ON THE CHARACTERIZATION OF THE HARMONIC PSEUDOSPHERES VIA KURAN'S FUNCTIONS AND SINGLE-LAYER POTENTIALS

SULLA CARATTERIZZAZIONE DELLE PSEUDOSFERE ARMONICHE MEDIANTE FUNZIONI DI KURAN E POTENZIALI DI SEMPLICE STRATO

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Abstract. We present some characterizations of the harmonic pseudospheres in terms of the so called Kuran's functions and of the single-layer potentials. Our characterizations apply to *solid*, *harmonically stable* domains.

Sunto. Presentiamo alcune caratterizzazioni delle pseudosfere armoniche in termini delle cosiddette funzioni di Kuran e dei potenziali di semplice strato. Le nostre caratterizzazioni valgono, in particolare, per domini *solidi*, *armonicamente stabili*.

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Basic notation.

If D is an open subset of $\mathbb{R}^n$, we write

$\overline{D} :=$ closure of D

$|D| :=$ Lebesgue measure of D

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$|\partial D| :=$ $(n-1)$-Hausdorff measure of ∂D

$\sigma :=$ $(n-1)$-Hausdorff measure

$B(x_0, r) :=$ Euclidean ball in $\mathbb{R}^n$ with center x_0 and radius $r > 0$

$\mathcal{H}(D) :=$ set of the harmonic functions in D

$\mathcal{H}(\overline{D}) :=$ set of the harmonic functions in some open set containing $\overline{D}$

$C(\overline{D}) :=$ set of the real continuous functions in $\overline{D}$.

1. Introduction: past and recent history

Let D be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$, such that $|\partial D| < \infty$ and let $x_0 \in D$. We say that ∂D is a *harmonic pseudosphere* centered at x_0 if

$$u(x_0) = \fint_{\partial D} u \, d\sigma \qquad \forall u \in \mathcal{H}(D) \cap C(\overline{D}). \tag{1}$$

By the classical Gauss Mean Value Theorem every Euclidean sphere $\partial B = \partial B(x_0, r)$ is a harmonic pseudosphere centered at x_0. Keldysch and Lavrentieff - in 1937 - proved the existence of harmonic pseudospheres in $\mathbb{R}^2$ which are *not circles*, [3]. Many years later, in 1991, Lewis and Vogel proved that in every Euclidean space $\mathbb{R}^n$, $n \geq 3$, there exist harmonic pseudospheres, homeomorphic to $\partial B(0,1)$ via bi-Hölder continuous homeomorphisms, which are not Euclidean spheres [4]. Today it is quite well known that harmonic pseudospheres, with even mild regularity properties, actually are Euclidean spheres: see e.g. the recent paper [1] and the references therein.

Aim of this note is to show that harmonic pseudospheres can be characterized in several ways: in particular by using single-layer potentials, and the harmonic maps called, in [1], Kuran's functions.

2. The Kuran's function

Let $x_0, \alpha \in \mathbb{R}^n$, $x_0 \neq \alpha$, $n \geq 3$. We call (x_0, α)-Kuran function the map

$$k_{x_0,\alpha}(x) = 1 + h_{x_0,\alpha}(x), \qquad x \neq \alpha,$$

where

$$h_{x_0,\alpha}(x) := |\alpha - x_0|^{n-2}\frac{|x-x_0|^2 - |\alpha - x_0|^2}{|x-\alpha|^n}.$$

One immediately verifies that

$$h_{x_0,\alpha}(x_0) = -1,$$

so that

$$k_{x_0,\alpha}(x_0) = 0.$$

The function $h_{x_0,\alpha}$, up to a multiplicative constant, is the Poisson kernel of the ball $B(x_0, r)$ with $r = |\alpha - x_0|$. Therefore $k_{x_0,\alpha}$ is harmonic in $\mathbb{R}^n \setminus \{\alpha\}$ since it is harmonic in $B(x_0, |x_0 - \alpha|)$ and real analytic in $\mathbb{R}^n \setminus \{\alpha\}$.

If $x_0 = 0$ we will use the notations h_α and k_α instead of $h_{x_0,\alpha}$ and $k_{x_0,\alpha}$ respectively.

