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Vector valued Fourier multipliers and applications

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Abstract

In questo seminario sono illustrati alcuni recenti sviluppi della teoria dei moltiplicatori di Fourier negli spazi L^p a valori in spazi di Banach. Seguono alcune applicazioni a problemi al contorno di tipo ellittico e a problemi misti di tipo parabolico.

In this seminar I shall illustrate some fairly new developments of the theory of Fourier multipliers in Banach spaces. Only the L^p case will be treated. Concerning vector valued Fourier multipliers in spaces of Hölder continuous functions,or, more generally, in Besov spaces, we refer to [1], and also to [8]. These results have supplied new proofs of known facts, in particular in the field of maximal regularity for parabolic problems, and have also been the source of new discoveries. In a joint paper with A. Favini and Y. Yakubov ([6]), these techniques are employed to study some elliptic and parabolic systems in cylindrical spaces domains. These applications will be illustrated in the last part of the seminar.

To start with, I recall one version of the classical Mikhlin's multiplier theorem (see [12], Theorem IV.3):

Theorem 1. *Let $m \in C^{[\frac{n}{2}]+1}(\mathbb{R}^n \setminus \{0\})$, be such that for every $\alpha \in \mathbb{N}_0^n$, with $|\alpha| \leq [\frac{n}{2}]+1$,*

$$|\xi|^{|\alpha|}|\partial^\alpha m(\xi)| \leq C, \quad \xi \in \mathbb{R}^n \setminus \{0\}, \tag{1}$$

for some $C \in \mathbb{R}^+$. Then m is a Fourier multiplier for $L^p(\mathbb{R}^n)$, for every $p \in (1, \infty)$.

This means that the linear operator $f \to \mathcal{F}^{-1}(m\mathcal{F}f)$, defined, for example, for $f \in \mathcal{S}(\mathbb{R}^n)$, can be extended to a linear bounded operator in $L^p(\mathbb{R}^n)$.

A variation of Theorem 1 is the following result, which can be obtained as a particular case of Theorem IV.6' in [12]:

Theorem 2. *Let $m \in C^n(\mathbb{R}^n \setminus \{0\})$ be such that, for every $\alpha \in \mathbb{N}_0^n$, with $\alpha \leq (1, ..., 1)$, for some $C \in \mathbb{R}^+$, for every $\xi \in \mathbb{R}^n \setminus \{0\}$,*

$$|\xi^\alpha D^\alpha m(\xi)| \leq C. \tag{2}$$

Then m is a Fourier multiplier for $L^p(\mathbb{R}^n)$, for every $p \in (1, \infty)$.

It is clear that, on one side, Theorem 1 requires less regularity, on the other hand, condition (2) is weaker than condition (1). An example of Fourier multiplier which is covered by Theorem 2, but not by Theorem 1, is (for $n = 2$)

$$m(\xi_1, \xi_2) = \frac{\xi_1}{\xi_1 + i\xi_2^2},$$

which is connected with parabolic equations.

The most basic example of Fourier multiplier, which is covered by both Theorems 1 and 2, is (in dimension one) the function $m(\xi) = sgn(\xi)$. The corresponding operator is called Hilbert transform. It turns out that the Hilbert transform can be used as a starting point to extend Theorems 1 and 2 to the vector valued case, where the space of complex valued functions $L^p(\mathbb{R}^n)$ is replaced by spaces $L^p(\mathbb{R}^n; E)$, with E Banach space. In this vector valued case, integration will be always intended in the sense of Bochner. So we shall consider the class of Banach spaces E, such that the Hilbert transform is a bounded operator in $L^p(\mathbb{R}; E)$, for $p \in (1, \infty)$.

Definition 1. *Let E be a complex Banach space. We shall say that E is UMD if the Hilbert transform $f \to \mathcal{F}^{-1}(sgn(\xi)\mathcal{F}f)$ is a bounded operator in $L^2(\mathbb{R}; E)$.*

Owing to a well known result (see [7], Theorem 3.4), if E is UMD, the Hilbert transform is a bounded operator in $L^p(\mathbb{R}; E)$, for every $p \in (1, \infty)$. The prototype of UMD space is given by L^p spaces: one can show that, if μ is a positive measure and $q \in (1, \infty)$, $L^q(\mu)$ is UMD. Moreover, closed subspaces of UMD spaces are UMD spaces. This implies, for example, that standard Sobolev spaces $W^{m,q}(\Omega)$, with $1 < q < \infty$ are UMD.

The second key notion to extend Theorems 1 and 2 is the notion of $R-$boundedness. To define it, we introduce a probability space Ω, with probability measure P, and a class of random variables $\{r_n : n \in \mathbb{N}\}$, such that

(A1) for every $n \in \mathbb{N}$, $P(r_n = 1) = P(r_n = -1) = \frac{1}{2}$;

(A2) the random variables r_n are independent.

