ELLIPTIC SYSTEMS WITH BOUNDED SOLUTIONS AND DOUBLE PHASE FUNCTIONALS

SISTEMI ELLITTICI CON SOLUZIONI LIMITATE E FUNZIONALI DOPPIA FASE

FRANCESCO LEONETTI

ABSTRACT. We study boundedness of weak solutions to elliptic systems of partial differential equations in divergence form, under the so-called p, q growth conditions. Examples are obtained by writing the Euler system of some double phase functionals.

SUNTO. Studiamo la limitatezza delle soluzioni deboli di sistemi di equazioni differenziali di tipo ellittico in forma di divergenza, sotte le cosiddette condizioni di crescita p, q. Esempi sono ottenuti scrivendo l'equazione di Eulero di alcuni funzionali doppia fase.

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1. ELLIPTIC SYSTEMS WITH BOUNDED SOLUTIONS

We consider the system of partial differential equations in divergence form

(1)
$$\begin{cases} -\sum_{i=1}^{n} D_i \left[A_i^1(x, Du(x)) \right] &= 0, \\ -\sum_{i=1}^{n} D_i \left[A_i^2(x, Du(x)) \right] &= 0, \\ \dots\dots\dots\dots\dots\dots & \\ -\sum_{i=1}^{n} D_i \left[A_i^m(x, Du(x)) \right] &= 0, \end{cases}$$

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where $x \in \Omega \subset \mathbb{R}^n$, Ω is bounded and open, $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and $u^1, u^2, ..., u^m$ are the components of u, so that, $u = (u^1, u^2, ..., u^m)$. We assume that u is a weak solution of the previous system, $A(x, z)$ is measurable with respect to x and continuous with respect to z. We ask what are sufficient conditions on $A(x, z)$ that force u to be bounded. We first consider the scalar case $m = 1$, that is, the case in which the system consists of only one equation

$$-\sum_{i=1}^{n} D_i \left[A_i^1(x, Du(x)) \right] = 0, \tag{2}$$

where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and, for some constants $p \in (1, +\infty)$, $c_1, \nu \in (0, +\infty)$, $c_2 \in [0, +\infty)$,

$$\left| A_i^1(x, z) \right| \leq c_1(|z|^{p-1} + 1), \tag{3}$$

$$\nu |z|^p - c_2 \leq \sum_{i=1}^{n} A_i^1(x, z) z_i. \tag{4}$$

Let us assume that $u \in W^{1,p}(\Omega; \mathbb{R})$ is a weak solution of equation (2); then

$$u \in C^{0,\alpha}_{loc}(\Omega; \mathbb{R}).$$

This is the celebrated regularity result obtained by De Giorgi, Nash, Moser, at the end of '50, see [55], [108], [107].

In the vectorial case $m \geq 2$, singular solutions to elliptic systems may appear, as De Giorgi (1968) shows in [56]; see also [101], [75], [67], [110], [72], [85], [113], [74], [84], [88], [78], [77], [115], [68], [105]. See also the reviews [102], [103], [87].

Let us come back to the scalar case $m = 1$ and let us note that the exponent p of the growth condition from above (3) is the same exponent in the growth condition from below (4) and it is the same exponent of the Sobolev space $W^{1,p}(\Omega)$ to which the weak solution

u is assumed to belong. If we allow a bigger exponent q in the growth condition from above, namely

$$\left|A_i^1(x,z)\right| \le c_1(|z|^{q-1}+1), \tag{5}$$

$$\nu|z|^p - c_2 \le \sum_{i=1}^{n} A_i^1(x,z) z_i, \tag{6}$$

$$p < q,$$

then, singular solutions may appear: Giaquinta, Marcellini, Hong, end of '80, see [73], [96], [82].

Some regularity can be obtained if q is close to p: [97], [98], [106], [69], [116], [14], [70], [94], [29], [30], [1], [114], [82], [11], [28], [43] [62], [63], [71], [25], [12], [92], [32], [38], [15], [26], [7], [36], [16], [23], [34], [57], [27], [39], [60], [21], [2], [3], [8], [9], [81], [46], [22], [20], [109], [17], [5], [99], [58], [59], [100], [48], [49], [52], [83] [54], [40]. See also the reviews [90], [102], [104],

Let us come back to the vectorial case $m \ge 1$, that is, we consider the system (1), where $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m$ and, for constants $q, p \in (1, +\infty)$, $c_1, \nu \in (0, +\infty)$, $c_2 \in [0, +\infty)$,

$$|A_i^\alpha(x,z)| \le c_1(|z|^{q-1}+1), \tag{7}$$

$$\nu|z^\alpha|^p - c_2 \le \sum_{i=1}^{n} A_i^\alpha(x,z) z_i^\alpha, \tag{8}$$

$$1 < p < q < \frac{p^*}{p}\frac{n}{n+1}, \qquad p < n. \tag{9}$$

Assume that

$$u \in W^{1,q}(\Omega; \mathbb{R}^m) \tag{10}$$

is a weak solution to system (1), then

$$u \in L^\infty_{loc}(\Omega; \mathbb{R}^m), \tag{11}$$

see Cupini-Leonetti-Mascolo (2022): [37].

