EXACT CONTROLLABILITY FOR NONLOCAL SEMILINEAR DIFFERENTIAL INCLUSIONS

CONTROLLABILITÀ ESATTA PER INCLUSIONI DIFFERENZIALI SEMILINEARI CON CONDIZIONI INIZIALI NONLOCALI

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Abstract. In this paper, we investigate the controllability of a class of semilinear differential inclusions in Hilbert spaces. Assuming the exact controllability of the associated linear problem, we establish sufficient conditions for achieving the exact controllability of the nonlinear problem. In infinite-dimensional spaces, the compactness of the evolution operator and the linear controllability condition are often incompatible. To address this, we avoid the compactness assumption on the semigroup by employing two distinct approaches: one based on weak topology, and the other on the concept of Gelfand triples. Furthermore, the problem we consider is that of nonlocal controllability, where the solution satisfies a nonlocal initial condition that depends on the behaviour of the solution over the entire time interval.

Sunto. In questo lavoro, studiamo la controllabilità di una classe di inclusioni differenziali semilineari in spazi di Hilbert. Assumendo la controllabilità esatta del problema lineare associato, forniamo condizioni sufficienti per ottenere la controllabilità esatta anche nel caso non lineare. Negli spazi di dimensione infinita, la compattezza dell'operatore di evoluzione e la condizione di controllabilità lineare risultano spesso incompatibili. Per superare questo ostacolo, evitiamo di imporre l'assunzione di compattezza sul semigruppo, adottando due approcci distinti: uno basato sulla topologia debole e l'altro sul concetto di tripla di Gelfand. Inoltre, consideriamo il problema della controllabilità non locale, in cui la soluzione soddisfa una condizione iniziale non locale che dipende dal comportamento della soluzione su tutto l'intervallo di tempo.

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1. Introduction

This work deals with the study of the exact controllability for semilinear differential equations, with a general nonlocal condition, of the form

$$(\mathcal{P}) \begin{cases} y'(t) \in Ay(t) + F(t, y(t)) + Bu(t), \text{ a.e. } t \in [0, T] \\ y(0) = y_0 + g(y) \end{cases}$$

with $0 < T < +\infty$, in an infinite dimensional Hilbert space $(H, \|\cdot\|)$. We assume that $A : D(A) \subseteq H \to H$ is a linear operator generating a C_0-semigroup $\{S(t)\}_{t\geq 0}$, $F : [0, T] \times H \multimap H$ and $g : C([0, T]; H) \to H$ are given multivalued and single valued maps respectively. Moreover, the operator $B : U \to H$, where U is a Hilbert space, is linear and bounded and we assume that the control term u belongs to $L^2([0, T]; U)$. The study of nonlocal Cauchy problems in Banach spaces begins in 1991 with the work of Byszewski (see [8]). In that paper, the author examines the so-called multipoint initial condition, i.e.

$$(1) \qquad y(0) = \sum_{i=1}^{n} \alpha_i y(t_i) + y_0, \quad y_0 \in H, \alpha_i \in \mathbb{R} \setminus \{0\}, t_i \in [0, T],\ i = 1, \dots, n,$$

to study kinematics, determining the evolution $t \to y(t)$ of the position of a physical object where the values of $y(0)$, $y(t_i)$, $i = 1, \dots, n$ are unknown, but a relationship described by (1) is given. As a result, the condition (1), along with more general nonlocal initial conditions, allows for the modeling of physical problems that cannot be addressed through classical Cauchy problems. Later on, nonlocal problems for a semilinear differential equation with a C_0-semigroup generator have been extensively studied for their interest in several contexts, see for instance [3] and the reference therein. Differential inclusions are a valuable tool for describing a variety of optimal control problems (see, e.g., [1] and Section 5.2.2 in the monography [14]). Exact controllability problems are also highly important in applications. For instance, they are instrumental in addressing the controllability of size-structured population equations (see [7]). Studying controllability for differential inclusions allows for the integration of these two different types of control problems, for instance, achieving the dual goal of minimizing a given cost functional while simultaneously reaching a specified set or position. We refer to [2] for a wide overview on the controllability results of the problem (1). Among the extended literature on the subject, concerning the nonlocal controllability problem, we would like to mention the contributions [9], where in the linear part is considered a family of operators $\{A(t)\}$ generating an evolution system, the multivalued map F satisfies a lower semicontinuous Scorza-Dragony property and g is a compact map; the paper [13] where the exact controllability is studied for an impulsive differential inclusion with a compact nonlocal initial condition and the recent contribution [20], where the approximate controllability is obtained for a semilinear equation with Lipshitz forcing term and a multivalued initial condition.