For example, we can consider the Rademacher sequence: in this case, $\Omega = [0, 1)$, P is the Lebesgue measure, if $n \in \mathbb{N}$,

$$r_n(t) = (-1)^k \quad \text{if } \tfrac{k-1}{2^n} \leq t < \tfrac{k}{2^n}, \quad 1 \leq k \leq 2^n.$$

If E and F are complex Banach spaces, we shall indicate with $\mathcal{L}(E, F)$ the Banach space of linear, bounded operators from E to F. Now we are able to define $R-$boundedness:

Definition 2. *Let E and F be complex Banach spaces and $\tau \subseteq \mathcal{L}(X, Y)$. We shall say that τ is $R-$bounded if there exists $C > 0$ such that, $\forall n \in \mathbb{N}$, $\forall T_1, ..., T_n \in \tau$,*

$\forall x_1, ..., x_n \in X$,

$$(3) \qquad \| \sum_{k=1}^{n} r_k T_k x_k \|_{L^2(\Omega;F)} \leq C \| \sum_{k=1}^{n} r_k x_k \|_{L^2(\Omega;E)},$$

with $(r_n)_{n\in\mathbb{N}}$ *fulfilling (A1)-(A2).*

Clearly (3) is equivalent to

$$(4) \quad \sum_{\epsilon_1\in\{-1,1\}} ... \sum_{\epsilon_n\in\{-1,1\}} \|\epsilon_1 T_1 x_1 + ...\epsilon_n T_n x_n\|_F^2 \leq C \sum_{\epsilon_1\in\{-1,1\}} ... \sum_{\epsilon_n\in\{-1,1\}} \|\epsilon_1 x_1 + ...\epsilon_n x_n\|_E^2,$$

with C independent of n, $T_1, ..., T_n \in \tau$, $x_1, ..., x_n \in E$. It is easily seen that, if τ is $R-$bounded, it is also bounded in $\mathcal{L}(E, F)$ and the converse holds if E and F are Hilbert spaces. The class of translation operators $\{T(t) : t \in \mathbb{R}\}$

$$[T(t)f](x) := f(x + t)$$

in $L^p(\mathbb{R})$, with $p \neq 2$, provides an example of a bounded family which is not $R-$bounded. A very important example of $R-$bounded family is given by the following theorem, which is usualy called Kahane's contraction principle:

Theorem 3. *(Kahane's contraction principle) Let E be a Banach space. Then, for ever $\Lambda \in \mathbb{R}^+$, $\{x \to \lambda x : |\lambda| \leq \Lambda\}$ is $R-$bounded in $\mathcal{L}(E)$.*

It is worth mentioning some well known facts concerning a family of random variables $(r_n)_{n\in\mathbb{N}}$ satisfying $(A1) - (A2)$. The first important fact is Khinchine's inequality: if $p \in [1, \infty)$, there exists $C_p \geq 1$, such that, $\forall n \in \mathbb{N}$, $\forall a_1, ..., a_n$ in $\mathbb{C}$,

$$(5) \qquad C_p^{-1} (\sum_{k=1}^{n} |a_k|^2)^{1/2} \leq \| \sum_{k=1}^{n} r_k a_k \|_{L^p(\Omega)} \leq C_p (\sum_{k=1}^{n} |a_k|^2)^{1/2}.$$

An inequality of the form (5) does not hold in a general Banach space E (unless E is a Hilbert space). However, the following fact (which in the case of $E = \mathbb{C}$ follows immediately from Khinchine's inequality) holds:

Theorem 4. *(Kahane's inequality) Let E be a Banach space, let $p \in [1, \infty)$ and let $(r_n)_{n\in\mathbb{N}}$ be a sequence of random variables satisfying $(\alpha_1) - (\alpha_2)$. Then, there exists $C(E, p) \geq 1$,* *such that,* $\forall n \in \mathbb{N}$, $\forall x_1, ..., x_n \in E$,

$$C(E,p)^{-1}\|\sum_{k=1}^{n} r_k x_k\|_{L^2(\Omega;E)} \le \|\sum_{k=1}^{n} r_k x_k\|_{L^p(\Omega;E)} \le C(E,p)\|\sum_{k=1}^{n} r_k x_k\|_{L^2(\Omega;E)}. \tag{6}$$

Kahane's inequality implies that in (3) and (4) we can replace the exponent 2 with any $p \in [1,\infty)$.

Now, we are able to state the following partial generalization of Theorem 1, essentially due to L. Weis (see [9], Theorem 4.6):

Theorem 5. *Let E and F be UMD spaces, $n \in \mathbb{N}$, and let $m \in C^n(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(E,F))$. We assume that*

$$\{|\xi|^{|\alpha|} D^\alpha m(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \alpha \in \mathbb{N}_0^n, \alpha \le (1,...,1)\} \quad \text{is } R\text{-bounded in } \mathcal{L}(E,F). \tag{7}$$

Then, for every $p \in (1,\infty)$, m is a Fourier multiplier between $L^p(\mathbb{R}^n;E)$ and $L^p(\mathbb{R}^n;F)$.

Of course, the meaning of the conclusion is the following: that the linear operator $f \to \mathcal{F}^{-1}(m\mathcal{F}f)$, defined, for example, for $f \in \mathcal{S}(\mathbb{R}^n;E)$, can be extended to a linear bounded operator between $L^p(\mathbb{R}^n;E)$ and $L^p(\mathbb{R}^n;F)$.

The extension of Theorem 2 requires a further assumptions concerning the spaces E and F.

Definition 3. *Let E be a Banach space and let $(r_n)_{n\in\mathbb{N}}$ be a family of random variables satisfying $(A_1)-(A_2)$. We shall say that E has property (α) if there exists $C \in \mathbb{R}^+$, such that, $\forall N \in \mathbb{N}$, $\forall \alpha_{ij} \in \mathbb{C}$ with $|\alpha_{ij}| \le 1$, $\forall x_{ij} \in E$ $(1 \le i,j \le N)$,*

$$\int_{\Omega\times\Omega} \|\sum_{i=1}^{N}\sum_{j=1}^{N} r_i(u)r_j(v)\alpha_{ij}x_{ij}\| d(P\otimes P) \le C\int_{\Omega\times\Omega} \|\sum_{i=1}^{N}\sum_{j=1}^{N} r_i(u)r_j(v)x_{ij}\| d(P\otimes P).$$

One can prove that, if $q \in (1,\infty)$ and μ is an arbitrary measure (in some set), $L^q(\mu)$ has property (α). There exist UMD spaces without property (α) (see [10]).