In order to get bounded solutions, we need to keep away De Giorgi's counterexample, [56]. So, we must assume some structure condition that excludes De Giorgi's system. In the previous result, such a structure condition is (8): let us call it "componentwise coercivity". It says that the α row of the system $A_i^\alpha(x, Du)$ contains the gradient of all the components of u, but, when multiplied by the gradient of the α component of u, from below we can see only the gradient of the α component of u. Componentwise coercivity allows us to arrive at Caccioppoli inequality on superlevel sets for the scalar function u^α, so that we can use De Giorgi's iteration for the scalar function u^α. Such a componentwise coercivity has been used by Bjorn (2001), Leonetti-Petricca (2011), Softova (2018), Palagacev-Softova (2020), Cupini-Leonetti-Mascolo (2022): [13], [93], [112], [111], [37].
Let us also mention Zhou (2000), [119], where a componentwise sign condition was assumed.

Examples of $A(x, z)$ enjoying componentwise coercivity are the Euler systems of some double phase functionals, see next section.

2. Double phase functionals

Example 1. Let us consider the variational integral

$$\mathcal{F}_1(u) = \int_\Omega (|Du(x)|^p + a(x)|Du(x)|^q)\, dx, \tag{12}$$

where

$$2 \le p < q, \qquad 0 \le a(x) \le M.$$

When $a(x) = 0$ then $|Du|^p + a(x)|Du|^q = |Du|^p$ and we are in the p-phase. On the other hand, if $a(x) > 0$ then, for large $|Du|$, we have that $|Du|^p + a(x)|Du|^q \approx |Du|^q$, so, we are in the q-phase. Functional $\mathcal{F}_1$ has been considered by Zhikov (1995), Esposito-Leonetti-Mingione (2004), Fonseca-Maly-Mingione (2004): [118], [64], [65]. Since then, a lot of people have been studying such a functional. Let us consider

$$f(x,z) = |z|^p + a(x)|z|^q$$

and

$$A_i^\alpha(x,z) = \frac{\partial f}{\partial {z_i}^\alpha}(x,z) = p|z|^{p-2} z_i^\alpha + a(x)q|z|^{q-2} z_i^\alpha.$$

Let us check componentwise coercivity:

$$\sum_{i=1}^n A_i^\alpha(x,z) z_i^\alpha = p|z|^{p-2}|z^\alpha|^2 + a(x)q|z|^{q-2}|z^\alpha|^2 \geq p|z^\alpha|^{p-2}|z^\alpha|^2 = p|z^\alpha|^p,$$

where we have lowered $|z|^{p-2}$ with $|z^\alpha|^{p-2}$ since $p \geq 2$. The Euler system of functional $\mathcal{F}_1$ appears to be

$$\begin{cases} -\sum_{i=1}^n D_i \left[p|Du|^{p-2} D_i u^1 + a(x)q|Du|^{q-2} D_i u^1 \right] &= 0, \\ -\sum_{i=1}^n D_i \left[p|Du|^{p-2} D_i u^2 + a(x)q|Du|^{q-2} D_i u^2 \right] &= 0, \\ \qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots & \\ -\sum_{i=1}^n D_i \left[p|Du|^{p-2} D_i u^m + a(x)q|Du|^{q-2} D_i u^m \right] &= 0. \end{cases}$$

It should be noted that $f(x,z) = g(x,|z|)$ with $g(x,t) = t^p + a(x)t^q$. Let us remark that $\frac{\frac{\partial g}{\partial t}(x,t)}{t} \geq pt^{p-2}$.

The previous calculations, for checking componentwise coercivity, apply to general Uhlenbeck structure:

$$f(x,z) = g(x,|z|),$$

where $\frac{\frac{\partial g}{\partial t}(x,t)}{t} \geq \nu t^{p-2}$ with $p \geq 2$ and $\nu > 0$. Indeed,

$$A_i^\alpha(x,z) = \frac{\partial f}{\partial {z_i}^\alpha}(x,z) = \frac{\partial g}{\partial t}(x,|z|)\frac{z_i^\alpha}{|z|},$$

so that

$$\sum_{i=1}^{n} A_i^\alpha(x,z) z_i^\alpha = \frac{\partial g}{\partial t}(x,|z|)\frac{|z^\alpha|^2}{|z|} \geq \nu |z|^{p-2}|z^\alpha|^2 \geq \nu |z^\alpha|^p,$$

so that componentwise coercivity holds true also for this more general structure.