Given $u \in L^2([0,T];U)$, we look for mild solutions of problem $(\mathcal{P})$, that is to say for continuous functions $y : [0,T] \to H$ that satisfy the integral equation

$$y(t) = S(t)(y_0 + g(y)) + \int_0^t S(t-s)f(s)\, ds + \int_0^t S(t-s)Bu(s)\, ds, \tag{2}$$

for each $t \in [0,T]$, where $f \in L^1([0,T];H)$ is a selection of the multivalued map F, i.e. $f(s) \in F(s, y(s))$ for a.e. $s \in [0,T]$. We say that problem $(\mathcal{P})$ is controllable if every initial condition $y_0 \in H$ can be steered at time T to any $y_1 \in H$, i.e. if $y(0) = y_0 + g(y)$ and $y(T) = y_1$, by some admissible control u (see Definition 3.2). So, we will study conditions under which there exists a mild solution $y(\cdot)$ of $(\mathcal{P})$ reaching a given state at the final time T.

In the study of the controllability of $(\mathcal{P})$, a pivotal role is played by the linear operator $G : L^2([0,T];U) \to H$ defined by

$$G(u) = \int_0^T S(T-s)Bu(s)\, ds,$$

as it provides a representation of the control function u that satisfies the controllability condition. In order to characterize the control function, we introduce the following map $y_q : [0,T] \to H$ defined as

$$y_q(t) = S(t)(y_0 + g(q)) + \int_0^t S(t-s)f(s)\, ds + \int_0^t S(t-s)Bu(s)\, ds, \tag{3}$$

where $q : [0,T] \to H$ is any continuous function, $f \in L^1([0,T];H)$, with $f(t) \in F(t, q(t))$ for a.e. $t \in [0,T]$. Let $y_1 \in H$, if the operator G admits a right inverse, denoted by

$\widetilde{G}^{-1} : H \to L^2([0,T];U)$, then the function $u \in L^2([0,T];U)$ given by

$$u = \widetilde{G}^{-1}\bigg(y_1 - S(T)(y_0 + g(q)) - \int_0^T S(T-s)f(s)\, ds\bigg),$$

is such that $y_q(T) = y_1$. It is known that every surjective, bounded, linear operator defined in a Hilbert space admits a right inverse of minimal norm, see Proposition 2.2 in [17] for a detailed proof.