Now we are in position to state the following generalization of Theorem 2:

Theorem 6. *Let E and F be UMD Banach spaces with property (α), and let $m \in C^n(\mathbb{R}^n \setminus \{0\}; \mathcal{L}(E,F))$. We assume that*

$$\{\xi^\alpha D^\alpha m(\xi) : \xi \in \mathbb{R}^n \setminus \{0\}, \alpha \in \mathbb{N}_0^n, \alpha \le (1,...,1)\} \quad \text{is } R\text{-bounded in } \mathcal{L}(E,F). \tag{8}$$

Then, for every $p \in (1,\infty)$, m *is a Fourier multiplier between* $L^p(\mathbb{R}^n;E)$ *and* $L^p(\mathbb{R}^n;F)$.

For a proof, see [9], Theorem 4.13.

Now we want to give some idea of the proof of Theorem 5. For the sake of simplicity, we shall limit ourselves to consider the case $n=1$. The first step is the following

Theorem 7. *(Stekhlin's theorem) Let* E *be a* UMD *space, let* F *be a complex Banach space, and* $m \in C^1(\mathbb{R};\mathcal{L}(E,F))$, *be such that:*

(a) $\lim_{\xi\to-\infty} m(\xi) = 0$ *(in* $\mathcal{L}(E,F)$*);*

(b) $m' \in L^1(\mathbb{R};\mathcal{L}(E,F))$.

Then:

(I) for every $p \in (1,\infty)$, m *is a Fourier multiplier between* $L^p(\mathbb{R};E)$ *and* $L^p(\mathbb{R};F)$*;*

(II) let $\{m_i : i \in I\}$ *be a subset of* $C^1(\mathbb{R};\mathcal{L}(E,F))$, *such that, for every* i, m_i *satisfies (a), and* $m_i'(\xi) = g_i(\xi)\nu_i(\xi)$, *with* $\|g_i\|_{L^1(\mathbb{R})} \leq 1$ *and* $\{\nu_i(\xi) : i \in I, \xi \in \mathbb{R}\}$ $R-$*bounded in* $\mathcal{L}(E;F)$. *Then, the set of operators* $\{f \to \mathcal{F}^{-1}(m_i\mathcal{F}f) : i \in I\}$ *is* $R-$*bounded in* $\mathcal{L}(L^p(\mathbb{R};E);L^p(\mathbb{R};F))$ *for every* $p \in (1,\infty)$.

Sketch of the proof. Using the fact that E is UMD, one can easily show that, for every $s \in \mathbb{R}$, the characteristic function of $[s,\infty)$ χ_s is a Fourier multiplier in $L^p(\mathbb{R};E)$. Moreover,

$$\|\mathcal{F}^{-1}(\chi_s\mathcal{F}f)\|_{L^p(\mathbb{R};E)} \leq C\|f\|_{L^p(\mathbb{R};E)},$$

for some $C \in \mathbb{R}^+$, independent of $s \in \mathbb{R}$ and $f \in L^p(\mathbb{R};E)$ (in fact, one could show that the family of operators $\{f \to \mathcal{F}^{-1}(\chi_s\mathcal{F}f) : s \in \mathbb{R}\}$ is $R-$bounded in $\mathcal{L}(L^p(\mathbb{R};E))$). Now we observe that, $\forall\xi \in \mathbb{R}$,

$$m(\xi) = \int_{-\infty}^{\xi} m'(s)ds = \int_{\mathbb{R}} m'(s)\chi_s(\xi)ds,$$

so that

$$\mathcal{F}^{-1}(m\mathcal{F}f) = \int_{\mathbb{R}} m'(s)\mathcal{F}^{-1}[\chi_s\mathcal{F}f]ds,$$

and, by Minkowski inequality,

$$\|\mathcal{F}^{-1}(m\mathcal{F}f)\|_{L^p(\mathbb{R};F)} \leq \int_{\mathbb{R}} \|m'(s)\mathcal{F}^{-1}[\chi_s\mathcal{F}f]\|_{L^p(\mathbb{R};F)}ds$$

$$\leq \int_{\mathbb{R}} \|m'(s)\|_{\mathcal{L}(E,F)}\|\mathcal{F}^{-1}[\chi_s\mathcal{F}f]\|_{L^p(\mathbb{R};E)}ds \leq C\int_{\mathbb{R}} \|m'(s)\|_{\mathcal{L}(E,F)}ds\|f\|_{L^p(\mathbb{R};E)}.$$

We omit the proof of (II).

Another crucial tool is the Littlewood-Paley decomposition. For any $k \in \mathbb{Z}$, we set

$$I_k := (-2^k, -2^{1-k}] \cup [2^{k-1}, 2^k).$$

Using the fact that the Hilbert transform is a bounded operator, it is easily seen that, if E is UMD, for every k the characteristic funzione χ_{I_k} is a Fourier multiplier for $L^p(\mathbb{R}; E)$, for every $p \in (1, \infty)$, if E is a UMD Banach space. Firstly, we consider the case $E = \mathbb{C}$. Given $f \in L^p(\mathbb{R})$, we observe that, at least in the sense of distributions,

$$f = \sum_{k=-\infty}^{\infty} \mathcal{F}^{-1}[\chi_{I_k} \mathcal{F} f].$$

This is the classical Littlewood-Paley decomposition of f in the one dimensional case. The following classical fact holds (see [12], chapter IV): there exists $C_p \geq 1$, such that, for every $f \in L^p(\mathbb{R})$,

$$C_p^{-1} \|f\|_{L^p(\mathbb{R})}^p \leq \int_{\mathbb{R}} (\sum_{k=-\infty}^{\infty} |\mathcal{F}^{-1}[\chi_{I_k} \mathcal{F} f](x)|^2)^{p/2}) dx \leq C_p \|f\|_{L^p(\mathbb{R})}^p, \tag{9}$$

or, in short notation,

$$\|f\|_{L^p(\mathbb{R})}^p \sim \int_{\mathbb{R}} (\sum_{k=-\infty}^{\infty} |\mathcal{F}^{-1}[\chi_{I_k} \mathcal{F} f](x)|^2)^{p/2}) dx. \tag{10}$$