3. The upper Kuran gap

Let $D \subseteq \mathbb{R}^n$, $n \geq 3$, be a bounded open set with $|\partial D| < \infty$ and let x_0 be a point of D. We define

$$\begin{aligned}\overline{\mathcal{K}}(\partial D, x_0) &= \textit{upper Kuran gap of } \partial D \textit{ w.r.t. } x_0 \\ &:= \sup_{\alpha \in \mathbb{R}^n \setminus \overline{D}} \left| k_{x_0,\alpha}(x_0) - \fint_{\partial D} k_{x_0,\alpha}(x)\, d\sigma(x) \right|. \end{aligned} \tag{2}$$

Since $k_{x_0,\alpha}(x_0) = 0$, we have

$$\overline{\mathcal{K}}(\partial D, x_0) = \sup_{\alpha \in \mathbb{R}^n \setminus \overline{D}} \left| \fint_{\partial D} k_{x_0,\alpha}(x)\, d\sigma(x) \right|.$$

We want to stress that the upper Kuran gap only depends on the α's close to ∂D since $k_{x_0,\alpha}(x)$ goes to zero as $|\alpha| \to \infty$, uniformly w.r.t. $x \in \partial D$. Due to the symmetry properties of the Kuran's functions, it is easy to show that $\overline{\mathcal{K}}(\partial D, x_0)$ is invariant w.r.t.

translations, dilations and *rotations around* x_0.

We explicitly remark that

$$\overline{\mathcal{K}}(\partial D, x_0) = 0$$

if ∂D is a harmonic pseudosphere centered at x_0. Indeed, being $k_{x_0,\alpha}$ harmonic in $\mathbb{R}^n \setminus \{\alpha\}$, if ∂D is a harmonic pseudosphere with center at x_0, one has

$$k_{x_0,\alpha}(x_0) = \fint_{\partial D} k_{x_0,\alpha}(x)\, d\sigma(x), \qquad \forall \alpha \in \mathbb{R}^n \setminus \overline{D}.$$

Therefore, by the very definition of upper Kuran gap (see (2)), $\overline{\mathcal{K}}(\partial D, x_0) = 0$. We will show that for a wide class of open sets D, if

$$\overline{\mathcal{K}}(\partial D, x_0) = 0,$$

then ∂D is a harmonic pseudosphere with center at x_0.

4. Our main Theorem

Aim of this section is to prove the following theorem, the main result of this note.

Theorem 4.1. *Let D be a bounded open set of $\mathbb{R}^n$ with $|\partial D| < \infty$, and let $x_0 \in D$. Assume $n \geq 3$ and*

$$\mathbb{R}^n \setminus \overline{D} \quad \textit{connected.} \tag{3}$$

Then the following statements are equivalent

(i) $\overline{\mathcal{K}}(\partial D, x_0) = 0$;
(ii) $\fint_{\partial D} \Gamma(x - y)\, d\sigma(x) = \Gamma(x_0 - y)$ *for every* $y \notin \overline{D}$;
(iii) $u(x_0) = \fint_{\partial D} u(x)\, d\sigma(x)$ *for every* $u \in \mathcal{H}(\overline{D})$.

Note: Condition (ii) is usually called *single-layer potential property.*

Proof of Theorem 4.1. The equivalence

$$(ii) \;\Leftrightarrow\; (iii)$$

is proved in [1], Lemma 4.1. From the very definition of $\overline{\mathcal{K}}(\partial D, x_0)$ it immediately follows that

$$(iii) \;\Rightarrow\; (i).$$

Then we only have to prove that

$$(i) \;\Rightarrow\; (ii). \tag{4}$$

Due to the translation invariance of both (i) and (ii), it is enough to prove (4) in the case $x_0 = 0$. It is convenient to split the proof into several steps.

Step I. From our assumption

$$\overline{\mathcal{K}}(\partial D, 0) = 0. \tag{5}$$

This is equivalent to saying that

$$\fint_{\partial D} k_\alpha(x)\, d\sigma(x) = k_\alpha(0) = 0 \qquad \forall \alpha \in \mathbb{R}^n \setminus \overline{D}.$$

Hence, keeping in mind that $k_\alpha = 1 + h_\alpha$,

$$\fint_{\partial D} h_\alpha(x)\, d\sigma(x) = -1 \qquad \forall \alpha \in \mathbb{R}^n \setminus \overline{D}.$$

More explicitly, for every $\alpha \in \mathbb{R}^n \setminus \overline{D}$,

$$|\alpha|^{n-2} \fint_{\partial D} \frac{|x|^2 - |\alpha|^2}{|x-\alpha|^n}\, d\sigma(x) = -1.$$

Then, since

$$|x|^2 - |\alpha|^2 = |x - \alpha + \alpha|^2 - |\alpha|^2 = |x-\alpha|^2 + 2\langle x - \alpha, \alpha\rangle,$$

we get

$$|\alpha|^{n-2} \fint_{\partial D} \left(\frac{1}{|x-\alpha|^{n-2}} + 2\frac{\langle x-\alpha, \alpha\rangle}{|x-\alpha|^n} \right) d\sigma(x) = -1,$$

or, equivalently,

$$\fint_{\partial D} \frac{1}{|x-\alpha|^{n-2}}\, d\sigma(x) + 2 \fint_{\partial D} \frac{\langle x-\alpha, \alpha\rangle}{|x-\alpha|^n}\, d\sigma(x) = -\left(\frac{1}{|\alpha|}\right)^{n-2} \tag{6}$$

for every $\alpha \in \mathbb{R}^n \setminus \overline{D}$.