REMARK. What happens to $f(x,z) = |z|^p + a(x)|z|^q$ when $1 < p < 2$? Componentwise coercivity fails in the vectorial case $m \geq 2$ when $1 < p < 2$. Let us consider this more general density

$$f(x,z) = (\mu + |z|^2)^{p/2} + a(x)|z|^q,$$

where $\mu \geq 0$, $a(x) = 0$ at some point x, $1 < p < 2$ and $m \geq 2$. The most important cases are $\mu = 0$, that gives $f(x,z) = |z|^p + a(x)|z|^q$, and $\mu = 1$, that gives $f(x,z) = (1 + |z|^2)^{p/2} + a(x)|z|^q$. Then

$$A_i^\alpha(x,z) = \frac{\partial f}{\partial {z_i}^\alpha}(x,z) = p(\mu + |z|^2)^{(p-2)/2} z_i^\alpha + a(x) q |z|^{q-2} z_i^\alpha.$$

We take x verifying $a(x) = 0$; then

$$A_i^\alpha(x,z) = p(\mu + |z|^2)^{(p-2)/2} z_i^\alpha$$

and

$$\sum_{i=1}^{n} A_i^\alpha(x,z) z_i^\alpha = p(\mu + |z|^2)^{(p-2)/2} |z_i^\alpha|^2.$$

By contradiction, let us assume that

$$\nu |z_i^\alpha|^{\tilde{p}} - c \leq \sum_{i=1}^{n} A_i^\alpha(x,z) z_i^\alpha \tag{13}$$

for some positive constants $\nu, \tilde{p}, c$. Since $m \geq 2$, then, for $\alpha \in \{1, ..., m\}$, there exists $\beta \in \{1, ..., m\} \setminus \{\alpha\}$. Let us take the matrix z as follows

$${z_1}^\alpha = t > 0, \qquad z_1^\beta = t^s, \qquad z_i^\gamma = 0 \text{ otherwise},$$

where $s > 1$ has to be choosen later. Then,

$$|z^\alpha| = t, \qquad |z|^2 = t^2 + t^{2s} \qquad (\mu + |z|^2)^{(p-2)/2} = (\mu + t^2 + t^{2s})^{(p-2)/2}.$$

Now, (13) becomes

$$\nu t^{\tilde{p}} - c \le p(\mu + t^2 + t^{2s})^{(p-2)/2} t^2 = \frac{pt^2}{(\mu + t^2 + t^{2s})^{(2-p)/2}}.$$

We divide both sides by $t^{\tilde{p}}$ and we get

$$\nu - \frac{c}{t^{\tilde{p}}} \le \frac{pt^{2-\tilde{p}}}{(\mu + t^2 + t^{2s})^{(2-p)/2}} = \frac{pt^{2-\tilde{p}}}{t^{s(2-p)}(\frac{\mu}{t^{2s}} + \frac{1}{t^{2s-2}} + 1)^{(2-p)/2}}. \tag{14}$$

We would like that $s(2-p) > 2 - \tilde{p}$: since $p < 2$ we have $2 - p > 0$ and this means that $s > \frac{2-\tilde{p}}{2-p}$. We choose $s > \max\{1; \frac{2-\tilde{p}}{2-p}\}$ and we let $t \to +\infty$: (14) becomes

$$\nu \le 0.$$

This gives us the desired contradiction since ν was assumed to be positive: $\nu > 0$. The previous calculations were performed at points x with $a(x) = 0$. This means that also the standard vectorial p-Laplacian does not enjoy componentwise coercivity when $1 < p < 2$. More precisely, neither the system

$$\begin{cases} -\sum_{i=1}^n D_i \left[p|Du|^{p-2} D_i u^1\right] &= 0, \\ -\sum_{i=1}^n D_i \left[p|Du|^{p-2} D_i u^2\right] &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ -\sum_{i=1}^n D_i \left[p|Du|^{p-2} D_i u^m\right] &= 0, \end{cases}$$

nor the one

$$\begin{cases} -\sum_{i=1}^n D_i \left[p(1 + |Du|^2)^{(p-2)/2} D_i u^1\right] &= 0, \\ -\sum_{i=1}^n D_i \left[p(1 + |Du|^2)^{(p-2)/2} D_i u^2\right] &= 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ -\sum_{i=1}^n D_i \left[p(1 + |Du|^2)^{(p-2)/2} D_i u^m\right] &= 0, \end{cases}$$

enjoy the componentwise coercivity when $1 < p < 2$. Are there examples of subquadratic systems enjoying componentwise coercivity? Here is the following one in which we split the components:

$$
\begin{cases}
-\sum_{i=1}^{n} D_i \left[p|Du^1|^{p-2} D_i u^1\right] &= 0, \\
-\sum_{i=1}^{n} D_i \left[p|Du^2|^{p-2} D_i u^2\right] &= 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots & \\
-\sum_{i=1}^{n} D_i \left[p|Du^m|^{p-2} D_i u^m\right] &= 0.
\end{cases}
\tag{15}
$$