By Khinchine's inequality, if $(r_k)_{k \in \mathbb{Z}}$ satisfies $(A_1) - (A_2)$,

$$\begin{aligned} \int_{\mathbb{R}} (\textstyle\sum_{k=-\infty}^{\infty} |\mathcal{F}^{-1}[\chi_{I_k} \mathcal{F} f](x)|^2)^{p/2}) dx &\sim \int_{\mathbb{R}} \| \textstyle\sum_{k=-\infty}^{\infty} r_k \mathcal{F}^{-1}[\chi_{I_k} \mathcal{F} f](x) \|_{L^p(\Omega)}^p dx \\ &= \| \textstyle\sum_{k=-\infty}^{\infty} r_k \mathcal{F}^{-1}[\chi_{I_k} \mathcal{F} f] \|_{L^p(\Omega; L^p(\mathbb{R}))}^p. \end{aligned} \tag{11}$$

In conclusion,

$$\|f\|_{L^p(\mathbb{R})} \sim \| \sum_{k=-\infty}^{\infty} r_k \mathcal{F}^{-1}[\chi_{I_k} \mathcal{F} f] \|_{L^p(\Omega; L^p(\mathbb{R}))}. \tag{12}$$

In this form, Littlewood-Paley decomposition has been extended to the vector valued case by J. Bourgain (see [2]). The following theorem holds:

Theorem 8. *Let E be a UMD space, $p \in (1,\infty)$ and let $(r_k)_{k\in\mathbb{Z}}$ be a class of random variables satisfying $(A_1)-(A_2)$. Then*

$$\|f\|_{L^p(\mathbb{R};E)} \sim \|\sum_{k=-\infty}^{\infty} r_k \mathcal{F}^{-1}[\chi_{I_k}\mathcal{F}f]\|_{L^p(\Omega;L^p(\mathbb{R};E))}. \tag{13}$$

Now we are able to prove Theorem 5 in case $n=1$:

Proof of Teorem 5 in case $n=1$. In this case, we have $m \in C^1(\mathbb{R};\mathcal{L}(E,F))$, with $\{m(\xi):\xi\in\mathbb{R}\setminus\{0\}\}\cup\{\xi m'(\xi):\xi\in\mathbb{R}\setminus\{0\}\}$ $R-$bounded in $\mathcal{L}(E;F)$. Take $f\in\mathcal{S}(\mathbb{R};E)$. We want to prove an estimate of the form

$$\||\mathcal{F}^{-1}[m\mathcal{F}f]\|_{L^p(\mathbb{R};F)} \le C\|f\|_{L^p(\mathbb{R};E)}, \tag{14}$$

with $C\in\mathbb{R}^+$ independent of f. As F is UMD, by Theorem 8, we can try to estimate

$$\|\sum_{k=-\infty}^{\infty} r_k\mathcal{F}^{-1}[\chi_{I_k}m\mathcal{F}f]\|_{L^p(\Omega;L^p(\mathbb{R};F))}.$$

We fix $g\in\mathcal{D}(\mathbb{R})$, such that $g(\xi)=1$ if $1/2\le|\xi|\le 1$, $g(\xi)=0$ if $|\xi|\le 1/4$ or $|\xi|\ge 2$. For $k\in\mathbb{Z}$, we set

$$g_k(\xi) := g(2^{-k}\xi),$$

in such a way that $g_k(\xi)=1$ if $\xi\in I_k$. So, as $\chi_{I_k}=g_k\chi_{I_k}$,

$$\|\sum_{k=-\infty}^{\infty} r_k\mathcal{F}^{-1}[\chi_{I_k}m\mathcal{F}f]\|_{L^p(\Omega;L^p(\mathbb{R};F))} = \|\sum_{k=-\infty}^{\infty} r_k\mathcal{F}^{-1}[mg_k\mathcal{F}\mathcal{F}^{-1}(\chi_{I_k}\mathcal{F}f)]\|_{L^p(\Omega;L^p(\mathbb{R};F))}.$$

Now, for each $k\in\mathbb{Z}$, we set

$$m_k(\xi) := g_k(\xi)m(\xi).$$

It is clear that, for each $k\in\mathbb{Z}$, m_k satisfies the assumptions of Stekhlin's theorem. Moreover,

$$m_k'(\xi) = g_k'(\xi)m(\xi)+g_k(\xi)m'(\xi) = g_k'(\xi)m(\xi)+\xi^{-1}g_k(\xi)\xi m'(\xi)$$

We observe that

$$\int_{\mathbb{R}}|g_k'(\xi)|d\xi = \int_{\mathbb{R}}|g'(\xi)|d\xi, \int_{\mathbb{R}}|\xi^{-1}g_k(\xi)|d\xi = \int_{\mathbb{R}}|\xi^{-1}|g(\xi)|d\xi.$$