In order to prove the existence of at least one solution of the problem $(\mathcal{P})$ we make use of the Gliksberg - Fan fixed point Theorem (see Theorem 2.3). This Theorem guarantees the existence of a fixed point of an upper semicontinuous multimap with closed, convex values, defined on a compact convex subset of an Hausdorff locally convex topological vector space. So, we introduce a suitable multioperator whose fixed points are the sought solutions. As can be seen from the statement of the Gliksberg - Fan fixed point Theorem, one of the key assumptions is the compactness of the multioperator. This aim can be achieved, for instance, assuming the compactness of the semigroup $\{S(t)\}_{t\geq 0}$. However, as it was pointed out by Triggiani in [21] and [22], in infinite dimensional Banach spaces the compactness of the semigroup $\{S(t)\}_{t\geq 0}$ generated by the operator A or the compactness of the control operator B is in contradiction with the exact controllability of problem $(\mathcal{P})$ while using control $u \in L^p([0,T];U)$, for $p > 1$. To address this issue, several methods have been developed. For instance, one approach relies on regularity assumptions with respect to a measure of non-compactness imposed on the nonlinear term, see [19] and [5]; another is based on weak topology, see [4]; and a third approach uses Gelfand triples, see [17]. We also mention the recent work [18] where the controllability for a semilinear equation with non local initial conditions is proven. Specifically, in the mentioned paper the two approaches, the one introduced for this type of problems in [4] based on weak topology, and the one introduced in [19] based on the regularity with respect to a measure of non compactness, are considered. In particular, by employing an appropriate approximation technique and leveraging the approach based on weak topology, the study successfully addressed the case of a uniformly convex Banach space U, not necessarily a Hilbert space, thus, as a consequence, the case of an operator G with a nonlinear inverse.

In the present paper, employing either weak topology techniques or Gelfand triple techniques, where the Hilbert space H is compactly embedded in a Banach space $(E, \|\cdot\|_E)$, we introduce sufficient conditions on the multimap F and the operator g that guarantee controllability for semilinear equations without requiring compactness of the semigroup or control operator. The first approach was applied to obtain the exact controllability for semilinear differential inclusions with Cauchy initial conditions in [4] and in [18] for semilinear differential equations with nonlocal initial conditions, while the second approach was introduced in [17] for the controllability of a Cauchy initial problem.

Compared to the cited results [4], [17] and [18], the novelty of this work lies in simultaneously addressing nonlocal initial conditions and a multivalued nonlinearity. Moreover, we avoid the use of a finite dimensional approximation exploited in [17], allowing to consider the problem $(\mathcal{P})$ in a non separable Hilbert space. We emphasize that no compactness assumption is imposed on the map g, which enables us to address a wide range of typical nonlocal initial conditions, including those in (1), as well as periodic, antiperiodic, $(y(0) = \pm y(T))$ and mean value conditions $y(0) = \dfrac{1}{T}\displaystyle\int_0^T y(t)\,dt$.

The paper is organized as follows. In Section 2 we recall some notions and preliminary results from the theory of functional analysis and we present some properties about multivalued maps and semigroups. The existence and controllability results are Theorems 3.1, 3.2, 3.3 and 3.4 in Section 3.

2. Preliminaries

Given a Hilbert space $(H, \|\cdot\|)$, by H^ω we denote the Hilbert space H endowed with the weak topology and, given a sequence $\{x_n\}_{n\in\mathbb{N}} \subset H$ we write $x_n \to x_0$ and $x_n \rightharpoonup x_0$, with $x_0 \in H$, to denote the strong and weak convergence in H respectively. Moreover, we denote with nB the closed ball of H centered at the origin and of radius n and, for a set $D \subset H$, the symbol $\overline{D}^\omega$ denotes the weak closure of D. We denote by $C([0,T]; H)$ the space of all continuous functions $y : [0,T] \to H$ with norm

$$\|y\|_0 = \max_{t\in[0,T]} \|y(t)\|$$

and, for $1 \leq p \leq \infty$, by $L^p([0,T];H)$ we denote the space of equivalence classes of functions $y:[0,T]\to H$ such that y is measurable in $[0,T]$ and $\|y\|_p < +\infty$, where $\|\cdot\|_p$ is the usual norm defined by

$$\|y\|_p = \left(\int_0^T \|y(t)\|^p \, dt\right)^{\frac{1}{p}}$$

when $1 \leq p < +\infty$ and

$$\|y\|_\infty = \operatorname*{ess\,sup}_{t\in[0,T]} \|y(t)\|$$

when $p=\infty$. We denote by $AC([0,T];H)$ the space of all absolutely continuous functions $y:[0,T]\to H$. We recall that a function $y:[0,T]\to H$ with values in a Hilbert space is absolutely continuous if and only if there exists $g\in L^1([0,T];H)$ such that

$$y(t) = y(0) + \int_0^t g(s)\, ds,\ t\in[0,T].$$

Moreover y is a.e. differentiable on $[0,T]$ and $y'(t)=g(t)$ for almost every $t\in[0,T]$.
We recall the following characterization of weak convergence in the space of continuous functions.