Indeed, in the previous system we have $A_i^\alpha(z) = p|z^\alpha|^{p-2} z_i^\alpha$ and componentwise coercivity can be easily checked as follows

$$
\sum_{i=1}^{n} A_i^\alpha(z) z_i^\alpha = p|z^\alpha|^{p-2}|z^\alpha|^2 = p|z^\alpha|^p.
$$

Note that the previous system is decoupled. The next one is no longer decoupled:

$$
\begin{cases}
-\sum_{i=1}^{n} D_i \left[p|Du^1|^{p-2} D_i u^1\right] - D_1 \left[\frac{2D_1u^1 D_1u^2 D_1u^2}{\left(1+\left(D_1u^1 D_1u^2\right)^2\right)^2}\right] &= 0, \\
-\sum_{i=1}^{n} D_i \left[p|Du^2|^{p-2} D_i u^2\right] - D_1 \left[\frac{2D_1u^1 D_1u^2 D_1u^1}{\left(1+\left(D_1u^1 D_1u^2\right)^2\right)^2}\right] &= 0.
\end{cases}
\tag{16}
$$

Here, $1 < p < 2$, $m = 2$ and $A_i^\alpha(z) = p|z^\alpha|^{p-2} z_i^\alpha + \frac{2z_1^1 z_1^2 z_1^{\hat{\alpha}}}{\left(1+\left(z_1^1 z_1^2\right)^2\right)^2}$, where

$$
\hat{\alpha} = \begin{cases} 2 & \text{if } \alpha = 1, \\ 1 & \text{if } \alpha = 2. \end{cases}
$$

Componentwise coercivity can be easily checked as follows

$$
\sum_{i=1}^{n} A_i^\alpha(z) z_i^\alpha = p|z^\alpha|^{p-2}|z^\alpha|^2 + \frac{2z_1^1 z_1^2 z_1^{\hat{\alpha}} z_1^\alpha}{\left(1+(z_1^1 z_1^2)^2\right)^2} = p|z^\alpha|^p + \frac{2(z_1^1 z_1^2)^2}{\left(1+(z_1^1 z_1^2)^2\right)^2} \geq p|z^\alpha|^p.
$$

On the other hand, system (16) does not enjoy Uhlenbeck structure. Indeed, we argue by contradiction: let assume Uhlenbeck structure, so that

$$A_i^\alpha(z) = g'(|z|)\frac{z_i^\alpha}{|z|}. \tag{17}$$

We first take the matrix $\tilde{z}$ as follows

$$\tilde{z}_1^1 = 1, \qquad \tilde{z}_j^\beta = 0 \text{ otherwise},$$

so that $|\tilde{z}| = |\tilde{z}^1| = 1$. Then, we take the matrix $\tilde{\tilde{z}}$ as follows

$$\tilde{\tilde{z}}_1^1 = \frac{1}{\sqrt{2}}, \qquad \tilde{\tilde{z}}_2^2 = \frac{1}{\sqrt{2}}, \qquad \tilde{\tilde{z}}_j^\beta = 0 \text{ otherwise},$$

so that $|\tilde{\tilde{z}}| = 1$ and $|\tilde{\tilde{z}}^1| = \frac{1}{\sqrt{2}}$. We first exploit (17) with $z = \tilde{z}$:

$$A_1^1(\tilde{z}) = g'(1).$$

Then, we exploit (17) with $z = \tilde{\tilde{z}}$:

$$A_1^1(\tilde{\tilde{z}}) = g'(1)\frac{1}{\sqrt{2}}.$$

These two conditions merge into

$$A_1^1(\tilde{\tilde{z}}) = g'(1)\frac{1}{\sqrt{2}} = A_1^1(\tilde{z})\frac{1}{\sqrt{2}}. \tag{18}$$

Now, we compute $A_1^1(\tilde{\tilde{z}})$ and $A_1^1(\tilde{z})$. We have $A_1^1(\tilde{\tilde{z}}) = p\left(\frac{1}{\sqrt{2}}\right)^{p-1}$ and $A_1^1(\tilde{z}) = p$. Then, (18) gives

$$p\left(\frac{1}{\sqrt{2}}\right)^{p-1} = p\frac{1}{\sqrt{2}},$$

that is $2^{\frac{1-p}{2}} = 2^{\frac{-1}{2}}$; this means $\frac{1-p}{2} = \frac{-1}{2}$, that is, $2 = p$: this is false, since we assumed $1 < p < 2$. This shows that we cannot have Uhlenbeck structure in system (16).