So, by Stekhlin's theorem,

(I) for each $k\in\mathbb{Z}$, m_k is a Fourier multiplier between $L^p(\mathbb{R};E)$ and $L^p(\mathbb{R};F)$;

(II) the set of operators $\{f \to \mathcal{F}^{-1}[m_k \mathcal{F} f] : k \in \mathbb{Z}\}$ is $R-$bounded in $\mathcal{L}(L^p(\mathbb{R}; E); L^p(\mathbb{R}; F))$. We deduce that

$$\| \sum_{k=-\infty}^{\infty} r_k \mathcal{F}^{-1}[m g_k \mathcal{F}\mathcal{F}^{-1}(\chi_{I_k} \mathcal{F} f)]\|_{L^p(\Omega; L^p(\mathbb{R}; F))} \leq C \| \sum_{k=-\infty}^{\infty} r_k \mathcal{F}^{-1}(\chi_{I_k} \mathcal{F} f)]\|_{L^p(\Omega; L^p(\mathbb{R}; E))},$$

and another application of Theorem 8 gives

$$\| \sum_{k=-\infty}^{\infty} r_k \mathcal{F}^{-1}(\chi_{I_k} \mathcal{F} f)]\|_{L^p(\Omega; L^p(\mathbb{R}; E))} \leq C \|f\|_{L^p(\mathbb{R}; E)}.$$

At first sight, one gets the impression that $R-$boundedness is just a useful technical tool to obtain some version of Mikhlin's multiplier in Banach spaces. In fact, one can show that it is almost necessary, as the following result, due to P. Clement and J. Prüss ([3]), shows:

Theorem 9. *Let E and F be Banach spaces, $m \in L^\infty(\mathbb{R}; \mathcal{L}(E, F))$, and let $L(m)$ be the set of continuity points of m. Then, if m is a Fourier multiplier between $L^p(\mathbb{R}; E)$ and $L^p(\mathbb{R}; F)$ for some $p \in (1, \infty)$, $\{m(\xi) : \xi \in L(m)\}$ is $R-$bounded in $\mathcal{L}(E, F)$.*

Now we want to show a relevant application of these techniques. We shall prove a result of maximal regularity for the mixed Cauchy-Dirichlet problem for the heat equation. The problem is the following:

$$\begin{cases} D_t u(t,x) = \Delta_x u(t,x) + f(t,x), \quad t > 0, x \in O, \\ \\ u(t,x') = 0, \quad t > 0, \quad x' \in \partial O, \\ \\ u(0,x) = 0, \quad x \in O. \end{cases} \tag{15}$$

O is an open bounded subset of $\mathbb{R}^m$, lying on one side of its boundary ∂O, which is a submanifold of $\mathbb{R}^m$ of class C^2. We consider the Banach space $E = L^q(O)$, with $q \in (1, \infty)$, which is UMD, and introduce the following operator A:

$$\begin{cases} D(A) = W^{2,q}(O) \cap W_0^{1,q}(O), \\ Au = \Delta u. \end{cases} \tag{16}$$

It is well known that A is the infinitesimal generator of an analytic semigroup $\{T(t) : t \geq 0\}$ in E, exponentially decreasing at ∞. So, if, for some $p \in (1, \infty)$, $f \in L^p(\mathbb{R}^+; E)$, (15) has a unique mild solution, given by the variation of parameter formula:

$$u(t) = \int_0^t T(t-s) f(s) ds. \tag{17}$$

We want to show that u, given by (17), belongs, in fact, to $W^{1,p}(\mathbb{R}^+; L^q(O)) \cap L^p(\mathbb{R}^+; W^{2,q}(O))$. Employing the first equation in (15), it clearly suffices to show that $u \in L^p(\mathbb{R}^+; D(A))$. We observe that (17) can be written in the form

$$u = K * \tilde{f},$$

with

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if } t \geq 0, \\ \\ 0 & \text{otherwise}, \end{cases} \tag{18}$$

so that

$$u = K * \tilde{f} = \mathcal{F}^{-1}(\hat{K} \mathcal{F} f).$$

It is easily seen that

$$\hat{K}(\xi) = (i\xi - A)^{-1}, \quad \xi \in \mathbb{R}, \tag{19}$$

so that, at least formally,

$$Au = \mathcal{F}^{-1}(A(i\xi - A)^{-1} \mathcal{F} f),$$

and we can try to show that $m(\xi) = A(i\xi - A)^{-1}$ is a Fourier multiplier in $L^p(\mathbb{R}; E)$, applying Theorem 5. It is known (see, for example, [4]), that, for every $\theta \in [0, \pi)$, the set of operators $\{\lambda(\lambda - A)^{-1} : \lambda \in \mathbb{C} \setminus \{0\}, |Arg(\lambda)| \leq \theta\}$ is $R-$bounded in $\mathcal{L}(E)$ (similar results hold for a large class of elliptic boundary value problems). As

$$A(i\xi - A)^{-1} = i\xi(i\xi - A)^{-1} - 1,$$

$\{m(\xi) : \xi \in \mathbb{R}\}$ is $R-$bounded in $\mathcal{L}(E)$. Moreover,

$$\xi m'(\xi) = \xi^2 (i\xi - A)^{-2} + i\xi(i\xi - A)^{-1},$$

so that even $\{\xi m'(\xi) : \xi \in \mathbb{R} \setminus \{0\}\}$ is $R-$bounded in $\mathcal{L}(E)$. We conclude that m is a Fourier multiplier and

$$u \in W^{1,p}(\mathbb{R}^+; E) \cap L^p(\mathbb{R}^+; D(A)),$$

so that

$$u \in W^{1,p}(\mathbb{R}^+; L^q(O)) \cap L^p(\mathbb{R}^+; W^{2,q}(O)).$$

We recall that the case $p = q$ was proved by Solonnikov (see [11]). A different proof can be obtained applying Dore-Venni's theorem (see [5]).