Theorem 2.1. *(*[6, Theorem 4.3]*) A sequence of continuous functions $\{x_n\}_{n\in\mathbb{N}}\subset C([0,T];H)$ weakly converges to an element $x\in C([0,T];H)$ if and only if*

(i) *there exists $N>0$ such that $\|x_n(t)\|\leq N$, for every $n\in\mathbb{N}$ and for all $t\in[0,T]$;*
(ii) *$x_n(t)\rightharpoonup x(t)$ as $n\to\infty$, for every $t\in[0,T]$.*

It follows that $\{x_n\}_{n\in\mathbb{N}}\rightharpoonup x$ in $C([0,T];H)$ implies that $\{x_n\}_{n\in\mathbb{N}}\rightharpoonup x$ in $L^1([0,T];H)$.
The following proposition shows that every surjective, bounded, linear operator defined in a Hilbert space admits a right inverse of minimal norm.

Proposition 2.1. *(*[17, Proposition 2.2]*) Let H_1 and H_2 be two Hilbert spaces, let $G: H_1\to H_2$ be a surjective, bounded, linear operator. Then there exists a bounded linear operator $\widetilde{G}^{-1}: H_2\to H_1$ such that, for every $w\in H_2$, $G\circ\widetilde{G}^{-1}(w)=w$ and*

$$\|\widetilde{G}^{-1}(w)\| = \min\{\|u\| : G(u)=w\},$$

for all $w\in H_2$.

We recall some properties of the multimaps, we refer to [14] for details.

Let X and Y be two locally convex topological spaces. A multivalued map F is a correspondence which associates to every $x \in X$ a nonempty subset $F(x) \subseteq Y$. We write this correspondence as $F : X \multimap Y$.

Definition 2.1. A multivalued map $F : X \multimap Y$ is *upper semicontinuous (u.s.c. for short) at the point* $x \in X$ if, for every open set $W \subseteq Y$ such that $F(x) \subset W$, there exists a neighbourhood $V(x)$ of x with the property that $F(V(x)) \subset W$. It is *upper semicontinuous (u.s.c. for short)* if it is upper semicontinuous at every point $x \in X$.

Definition 2.2. A multimap $F : X \multimap Y$ is said to be

(a) *closed* if its graph Γ_F is a closed subset of the space $X \times Y$;
(b) *compact* if its range $F(x)$ is relatively compact in Y, i.e. $\overline{F(X)}$ is compact in Y.

Proposition 2.2. *A closed, compact multivalued map $F : X \multimap Y$ with compact values is upper semicontinuous.*

Now, we recall some results concerning the semigroup theory, for further details we refer the reader to [23].

Let $(X, \|\cdot\|_X)$ be a Banach space and let $\mathcal{L}(X)$ be the set of all linear bounded operators from X to X.

Definition 2.3. A one parameter family $\{S(t)\}_{t\geq 0}$ in $\mathcal{L}(X)$ is a *semigroup* of linear operators on X if

(i) $S(0) = I$;
(ii) $S(t+s) = S(t)S(s)$ for every $t, s \geq 0$.

