{USC}(\Omega)$ defined on an open subset Ω of $\mathbb{R}^N$ will be called *m-continuous in* Ω if

$$\lim_{r\to 0} m_r(u)(x) = u(x), \quad \text{for every } x \in \Omega.$$

Analogously, u is said *M-continuous in* Ω if $\lim_{r\to 0} M_r(u)(x) = u(x)$, for every $x \in \Omega$.

Notice that, from the very definition of M_r, it follows that m-continuity implies M-continuity. Also, we shall prove that any $u \in C(\Omega, \mathbb{R})$ is m-continuous (whence M-continuous). This is the reason why m- and M-continuity assumptions do not explicitly appear in characterizations of *continuous* subharmonic functions (see e.g., [13]).

We are ready to state the main result of this section.

Theorem 4.2. *Let Ω be an open subset of $\mathbb{R}^N$ and let $u : \Omega \to [-\infty, \infty)$ be an upper semicontinuous function which is finite in a dense subset of Ω.*

Let q_r, Q_r, ω_r be as in Definition 3.1. Let also $R(x) := \sup\{r > 0 \ : \ \Omega_r(x) \subseteq \Omega\}$. Then, the following conditions are equivalent:

($\underline{S}$) **Subharmonicity:** $u \in \underline{S}(\Omega)$ *with respect to* $\mathcal{L}$.

(1) m**-Submean Condition:** $u(x) \le m_r(u)(x)$*, for every* $x \in \Omega$ *and* $r \in (0, R(x))$.

(2) M**-Submean Condition:** $u(x) \le M_r(u)(x)$*, for every* $x \in \Omega$ *and* $r \in (0, R(x))$.

(3) **Blaschke-type Condition:** *It holds that*

$$\limsup_{r\to 0} \frac{m_r(u)(x) - u(x)}{q_r(x)} \ge 0, \quad \text{for every } x \in \Omega(u). \tag{4.1}$$

(4) **Privaloff-type Condition:** *It holds that*

$$\limsup_{r\to 0} \frac{M_r(u)(x) - u(x)}{Q_r(x)} \ge 0, \quad \text{for every } x \in \Omega(u). \tag{4.2}$$

(5) m**-Monotonicity:** *u is m-continuous in Ω, and $r \mapsto m_r(u)(x)$ is monotone increasing on $(0, R(x))$, for every $x \in \Omega$.*

(6) M**-Monotonicity:** *u is M-continuous in Ω, and $r \mapsto M_r(u)(x)$ is monotone increasing on $(0, R(x))$, for every $x \in \Omega$.*

(7) **Beckenbach-Radó-type Condition:** *u is m-continuous in Ω, and*

$$M_r(u)(x) \leq m_r(u)(x),$$

for every $x \in \Omega$ and every $r \in (0, R(x))$.

(8) **Reade-type Condition:** *u is m-continuous in Ω, and*

$$\liminf_{r \to 0} \frac{m_r(u)(x) - M_r(u)(x)}{\omega_r(x)} \geq 0, \quad \text{for every } x \in \Omega(u). \tag{4.3}$$

The proof of Theorem 4.2 is quite long, and it is basically founded on a general lemma, which we present in the following Section.

5 A Kozakiewicz-type Theorem

In this Section we present a result that, for the ordinary Laplace operator, traces back to a result due to W. Kozakiewicz [19].

We begin with some notation and definitions. Throughout this section, Ω will denote a fixed open subset of $\mathbb{R}^N$. We denote by $\mathcal{U}(\Omega)$ the set of functions $v : V \to [-\infty, \infty)$ defined on some open subset V of Ω, such that $v \in \text{USC}(V)$ and $v > -\infty$ on a dense subset of V. We denote by $\mathcal{D}(v)$ (or by $V(v)$, when the domain V is specified) the set where v takes on finite values. Finally, we denote by $\mathcal{F}(\Omega)$ the set of the real-extended valued functions $f : A \to [-\infty, \infty]$, defined on some subset A of Ω.

Definition 5.1 ($\mathcal{L}$-Kozakiewicz Operator)**.** With all the above notation, we say that a map

$$G : \mathcal{U}(\Omega) \longrightarrow \mathcal{F}(\Omega)$$

is an *$\mathcal{L}$-Kozakiewicz operator* in Ω if it satisfies the following four axioms:

(K.1) If $v \in \mathcal{U}(\Omega)$ then $G(v)$ is defined on $\mathcal{D}(v)$, that is $\mathcal{D}(G(v)) \supseteq \mathcal{D}(v)$.

(K.2) For every $h \in \mathcal{U}(\Omega)$ of class C^2 it holds that $G(h) = \mathcal{L}h$.

(K.3) For every $v, h \in \mathcal{U}(\Omega)$, with h of class C^2, defined on the same open set $V \subseteq \Omega$, it holds that $G(v + h) = G(v) + G(h)$.

(K.4) If $v \in \mathcal{U}(\Omega)$ and if x_0 is a local maximum point of v, then $G(v)(x_0) \leq 0$.

We stress that, in condition (K.4), we have $x_0 \in \mathcal{D}(v)$, otherwise $v(x) \leq v(x_0) = -\infty$ for every x in some neighborhood of x_0, contradicting the assumption that v is finite in a dense subset of V.

Remark 5.2. We remark that, thanks to assumption (ND) on $\mathcal{L}$, for every bounded open set $U \subset \mathbb{R}^N$, there exists $w \in C^2(U, \mathbb{R})$ such that

$$w > 0 \quad \text{and} \quad \mathcal{L}w < 0 \quad \text{in } U. \tag{5.1}$$

It suffices to take, if i is as in hypothesis (ND), $w(x) = M - \exp(\lambda\, x_i)$, where

$$\lambda > -\frac{\min_{\overline{U}} \sum_{j=1}^N \partial_j a_{j,i}}{\min_{\overline{U}} a_{i,i}} \quad \text{and} \quad M > \max_{x \in \overline{U}} \exp(\lambda\, x_i).$$

We are ready to state the following result.

Theorem 5.3 (of Kozakiewicz-type. I)**.** *Let $\Omega \subseteq \mathbb{R}^N$ be an open set and let G be an $\mathcal{L}$-Kozakiewicz operator in Ω.*

Let $u : \Omega \to [-\infty, \infty)$ be an u.s.c. function, finite in a dense subset of Ω, such that

$$G(u) \geq 0 \quad \textit{in } \Omega(u). \tag{5.2}$$

Then u is $\mathcal{L}$-subharmonic in Ω.

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The extended version with complete proofs will appear in a forthcoming paper.

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