We can also modify (15) by adding the q-phase $a(x)q|Du|^{q-2}D_iu^\alpha$ as follows

$$(19)\qquad \begin{cases} -\sum_{i=1}^n D_i\left[p|Du^1|^{p-2}D_iu^1 + a(x)q|Du|^{q-2}D_iu^1\right] &= 0,\\ \\ -\sum_{i=1}^n D_i\left[p|Du^2|^{p-2}D_iu^2 + a(x)q|Du|^{q-2}D_iu^2\right] &= 0,\\ \\ \qquad\qquad\qquad\qquad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\\ \\ -\sum_{i=1}^n D_i\left[p|Du^m|^{p-2}D_iu^m + a(x)q|Du|^{q-2}D_iu^m\right] &= 0. \end{cases}$$

This system is coupled when $q \neq 2$, since, in $|Du|^{q-2}$, we have all the components of u when $q-2\neq 0$. In system (19) we have $1<p<2$, $p<q$, $2\leq m$ and $0\leq a(x)\leq M$. Moreover, $A_i^\alpha(x,z) = p|z^\alpha|^{p-2}z_i^\alpha + a(x)q|z|^{q-2}z_i^\alpha$ and componentwise coercivity can be easily checked as follows

$$\sum_{i=1}^n A_i^\alpha(x,z)z_i^\alpha = p|z^\alpha|^{p-2}|z^\alpha|^2 + a(x)q|z|^{q-2}|z^\alpha|^2 \geq p|z^\alpha|^p.$$

Note that (19) does not enjoy Uhlenbeck structure. Indeed, calculations made for system (16) can be done also for (19), since they used the two matrices $\tilde{z}$ and $\tilde{\tilde{z}}$; such matrices forced $\frac{2z_1^1z_1^2z_1^{\hat{\alpha}}}{\left(1+\left(z_1^1z_1^2\right)^2\right)^2}$ to be zero. In (19), we take x such that $a(x)=0$: this forces $a(x)q|z|^{q-2}z_i^\alpha$ to be zero as well. Then, the same matrices $\tilde{z}$ and $\tilde{\tilde{z}}$ give us the desired contradiction.

Example 2. Let us consider the variational integral

$$(20)\qquad \mathcal{F}_2(u) = \int_\Omega \left[|Du(x)|^p + a(x)\sum_{\beta=1}^m\sum_{j=1}^n \left(D_ju^\beta(x)\right)^4\right]dx,$$

where

$$2\leq p<4=q, \qquad 0\leq a(x)\leq M.$$

Functional $\mathcal{F}_2$ is inspired by an example contained in [80], Hasto-Ok (2022). Let us consider the density of $\mathcal{F}_2$:

$$f(x,z) = |z|^p + a(x) \sum_{\beta=1}^{m} \sum_{j=1}^{n} \left(z_j^\beta\right)^4,$$

so that

$$A_i^\alpha(x,z) = \frac{\partial f}{\partial z_i{}^\alpha}(x,z) = p|z|^{p-2} z_i^\alpha + a(x) 4 \left(z_i^\alpha\right)^3.$$

Componentwise coercivity holds true: see [91]. On the contrary, Uhlenbeck structure does not hold true, see [91]. The Euler system of functional $\mathcal{F}_2$ appears to be as follows

$$\begin{cases} -\sum_{i=1}^n D_i \left[p|Du|^{p-2} D_i u^1 + a(x) 4 \left(D_i u^1\right)^3 \right] & = 0, \\ -\sum_{i=1}^n D_i \left[p|Du|^{p-2} D_i u^2 + a(x) 4 \left(D_i u^2\right)^3 \right] & = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots & \\ -\sum_{i=1}^n D_i \left[p|Du|^{p-2} D_i u^m + a(x) 4 \left(D_i u^m\right)^3 \right] & = 0. \end{cases}$$

Example 3. Let us consider the variational integral

$$\mathcal{F}_3(u) = \int_\Omega \left\{ |Du(x)|^p + a(x) \left(\max\{D_n u^1(x); 0\} \right)^q \right\} dx, \tag{21}$$

where

$$2 \le p < q, \qquad 0 \le a(x) \le M.$$

Functional $\mathcal{F}_3$ is inspired by an example in [117], Tang (1993). See also Esposito-Leonetti-Petricca (2019): [61].