If, in addition, it satisfies the following continuity condition at $t = 0$

$$\lim_{t\to 0^+} \|S(t) - I\|_{\mathcal{L}(X)} = 0,$$

the semigroup is called *uniformly continuous* and it is called C_0-*semigroup*, if the mapping $t \mapsto S(t)x$ is strongly continuous, for each $x \in X$ i.e.

$$\lim_{t\to 0} S(t)x = x \quad \forall\, x \in X.$$

Definition 2.4. Let $\{S(t)\}_{t\geq 0}$ be a C_0-semigroup defined on X. The linear operator A defined by

$$Ax = \lim_{t\to 0} \frac{S(t)x - x}{t} \quad \text{for} \quad x \in D(A),$$
$$D(A) = \left\{x \in X : \lim_{t\to 0} \frac{S(t)x - x}{t} \text{ exists in } X\right\},$$

is the *infinitesimal generator* of $\{S(t)\}_{t\geq 0}$.

Theorem 2.2. *A linear operator $A : D(A) \subseteq X \to X$ is the generator of a uniformly continuous semigroup if and only if $D(A) = X$ and $A \in \mathcal{L}(X)$.*

Given a bounded and linear operator $A : X \to X$, the uniformly continuous semigroup generated by A is defined by

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$$

for each $t \geq 0$. Let $S(t) = e^{At}$ for each $t \geq 0$, it is well known that

$$\frac{d}{dt}(S(t)) = AS(t) = S(t)A.$$

Remark 2.1. An uniformly continuous semigroup is a C_0-semigroup.

It is known that there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|S(t)\|_{\mathcal{L}(X)} \leq Me^{\omega t}, \quad \forall\, t \geq 0. \tag{4}$$

Finally, for sake of completeness, we recall some results that we will need in the main section.

Firstly we state the Glicksberg - Fan fixed point Theorem ([12], [16]).

Theorem 2.3. *Let X be a Hausdorff locally convex topological vector space, K a compact convex subset of X and $G : K \multimap K$ an upper semicontinuous multimap with closed, convex values. Then G has a fixed point $x_* \in K : x_* \in G(x_*)$.*

We mention also two results that are contained in the so called Eberlein-Smulian theory.

Theorem 2.4. [15, Theorem 1, p. 219] *Let Ω be a subset of a Banach space X. The following statements are equivalent:*

1. *Ω is relatively weakly compact;*

2. Ω *is relatively weakly sequentially compact.*

Corollary 2.1. [15, p. 219] *Let* Ω *be a subset of a Banach space* X*. The following statements are equivalent:*

1. Ω *is weakly compact;*
2. Ω *is weakly sequentially compact.*

We recall the Krein-Smulian Theorem.

Theorem 2.5. [10, p. 434] *The convex hull of a weakly compact set in a Banach space* E *is weakly compact.*

3. Problem statement

Let $(H, \|\cdot\|)$ be a Hilbert space with scalar product $(\cdot\ ,\cdot)$. We study the controllability problem for a system governed by inclusion $(\mathcal{P})$ under the following assumptions:

(A): $A : D(A) \subseteq H \to H$ is a linear operator generating a C_0-semigroup $\{S(t)\}_{t\geq 0}$;

(F): $F : [0,T] \times H \multimap H$ is a multivalued map such that:

- (F_0) the multimap $F : [0,T] \times H \multimap H$ has nonempty, bounded, closed, convex values;
- (F_1) the multimap $F(\cdot, c) : [0,T] \multimap H$ has a strongly measurable selection for every $c \in H$, i.e., there exists a strongly measurable function $f : [0,T] \to H$ such that $f(t) \in F(t,c)$ for a.e. $t \in [0,T]$;
- (F_2) the multimap $F(t,\cdot) : H \multimap H^{\omega}$ is sequentially closed for a.e. $t \in [0,T]$;
- (F_3) for every bounded subset $\Omega \subset H$ there exists a function $v_\Omega \in L^1([0,T]; \mathbb{R}_+)$ such that
$$\|F(t,w)\| = \sup_{x\in F(t,w)} \|x\| \leq v_\Omega(t)$$
for a.e. $t \in [0,T]$ and for each $w \in \Omega$;

(B): $B : U \to H$ is a bounded linear operator defined on a Hilbert space U and the control function $u(\cdot)$ belongs to the space $L^2([0,T]; U)$;

(g): $g : C([0,T]; H) \to H$ is a given map.