On the other hand, for every $\alpha \notin \overline{D}$,

$$\nabla_\alpha \fint_{\partial D} \frac{1}{|x-\alpha|^{n-2}}\, d\sigma(x) = (n-2) \fint_{\partial D} \frac{x-\alpha}{|x-\alpha|^n}\, d\sigma(x), \tag{7}$$

so that, if we define

$$u(\alpha) := \fint_{\partial D} \frac{1}{|x-\alpha|^{n-2}}\, d\sigma(x), \qquad \alpha \notin \overline{D}, \tag{8}$$

from (6) and (7) we obtain

$$\frac{1}{2} u(\alpha) + \frac{1}{n-2}\langle \alpha, \nabla u(\alpha)\rangle = -\frac{1}{2}\left(\frac{1}{|\alpha|}\right)^{n-2} \tag{9}$$

for every $\alpha \notin \overline{D}$.

So, we have proved that (5) implies (9).

Step II. If we denote

$$\gamma(\alpha) := \left(\frac{1}{|\alpha|}\right)^{n-2}, \qquad \alpha \in \mathbb{R}^n \setminus \{0\},$$

then

$$\frac{1}{2}\gamma(\alpha) + \frac{1}{n-2}\langle \alpha, \nabla\gamma(\alpha)\rangle = -\frac{1}{2}\gamma(\alpha) \tag{10}$$

for every $\alpha \neq 0$.

Indeed, if $\alpha \neq 0$,

$$\begin{aligned}\frac{1}{2}\gamma(\alpha) + \frac{1}{n-2}\langle \alpha, \nabla\gamma(\alpha)\rangle =& \frac{1}{2}\left(\frac{1}{|\alpha|}\right)^{n-2} + \frac{1}{n-2}\langle \alpha, (2-n)|\alpha|^{1-n}\frac{\alpha}{|\alpha|}\rangle \\ =& \frac{1}{2}\left(\frac{1}{|\alpha|}\right)^{n-2} - \ |\alpha|^{2-n} \\ =& -\frac{1}{2}\left(\frac{1}{|\alpha|}\right)^{n-2} = -\frac{1}{2}\gamma(\alpha).\end{aligned}$$

This proves (10).

Step III. Let us denote

$$w(\alpha) := u(\alpha) - \gamma(\alpha), \qquad \alpha \notin \overline{D}. \tag{11}$$

Since u satisfies (9) and γ satisfies (10), we have

$$\frac{1}{2}w(\alpha) + \frac{1}{n-2}\langle \alpha, \nabla w(\alpha)\rangle = 0 \tag{12}$$

for every $\alpha \notin \overline{D}$.

On the other hand, as it is easy to recognize,

$$u(\alpha) = \gamma(\alpha)(1 + \eta(\alpha))$$

with $\eta(\alpha) \to 0$ as $|\alpha| \to \infty$. This implies

$$w(\alpha) = \gamma(\alpha)\eta(\alpha) = \frac{\eta(\alpha)}{|\alpha|^{n-2}}. \tag{13}$$

Let us now fix $\alpha \in \mathbb{R}^n$, $|\alpha| > R$, where $R > 0$ is such that

$$\overline{D} \subseteq B(0, R).$$

Then, define

$$f_\alpha : [1, \infty[\to \mathbb{R}, \qquad f_\alpha(t) := w(t\alpha). \tag{14}$$

Due to (12) we have

$$f'_\alpha(t) = \langle \nabla w(t\alpha), \alpha \rangle = -\frac{1}{t}\frac{n-2}{2} f_\alpha(t),$$

therefore the function f_α solves the Cauchy problem

$$\begin{cases} y' = -\frac{1}{t}\frac{n-2}{2} y, \quad t \geq 1 \\ y(1) = w(\alpha). \end{cases} \tag{15}$$