Let us consider the density of $\mathcal{F}_3$:

$$f(x,z) = |z|^p + a(x) \left(\max\{z_n{}^1; 0\} \right)^q,$$

so that

$$A_i^\alpha(x,z) = \frac{\partial f}{\partial z_i{}^\alpha}(x,z) = p|z|^{p-2} z_i^\alpha + a(x)\ \delta^{\alpha 1}\delta_{in}\ q\left(\max\{z_n{}^1; 0\}\right)^{q-1},$$

where δ_{in} is Kronecker symbol: $\delta_{in} = 1$ if $i = n$ and $\delta_{in} = 0$ otherwise; in a similar manner $\delta^{\alpha 1}$ is defined. Componentwise coercivity holds true: see [91]. Does this example enjoy Uhlenbeck structure? No, see [91]. Euler system of $\mathcal{F}_3$ appears to be

$$\begin{cases} -\sum_{i=1}^n D_i\left[p|Du|^{p-2}D_i u^1\right] - D_n\left[a(x) q\left(\max\{D_n u^1; 0\}\right)^{q-1}\right] & = 0, \\ -\sum_{i=1}^n D_i\left[p|Du|^{p-2}D_i u^2\right] & = 0, \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ -\sum_{i=1}^n D_i\left[p|Du|^{p-2}D_i u^m\right] & = 0. \end{cases}$$

Example 4. Let us consider the variational integral

$$\mathcal{F}_4(u) = \int_\Omega \left\{|Du(x)|^p + a(x)\left(D_1 u^1(x) - D_2 u^1(x)\right)^4\right\} dx, \tag{22}$$

where

$$2 \le p < 4 = q, \qquad 0 \le a(x) \le M.$$

This functional has been inspired by an example in [31], Cianchi (2000); see also Esposito-Leonetti-Petricca (2019): [61].

Let us consider the density of $\mathcal{F}_4$:

$$f(x,z) = |z|^p + a(x)\left(z_1{}^1 - z_2{}^1\right)^4;$$

then

$$A_i^\alpha(x,z) = \frac{\partial f}{\partial z_i{}^\alpha}(x,z) = p|z|^{p-2} z_i^\alpha + a(x)\ \delta^{\alpha 1}\ 4\left(z_1{}^1 - z_2{}^1\right)^{4-1}[\delta_{i1} - \delta_{i2}].$$

Componentwise coercivity holds true: see [91]. Does this example enjoy Uhlenbeck structure? No, see [91]. Euler system of $\mathcal{F}_4$ appears to be

$$\begin{cases} -\sum_{i=1}^n D_i \left[p|Du|^{p-2}D_i u^1\right] - [D_1 - D_2]\left[a(x)4\left(D_1u^1 - D_2u^1\right)^3\right] & = 0, \\ \quad -\sum_{i=1}^n D_i \left[p|Du|^{p-2}D_i u^2\right] & = 0, \\ \qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ \quad -\sum_{i=1}^n D_i \left[p|Du|^{p-2}D_i u^m\right] & = 0. \end{cases}$$

Let us come back to the regularity result previously discussed and contained in Cupini-Leonetti-Mascolo (2022) [37]. We recall that $A(x, z)$ has q growth from above, see (7); moreover, the weak solution u is assumed to belong to $W^{1,q}$, compare with (10). Examples 1,...,4 discussed before were the Euler system of a variational integral

$$\mathcal{F}(u, \Omega) = \int_\Omega f(x, Du(x))dx,$$

where

$$\nu|z|^p \leq f(x, z) \leq c(|z|^q + 1)$$

and

$$z \to f(x, z) \quad \text{convex.}$$

Direct methods of the calculus of variations ensure that, after fixing a suitable boundary value, there exists a minimizer

$$u \in W^{1,p}(\Omega).$$

Note that p is the exponent of the growth condition from below. We ask the question: does the minimality property forces u to enjoy

$$u \in W^{1,q}_{loc}(\Omega)?$$

The answer is YES, provided some further restrictions are assumed. Namely,

$$\nu(\mu + |z|^2 + |\tilde{z}|^2)^{\frac{p-2}{2}} |z - \tilde{z}|^2 \le \langle \frac{\partial f}{\partial z}(x, z) - \frac{\partial f}{\partial z}(x, \tilde{z}); z - \tilde{z} \rangle$$

and

$$\left| \frac{\partial f}{\partial z}(x, z) - \frac{\partial f}{\partial z}(y, z) \right| \le c|x - y|^{\sigma}(1 + |z|^{q-1}).$$

Moreover, we assume that

$$1 < p < q < p\frac{n + \sigma}{n},$$

and

$$\mathcal{F}(u_k, B_R) \to \mathcal{F}(u, B_R)$$

for some $u_k \in W^{1,p}(B_R; \mathbb{R}^m) \cap W^{1,q}_{loc}(B_R; \mathbb{R}^m)$ with $u_k \to u$ weakly in $W^{1,p}(B_R; \mathbb{R}^m)$.

Under these assumptions, the minimizer u enjoys the following higher integrability of the gradient

$$u \in W^{1,q}_{loc}(\Omega; \mathbb{R}^m).$$

This is contained in Esposito-Leonetti-Mingione (2004): [64].