We look for mild solutions of problem $(\mathcal{P})$, that is to say for functions that satisfy the following definition.

Definition 3.1. Given $u \in L^2([0,T];U)$, a continuous map $y:[0,T]\to H$ is said to be a mild solution of the problem $(\mathcal{P})$ if there exists $f \in L^1([0,T];H)$, $f(s)\in F(s,y(s))$ for a.e. $s\in[0,T]$, such that the integral equation

$$y(t) = S(t)(y_0 + g(y)) + \int_0^t S(t-s)f(s)\ ds + \int_0^t S(t-s)Bu(s)\ ds \tag{5}$$

is verified for each $t \in [0,T]$.

Definition 3.2. We say that problem $(\mathcal{P})$ is nonlocally controllable on $[0,T]$ if for all y_0, y_1 in H there exists a control $u \in L^2([0,T];U)$ such that the corresponding mild solution $y(\cdot)$ of $(\mathcal{P})$ satisfies $y(0) = y_0 + g(y)$ and $y(T) = y_1$.

In order to prove the controllability of the problem $(\mathcal{P})$ we need the following assumption:

(G) the linear operator $G : L^2([0,T];U)\to H$ defined by

$$G(u) = \int_0^T S(T-s)Bu(s)\ ds$$

is onto.

By Propostion 2.1 we deduce that the linear operator G admits a right inverse of minimal norm.

Proposition 3.1. *If $G : L^2([0,T];U)\to H$ is the linear operator defined in (G), then there exists a bounded linear operator $\widetilde{G}^{-1} : H \to L^2([0,T];U)$ such that, for every $w\in H$, $G\circ \widetilde{G}^{-1}(w) = w$ and*

$$\|\widetilde{G}^{-1}(w)\|_{L^2([0,T];U)} = \min\{\|u\|_{L^2([0,T];U)} : G(u) = w\},$$

for all $w \in H$.

Given $q \in C([0,T];H)$, let us denote

$$\Sigma_q = \{f \in L^1([0,T];H) : f(t) \in F(t,q(t)) \text{ for a.e. } t\in[0,T]\}. \tag{6}$$

Under the considered assumptions, this set is nonempty, as the following assertion shows.

Proposition 3.2. *(*[4, Proposition 4.1]*) Let $F : [0,T]\times H \multimap H$ be a multimap satisfying properties (F_0), (F_1), (F_2) and (F_3). Then the set Σ_q is nonempty for any $q \in C([0,T];H)$.*

We denote with $S_1 : L^1([0,T];H) \to C([0,T];H)$ the linear and continuous operator

$$S_1 f(t) = \int_0^t S(t-s)f(s)\,ds, \quad t \in [0,T] \tag{7}$$

and with $S_2 : L^1([0,T];H) \to C([0,T];H)$ the linear and continuous operator

$$S_2 f(t) = \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(-\int_0^T S(T-\tau)f(\tau)\,d\tau\Big)(s)\,ds \quad t \in [0,T]. \tag{8}$$

Fix $n \in \mathbb{N}$, consider

$$Q_n = \{q \in C([0,T];H) : \|q(t)\| \le n \text{ for all } t \in [0,T]\} \tag{9}$$

the closed ball of radius n in $C([0,T];H)$ centered at the origin and define the solution multioperator $\mathcal{T}_n : Q_n \multimap C([0,T];H)$ as

$$\begin{aligned}\mathcal{T}_n(q) = \Big\{ y \in C([0,T];H) :& \\ & y(t) = S(t)(y_0 + g(q)) + S_1 f(t) \\ & \quad + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0+g(q))\Big)(s)\,ds \\ & \quad + S_2 f(t),\ t \in [0,T], f \in \Sigma_q \Big\}.\end{aligned} \tag{10}$$

It is easy to verify that the fixed points of the multioperator $\mathcal{T}_n$ are mild solutions of problem $(\mathcal{P})$ such that $y(T) = y_1$.