Finally, I want to illustrate some recent results that I have obtained in collaboration with A. Favini and Y. Yakubov (see [6]), where we employed some of the results and techniques which I have described.

We have started by considering a general abstract system of the form

$$(20) \qquad \begin{cases} (\lambda + \lambda_0)u(x) - u''(x) + B(x)u'(x) + A(x)u(x) = f(x), \quad x \in [0,1], \\ \\ u^{(m_r)}(r) = 0, \quad r \in \{0,1\}, \end{cases}$$

where $\lambda \in \mathbb{C}$, $\lambda_0 \geq 0$, and $m_r \in \{0,1\}$, with $B(x)$ and $A(x)$ unbounded operators in the Banach space E, $f \in L^p(0,1;E)$. The assumptions are the following:

(L1) E is a UMD Banach space, with norm $\|.\|$ and property (α).

(L2) For every $x \in [0,1]$ $(-\infty, 0] \subseteq \rho(A(x))$ (the resolvent set) and

$$\|(\lambda + A(x))^{-1}\|_{\mathcal{L}(E)} \leq M(x)(1+\lambda)^{-1}, \quad \lambda \geq 0.$$

(L3) The domains $D(A(x))$ and $D(A(x)^{1/2})$ are independent of $x \in [0,1]$. We shall indicate them with $D(A)$ and $D(A^{1/2})$.

(L4) $\forall x \in [0,1]$, $B(x) \in \mathcal{L}(D(A^{1/2}), E))$.

(L5) $\forall \lambda \in \mathbb{C}$, with $Re(\lambda) \geq 0$, $\forall \sigma \in \mathbb{R}$, $\forall x \in [0,1]$, the operator $\lambda + \sigma^2 + i\sigma B(x) + A(x)$ is a bijection between $D(A)$ and E, and $(\lambda + \sigma^2 + i\sigma B(x) + A(x))^{-1} \in \mathcal{L}(E)$; the families of operators $\{(\lambda + \sigma^2)(\lambda + \sigma^2 + i\sigma B(x) + A(x))^{-1} : \quad Re(\lambda) \geq 0, \sigma \in \mathbb{R}\}$ and $\{A(x)(\lambda + \sigma^2 + i\sigma B(x) + A(x))^{-1} : Re(\lambda) \geq 0, \sigma \in \mathbb{R}\}$ are $R-$bounded in $\mathcal{L}(E)$.

(L6) The maps $x \to A(x)$ and $x \to B(x)$ belong to, respectively, $C([0,1];\mathcal{L}(D(A),E))$ and $C([0,1];\ \mathcal{L}(D(A^{1/2}),E))$.

(L7) If $x \in \{0,1\}$, for every $\epsilon \in \mathbb{R}^+$ there exists $C(\epsilon) \in \mathbb{R}^+$, such that, $\forall u \in D(A^{1/2})$,

$$\|B(x)u\| \le \epsilon \|u\|_{D(A^{1/2})} + C(\epsilon)\|u\|.$$

Then, we can prove the following:

Theorem 10. *Consider system (20), with the assumptions (L1)-(L7) and let $p \in (1,\infty)$. We introduce the following operator $\mathcal{A}$, in the space $L^p(0,1;E)$ $(1 < p < \infty)$.*

$$\text{(21)} \qquad \begin{cases} D(\mathcal{A}) = \{u \in \cap_{i=0}^{2} W^{i,p}(0,1;D(A^{1-i/2})) : D_x^{m_r} u(r) = 0, r \in \{0,1\}\}, \\ \\ \mathcal{A}u(x) = -u''(x) + B(x)u'(x) + A(x)u(x). \end{cases}$$

Then, there exists $\lambda_0 \in \mathbb{R}$, such that $\{\lambda : \lambda \in \mathbb{C}, Re(\lambda) \ge 0\} \subseteq \rho(-\mathcal{A} - \lambda_0)$, and $\{\lambda(\lambda + \lambda_0 + \mathcal{A})^{-1} : Re(\lambda) \ge 0\}$ is $R-$bounded in $\mathcal{L}(L^p(0,1;E))$.

As a consequence, with arguments resembling the ones we employed to study system (15), one can show the following

Theorem 11. *Assume that the conditions (L1)-(L7) hold and consider the system*

$$\text{(22)} \quad \begin{cases} D_t u(t,x) = D_x^2 u(t,x) - B(x) D_x u(t,x) - A(x)u(t,x) + f(t,x), \quad t \in (0,T), x \in (0,1), \\ \\ D_x^{m_r} u(t,r) = 0, \quad t \in (0,T), r \in \{0,1\}, \\ \\ u(0,x) = u_0(x), \quad x \in (0,1). \end{cases}$$

Let $p, q \in (1,\infty)$. Then the following conditions are necessary and sufficient in order that (22) have a unique solution u belonging to $W^{1,q}(0,\ T;L^p(0,1;E)) \cap L^q(0,T;D(\mathcal{A}))$:

(I) $f \in L^q(0,T;L^p(0,1;E))$;

(II) $u_0 \in (L^p(0,1;E);D(\mathcal{A}))_{1-1/q,q}$ (the real interpolation space).