The unique solution of this problem is the function

$$f_\alpha(t) = w(\alpha) t^{-\frac{n-2}{2}}, \qquad t \geq 1.$$

Then, keeping in mind (14),

$$w(t\alpha) = w(\alpha) t^{-\frac{n-2}{2}} \qquad \forall t \geq 1. \tag{16}$$

On the other hand, by (13), we get

$$w(t\alpha) = \frac{\eta(t\alpha)}{t^{n-2}|\alpha|^{n-2}}.$$

Using this information in (16), we get

$$w(\alpha) = \frac{\eta(t\alpha)}{t^{\frac{n-2}{2}}|\alpha|^{n-2}} \qquad \forall t \geq 1.$$

Since $\eta(t\alpha) \to 0$, as $t \to \infty$ and $n \geq 3$, from this identity, letting t go to infinity, we obtain

$$w(\alpha) = 0 \qquad \forall \alpha : |\alpha| > R. \tag{17}$$

Let us now remark that u and γ are real analytic in $\mathbb{R}^n \setminus \overline{D}$. Hence w is real analytic in $\mathbb{R}^n \setminus \overline{D}$. Then, from (17), since $\mathbb{R}^n \setminus \overline{D}$ is connected we deduce

$$w = 0 \quad \text{in } \mathbb{R}^n \setminus \overline{D},$$

i.e.,

$$u = \gamma \quad \text{in } \mathbb{R}^n \setminus \overline{D}.$$

This identity, thanks to (8) and the definition of γ, implies

$$\fint_{\partial D} \frac{1}{|x-\alpha|^{n-2}}\, d\sigma(x) = \left(\frac{1}{|\alpha|}\right)^{n-2} \qquad \forall \alpha \notin \overline{D}$$

or, equivalently, keeping in mind that $x_0 = 0$,

$$\fint_{\partial D} \Gamma(x-\alpha)\, d\sigma(x) = \Gamma(\alpha) = \Gamma(x_0 - \alpha) \qquad \forall \alpha \notin \overline{D}.$$

This proves (*ii*) and completes the proof of our theorem. □

Remark 4.1. *The equivalence* (*ii*) $\Leftrightarrow$ (*iii*) *does not require the assumption* (3) *(see* [1], *Lemma 4.1). This assumption is obviously not required also for the implication* (*iii*) $\Rightarrow$ (*i*).

5. $\mathcal{H}$-stable domains and harmonic pseudospheres

Let D be a bounded open subset of $\mathbb{R}^n$, $n \geq 3$. We say that D is $\mathcal{H}$*-stable* if, for every

$$u \in \mathcal{H}(D) \cap C(\overline{D})$$

and for every $\varepsilon > 0$, there exists $v \in \mathcal{H}(\overline{D})$ such that

$$\sup_{\overline{D}} |u - v| < \varepsilon.$$

Then, by the very definition of harmonic pseudosphere - see (1) - we have the following proposition.

Proposition 5.1. *If D is a bounded $\mathcal{H}$-stable open set such that $|\partial D| < \infty$, and if $x_0 \in D$, then D is a harmonic pseudosphere centered at x_0 if and only if*

$$u(x_0) = \fint_{\partial D} u(x)\, d\sigma(x) \qquad \forall u \in \mathcal{H}(\overline{D}).$$

It is known that $\mathcal{H}$-stable domains can be characterized in terms of *thinness*. Let us recall the following theorem, whose proof can be found in [2], page 11.

Theorem 5.1. *Let D be a bounded open subset of $\mathbb{R}^n$ such that*

$$D = \mathrm{int}(\overline{D}).$$

Then D is $\mathcal{H}$-stable if and only if

$$\mathbb{R}^n \setminus D \quad \textit{and} \quad \mathbb{R}^n \setminus \overline{D}$$

are thin at the same points of ∂D.

We also want to recall that the thinness of a set at a point can be geometrically characterized in terms of Wiener series, see e.g. [2], page 6.

Putting together Theorem 5.1, Proposition 5.1 and Theorem 4.1, we get the following result.

Theorem 5.2. *Let $n \geq 3$ and let D be a solid open subset of $\mathbb{R}^n$, that is: D is bounded,*

$$D = \operatorname{int}(\overline{D}) \quad \textit{and} \quad \mathbb{R}^n \setminus \overline{D} \textit{ is connected.}$$

Moreover, assume that

$$\mathbb{R}^n \setminus D \quad \textit{and} \quad \mathbb{R}^n \setminus \overline{D}$$

are thin at the same points of ∂D.

Then, if $x_0 \in D$, the following statements are equivalent:

(i) *∂D is a harmonic pseudosphere centered at x_0;*

(ii) *$\overline{\mathcal{K}}(\partial D, x_0) = 0$;*

(iii) $\displaystyle ⨍_{\partial D} \Gamma(x - y)\, d\sigma(x) = \Gamma(x_0 - y) \quad \forall y \notin \overline{D}.$

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