Failure of the energy approximation

$$\mathcal{F}(u_k, B_R) \to \mathcal{F}(u, B_R)$$

for some $u_k \in W^{1,p}(B_R; \mathbb{R}^m) \cap W^{1,q}_{loc}(B_R; \mathbb{R}^m)$ with $u_k \to u$ weakly in $W^{1,p}(B_R; \mathbb{R}^m)$, gives rise to the Lavrentiev phenomenon:

$$\inf_{v \in u + W^{1,p}_0(B_R; \mathbb{R}^m)} \mathcal{F}(v, B_R) < \inf_{v \in [u + W^{1,p}_0(B_R; \mathbb{R}^m)] \cap W^{1,q}_{loc}(B_R; \mathbb{R}^m)} \mathcal{F}(v, B_R),$$

see the survey [10] and the recent contributions [118], [44], [66], [64], [45], [79], [19], [6], [18], [4].

Do the double phase functionals $\mathcal{F}_1(u,\Omega),\cdots,\mathcal{F}_4(u,\Omega)$ enjoy the energy approximation? YES!

$$\mathcal{F}_1(u,\Omega) = \int_\Omega (|Du(x)|^p + a(x)|Du(x)|^q)\,dx,$$

$$\mathcal{F}_2(u,\Omega) = \int_\Omega \left\{|Du(x)|^p + a(x)\sum_{\beta=1}^{m}\sum_{j=1}^{n}\left(D_j u^\beta(x)\right)^4\right\} dx,$$

$$\mathcal{F}_3(u,\Omega) = \int_\Omega \left\{|Du(x)|^p + a(x)\left(\max\{D_n u^1(x);0\}\right)^q\right\} dx,$$

$$\mathcal{F}_4(u) = \int_\Omega \left\{|Du(x)|^p + a(x)\left(D_1 u^1(x) - D_2 u^1(x)\right)^4\right\} dx,$$

$$a \in C^{0,\sigma}(\Omega), \quad q \leq p\frac{n+\sigma}{n},$$

$$\Downarrow$$

$$\mathcal{F}_i(u_k, B_R) \to \mathcal{F}_i(u, B_R)$$

for some $u_k \in W^{1,p}(B_R;\mathbb{R}^m) \cap W^{1,q}_{loc}(B_R;\mathbb{R}^m)$ with $u_k \to u$ weakly in $W^{1,p}(B_R;\mathbb{R}^m)$. For energy approximation, see [118], [64], [61], [47], [51], [86], [24], [53], [18], [50], [54].

Remark. Existence of $W^{1,q}$ solutions to general elliptic systems with p,q-growth is obtaned in [35]. Existence of solutions to one single equation with p,q-growth and with explicit dependence on u has been studied in the recent [41].

An interesting example

Example 5. Let us consider the variational integral

$$\mathcal{F}_5(u) = \int_\Omega \{|Du(x) - \hat{z}|^p + a(x)|Du(x)|^q\}\,dx, \tag{23}$$

where

$$\hat{z} \in \mathbb{R}^{m \times n}, \qquad \hat{z} \neq 0, \qquad 2 \leq p < q, \qquad 0 \leq a(x) \leq M.$$

This functional has been inspired by an example in [76], Guarnotta-Mosconi (2023). Let us consider the density of $\mathcal{F}_5$:

$$f(x, z) = |z - \hat{z}|^p + a(x)|z|^q,$$

so that

$$A_i^\alpha(x, z) = \frac{\partial f}{\partial z_i^\alpha}(x, z) = p|z - \hat{z}|^{p-2}(z_i^\alpha - \hat{z}_i^\alpha) + a(x)q|z|^{q-2} z_i^\alpha.$$

Does this example satisfy Uhlenbeck structure? No, see [91]. What about componentwise coercivity? The answer depends on p:

$$2 = p \qquad \Longrightarrow \qquad \text{Yes}$$

$$2 < p \qquad \Longrightarrow \qquad \text{No}$$

Let us check componentwise coercivity when $2 = p$. We write the density for $\mathcal{F}_5$ when $2 = p$:

$$f(x, z) = |z - \hat{z}|^2 + a(x)|z|^q,$$

so that

$$A_i^\alpha(x, z) = \frac{\partial f}{\partial z_i^\alpha}(x, z) = 2(z_i^\alpha - \hat{z}_i^\alpha) + a(x)q|z|^{q-2} z_i^\alpha.$$

Then

$$\sum_{i=1}^n A_i^\alpha(x, z) z_i^\alpha = 2\langle z^\alpha - \hat{z}^\alpha; z^\alpha \rangle + a(x)q|z|^{q-2}|z^\alpha|^2 \geq$$

$$2\langle z^\alpha - \hat{z}^\alpha; z^\alpha \rangle = 2|z^\alpha|^2 - 2\langle \hat{z}^\alpha; z^\alpha \rangle \geq 2|z^\alpha|^2 - 2|\hat{z}^\alpha||z^\alpha| \geq$$

$$2|z^\alpha|^2 - |\hat{z}^\alpha|^2 - |z^\alpha|^2 = |z^\alpha|^2 - |\hat{z}^\alpha|^2,$$