(III) In case $p = q$, (II) is equivalent to $u_0 \in B^{2(1-1/p)}_{p,p}(0,1;E) \cap L^p(0,1;(E,D(A))_{1-1/p,p})$, $u_0^{(m_r)}(r) = 0$ if $p > \frac{3}{2-m_r}$ ($r \in \{0,1\}$), where $B^{2(1-1/p)}_{p,p}(0,1;E)$ indicates the abstract Besov space.

We show a "concrete" system to which the previous results are applicable. The problem is the following:

(23)
$$\begin{cases} (\lambda + \lambda_0)u(x,y) - D_x^2 u(x,y) + B(x,y,D_y)D_x u(x,y) + A(x,y,D_y)u(x,y) = f(x,y), \\ (x,y) \in (0,1) \times O, \\ D_x^{(m_r)} u(r,y) = 0, \quad r \in \{0,1\}, y \in O, \\ B_j(y', \partial_y)u(x,y) = 0, \quad j \in \{1,...,m\}, x \in (0,1), y' \in \partial O, \end{cases}$$

with the following assumptions:

(N1) O is an open bounded subset of $\mathbb{R}^n$ lying on one side of ∂O, which is a submanifold of $\mathbb{R}^n$ of class C^{2m}, for certain $m \in \mathbb{N}$, $\epsilon \in \mathbb{R}^+$; for each $r \in \{0,1\}$, $m_r \in \{0,1\}$.

(N2) $A(x,y,D_y) = \sum_{|\alpha| \le 2m} a_\alpha(x,y)\partial_y^\alpha$ with a_α ($|\alpha| \le 2m$) belonging to $C([0,1] \times \overline{O})$, $a_\alpha(x,.) \in C^\epsilon(\overline{O})$ $\forall x \in [0,1]$ in case $|\alpha| = 2m$, for some $\epsilon \in \mathbb{R}^+$.

(N3) $B(x,y,D_y) = \sum_{|\alpha| \le m} b_\alpha(x,y)\partial_y^\alpha$, with b_α ($|\alpha| \le m$) belonging to $C([0,1] \times \overline{O})$, $b_\alpha(x,.) \in C^\epsilon(\overline{O})$ $\forall x \in [0,1]$ in case $|\alpha| = m$.

(N4) For $j = 1,...,m$, $B_j(y', \partial_y) = \sum_{|\alpha| \le m_j} b_{j,\alpha}(y)\partial_y^\alpha$ ($y' \in \partial O$) is a linear differential operator of order m_j ($0 \le m_j \le 2m-1$), with coefficients of class $C^{2m-m_j}(\partial O)$.

In the following we shall indicate with $A^\sharp(x,y,\partial_y)$, $B^\sharp(x,y,\partial_y)$, $B_j^\sharp(y',\partial_y)$ ($1 \le j \le m$) the parts of order (respectively) $2m$, m, m_j of $A(x,y,\partial_y)$, $B(x,y,\partial_y)$, $B_j(y',\partial_y)$ and we shall consider also the characteristic polynomials $A^\sharp(x,y,\zeta)$, $B^\sharp(x,y,\zeta)$, $B_j^\sharp(y',\zeta)$ ($\zeta \in \mathbb{C}^n$).

(N5) $b_\alpha(x,y) \equiv 0$ if $|\alpha| = m$, $x \in \{0,1\}$ and $y \in \overline{O}$.

(N6) $\forall (x,y) \in [0,1] \times \overline{O}$,

$$Re\{\sigma^2 + i\sigma B^\sharp(x,y,i\xi) + A^\sharp(x,y,i\xi)\} > 0, \quad \forall(\sigma,\xi) \in (\mathbb{R} \times \mathbb{R}^n) \setminus \{(0,0)\}. \tag{24}$$

For every $(x,y') \in [0,1] \times \partial O$ *we consider the o. d. e. system*

$$\begin{cases} (\lambda+\sigma^2)v(t) + i\sigma B^\sharp(x,y',i\eta + \nu(y')D_t)v(t) + A^\sharp(x,y',i\eta+\nu(y')D_t)v(t) = 0, \quad t \geq 0, \\ \\ B_j^\sharp(y', i\eta + \nu(y')D_t)v(0) = g_j, \quad j = 1,...,m, \end{cases} \tag{25}$$

with $\lambda \in \mathbb{C}$, $Re(\lambda) \geq 0$, $\sigma \in \mathbb{R}$, $y' \in \partial O$, $\eta \in T_{y'}(\partial O)$ *(the tangent space),* $\nu(y')$ *unit vector orthogonal to* ∂O *and pointing inside* O *. Then, if* $(\lambda,\sigma,\eta) \neq (0,0,0)$*, for any* $(g_1,...,g_m) \in \mathbb{C}^m$*, (25) has a unique solution tending to* 0*, as* $t \to \infty$*.*

The following result holds:

Theorem 12. *Assume that the assumptions (N1)-(N6) are satisfied and let* $p \in (1,+\infty)$*. Then:*

(I) there exists $\lambda_0 \geq 0$*, such that, if* $\lambda \in \mathbb{C}$*,* $Re(\lambda) \geq 0$*,* $f \in L^p((0,1)\times O)$*, the system (23) has a unique solution* u *belonging to* $\cap_{i=0}^2 W^{i,p}(0,1;W^{(2-i)m,p}(O))$*.*

(II) Define the following operator $\mathcal{A}$*:*

$$\begin{cases} D(\mathcal{A}) := \{u \in \cap_{j=0}^2 W^{2-j,p}(0,1;W^{jm,p}(O)) : D_x^{m_r}u(r,\cdot) \equiv 0, r \in \{0,1\}, \\ B_j(y',D_y)u(\cdot,y') = 0, y' \in \partial O\}, \\ \\ \mathcal{A}u(x,y) = -D_x^2 u(x,y) + B(x,y.D_y)D_x u(x,y) + A(x,y,D_y)u(x,y), \end{cases} \tag{26}$$

and think of $\mathcal{A}$ *as an unbounded operator in* $L^p(0,1;L^p(O)) = L^p((0,1)\times O)$*. Then, there exists* $\lambda_0 \in \mathbb{R}$*, such that* $\{\lambda : \lambda \in \mathbb{C}, Re(\lambda) \geq 0\} \subseteq \rho(-\mathcal{A}-\lambda_0)$*, and* $\{\lambda(\lambda+\lambda_0+\mathcal{A})^{-1} : Re(\lambda) \geq 0\}$ *is* $R-$*bounded in* $\mathcal{L}(L^p(0,1;L^p(O))$*.*

Sketch of the proof We set $E = L^p(\Omega)$. We already know that E is UMD with property (α). Next, we introduce the following notation: let $s \in \mathbb{N}$, $s \leq 2m$. We set

$$W_B^{s,p}(O) := \{u \in W^{s,p}(O) : B_j u = 0 \quad \text{if } 1 \leq j \leq m, \quad m_j + 1/p < s\}. \tag{27}$$

Analogously, if $0 < s \le 2m$, $1 \le p, q \le 2m$, we set

$$B^s_{p,q,B}(O) := \{u \in B^s_{p,q}(O) : B_j u = 0 \quad \text{if } 1 \le j \le m, \quad m_j + 1/p < s\}. \tag{28}$$

Then, we set

$$\begin{cases} A_0(x) : D(A) \to E, \\ \\ A_0(x)u := A(x, \cdot, \partial_y)u, \quad u \in D(A), \end{cases} \tag{29}$$

One can show, essentially applying the results of [4], that there exists $\mu_0 \ge 0$, such that, if we set

$$A(x) := A_0(x) + \mu_0, \tag{30}$$

$A(x)$ satisfies (L2) for every $x \in [0,1]$, in such a way that the fractional power $A(x)^{1/2}$ is defined. One can see also that, for every $x \in [0,1]$,

$$D(A(x)^{1/2}) = W^{m,p}_B(O), \tag{31}$$

so that even (L3) is satisfied. Now, for every $x \in [0,1]$, we set

$$\begin{cases} B(x) : D(A^{1/2}) \to E, \\ \\ B(x)u := B(x, \cdot, \partial_y)u, \quad u \in D(A^{1/2}). \end{cases} \tag{32}$$

The assumptions of regularity of the coefficients imply that (L6) holds. We omit the technical proof of (L5), which can be obtained increasing (if necessary) μ_0. Finally (L7) is a consequence of (N5). In fact, if $x \in \{0,1\}$, it implies that $B(x)$ can be extended to $W^{m-1,p}_B(O)$. From this, the estimate in (L7) follows.

We conclude that Theorem 10 is applicable.

Now we consider the "parabolic" problem

(33)
$$\begin{cases} \partial_t u(t,x,y) = \partial_x^2 u(t,x,y) - B(x,y,D_y)\partial_x u(t,x,y) - A(x,y,D_y)u(t,x,y) + f(t,x,y), \\ \\ (t,x,y) \in (0,T) \times (0,1) \times \Omega, \\ \\ \partial_x^{(m_r)} u(t,r,y) = 0, \quad t \in (0,T), r \in \{0,1\}, y \in \Omega, \\ \\ B_j(y', \partial_y)u(t,x,y') = 0, \quad j \in \{1,...,m\}, t \in (0,T), x \in (0,1), y' \in \partial\Omega \\ \\ u(0,x,y) = u_0(x,y), \quad (x,y) \in (0,1) \times \Omega. \end{cases}$$

For simplicity, we consider only the case $q = p$. Applying Theorems 12 and 11, one can show the following

Theorem 13. *We assume that the conditions (N1)-(N6) are fulfilled. Let* $p \in (1, \infty)$, *with* $2m[1 - 1/(2p) - m_r/2] - m_j \neq 1/p$, *for each* $r \in \{0,1\}$, $j \in \{1,...,m\}$, *and* $p \neq \frac{3}{2-m_r}$, *for each* $r \in \{0,1\}$. *Then the following conditions are necessary and sufficient, in order that (33) have a unique solution* u *in the space* $W^{1,p}(0,T; L^p((0,1) \times O)) \cap \cap_{i=0}^{2} L^p(0,T; W^{i,p}(0,1; W^{(1-i/2)2m,p}(O))$:

(I) $f \in L^p(0,T; L^p((0,1) \times O))$;

(II) $u_0 \in B_{p,p}^{2(1-1/p)}(0,1; L^p(O)) \cap L^p(0,1; B_{p,B}^{2m(1-1/p)}(O))$, $\partial_x^{m_r} u_0(r,\cdot) = 0$, *in case* $p > \frac{3}{2-m_r}$ *(*$r \in \{0,1\}$*).*

Sketch of the proof We can apply Theorem 11. We must characterize the space $B_{p,p}^{2(1-1/p)}(0,1;E) \cap L^p(0,1;(E,D(A))_{1-1/p,p})$. One can show that, if $2m[1 - 1/(2p) - m_r/2] - m_j \neq 1/p$, for each $r \in \{0,1\}$, $j \in \{1,...,m\}$, and $p \neq \frac{3}{2-m_r}$, for each $r \in \{0,1\}$, it coincides with $B_{p,p}^{2(1-1/p)}(0,1;L^p(O)) \cap L^p(0,1;B_{p,B}^{2m(1-1/p)}(O))$.

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