so that $\mathcal{F}_5$ enjoys componentwise coercivity when $2 = p$. The Euler system of functional $\mathcal{F}_5$, with $p = 2$, appears to be

$$\begin{cases} -\sum_{i=1}^n D_i \left[2(D_i u^1 - \hat{z}_i^1) + a(x)q|Du|^{q-2} D_i u^1\right] & = 0, \\ -\sum_{i=1}^n D_i \left[2(D_i u^2 - \hat{z}_i^2) + a(x)q|Du|^{q-2} D_i u^2\right] & = 0, \\ \qquad\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots & \\ -\sum_{i=1}^n D_i \left[2(D_i u^m - \hat{z}_i^m) + a(x)q|Du|^{q-2} D_i u^m\right] & = 0. \end{cases}$$

Let us check that componentwise coercivity fails when $2 < p$, $a(x) = 0$ at some $x \in \Omega$, $\hat{z} \neq 0$ and $m \geq 2$. Now the density is

$$f(x,z) = |z - \hat{z}|^p + a(x)|z|^q,$$

so that

$$A_i^\alpha(x,z) = \frac{\partial f}{\partial z_i^\alpha}(x,z) = p|z-\hat{z}|^{p-2}(z_i^\alpha - \hat{z}_i^\alpha) + a(x)q|z|^{q-2} z_i^\alpha.$$

We work with x such that $a(x) = 0$ and with α such that $\hat{z}^\alpha \neq 0$:

$$\sum_{i=1}^n A_i^\alpha(x,z) z_i^\alpha = p|z-\hat{z}|^{p-2}\langle z^\alpha - \hat{z}^\alpha; z^\alpha\rangle.$$

Now we choose z as follows

$$z^\alpha = \frac{1}{2}\hat{z}^\alpha, \qquad z^\beta = \hat{z}^\beta + tv, \quad \text{for } \beta \neq \alpha,$$

where $|v| = 1$ and $t \in \mathbb{R}$. It turns out that

$$\langle z^\alpha - \hat{z}^\alpha; z^\alpha\rangle = -\frac{1}{4}|\hat{z}^\alpha|^2 < 0$$

and

$$|z-\hat{z}|^{p-2} = \left(\frac{1}{4}|\hat{z}^\alpha|^2 + (m-1)t^2\right)^{\frac{p-2}{2}} \to \infty \qquad \text{when } t \to \infty,$$

since $2 < p$ and $2 \leq m$. Putting together the previous informations, we get

$$\sum_{i=1}^{n} A_i^\alpha(x,z) z_i^\alpha = p|z-\hat{z}|^{p-2}\langle z^\alpha - \hat{z}^\alpha; z^\alpha\rangle =$$

$$p\left(\frac{1}{4}|\hat{z}^\alpha|^2 + (m-1)t^2\right)^{\frac{p-2}{2}} \left(-\frac{1}{4}|\hat{z}^\alpha|^2\right) \to -\infty.$$

If α and z are as before, the inequality

$$\nu |z^\alpha|^{\tilde{p}} - c_2 \leq \sum_{i=1}^{n} A_i^\alpha(x,z) z_i^\alpha$$

does not hold true, since the left hand side is the fixed number $\frac{\nu}{2^{\tilde{p}}}|\hat{z}^\alpha|^{\tilde{p}} - c_2$ and the right hand side tends to $-\infty$. This shows that componentwise coercivity fails when $2 < p$, $a(x) = 0$ at some $x \in \Omega$, $\hat{z} \neq 0$ and $m \geq 2$. The Euler system of functional $\mathcal{F}_5$, with $p > 2$, appears to be

$$\begin{cases} -\sum_{i=1}^{n} D_i \left[p|Du-\hat{z}|^{p-2}(D_i u^1 - \hat{z}_i^1) + a(x) q |Du|^{q-2} D_i u^1\right] = 0, \\ -\sum_{i=1}^{n} D_i \left[p|Du-\hat{z}|^{p-2}(D_i u^2 - \hat{z}_i^2) + a(x) q |Du|^{q-2} D_i u^2\right] = 0, \\ \qquad\qquad\qquad \dots\dots\dots\dots\dots\dots \\ -\sum_{i=1}^{n} D_i \left[p|Du-\hat{z}|^{p-2}(D_i u^m - \hat{z}_i^m) + a(x) q |Du|^{q-2} D_i u^m\right] = 0. \end{cases}$$

It would be nice to study boundedness of weak solutions to such a system, maybe using a technique not based on componentwise coercivity. Let us remark that boundedness of solutions to some elliptic systems has been studied in [38], see also [42]. Let us note that assumptions in [38] do not fit the previous system, see [91].

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Università degli Studi dell'Aquila, DISIM, Via Vetoio snc, Coppito, 67100 – L'Aquila, Italy

Email address: `francesco.leonetti@univaq.it`