3.1. **Existence via weak topology.** In this section we prove the existence of at least one mild solution of problem $(\mathcal{P})$ exploiting a method based on weak topology. To this aim we have to strengthen the assumptions on the regularity of the multimap F and the map g. Namely, we assume that:

- (F_2') the multimap $F(t,\cdot) : H^\omega \multimap H^\omega$ is sequentially closed for a.e. $t \in [0,T]$;
- (g') $g : C([0,T];H) \to H$ is a weakly sequentially continuous operator mapping bounded sets into bounded sets.

Remark 3.1. Notice that, since condition (F_2') implies (F_2), the set Σ_q is nonempty for every $q \in C([0,T];H)$ (see Proposition 3.2).

Now, we describe the properties of the multioperator $\mathcal{T}_n : Q_n \multimap C([0,T];H)$ needeed to apply the Gliksberg - Fan fixed point Theorem (see Theorem 2.3). We recall that in this subsection we require assumptions (A), (F_0), (F_1), (F_2'), (F_3), (B), (G) and (g').

Proposition 3.3. *The multioperator $\mathcal{T}_n : Q_n \multimap C([0,T];H)$ has a weakly sequentially closed graph.*

Proof. Let $\{q_k\}_{k\in\mathbb{N}} \subset Q_n$ be a sequence such that $q_k \rightharpoonup q$ in $C([0,T];H)$ and let $\{y_k\}_{k\in\mathbb{N}} \subset C([0,T];H)$ satisfying $y_k \in \mathcal{T}_n(q_k)$ for all $k \in \mathbb{N}$ and such that $y_k \rightharpoonup y$ in $C([0,T];H)$. We shall prove that $y \in \mathcal{T}_n(q)$.
Since $q_k \rightharpoonup q$ in $C([0,T];H)$, by assumption (g'), we have

$$g(q_k) \rightharpoonup g(q) \tag{11}$$

and, by the linearity and continuity of the operator $S(t)$, for every $t \geq 0$, we get

$$S(t)(y_0 + g(q_k)) \rightharpoonup S(t)(y_0 + g(q)) \tag{12}$$

for every $t \in [0,T]$ and, in particular,

$$g_k = y_1 - S(T)(y_0 + g(q_k)) \rightharpoonup y_1 - S(T)(y_0 + g(q)) =: g_0. \tag{13}$$

The fact that $y_k \in \mathcal{T}_n(q_k)$ means that there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$, $f_k \in \Sigma_{q_k}$, such that

$$\begin{aligned} y_k(t) &= S(t)(y_0 + g(q_k)) + S_1 f_k(t) \\ &\quad + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0 + g(q_k))\Big)(s)\, ds + S_2 f_k(t) \end{aligned} \tag{14}$$

for every $t \in [0,T]$. Since $q_k \in Q_n$ for all $k \in \mathbb{N}$, according to (F_3), there exists a function $v_n \in L^1([0,T];\mathbb{R}_+)$ such that

$$\|f_k(t)\| \leq v_n(t)$$

for a.e. $t \in [0,T]$ and every $k \in \mathbb{N}$, i.e. $\{f_k\}_{k\in\mathbb{N}}$ is bounded and uniformly integrable and $\{f_k(t)\}_{k\in\mathbb{N}}$ is bounded in H for a.e. $t \in [0,T]$. Hence, by the reflexivity of the space H and by Dunford-Pettis Theorem (see [10, p. 294]), we have that there exists a subsequence, still denoted as the sequence, and a function f_0 such that $f_k \rightharpoonup f_0$ in $L^1([0,T];H)$. Therefore, by the linearity and continuity of the operators S_1 and S_2, we have

$$S_1 f_k \rightharpoonup S_1 f_0 \tag{15}$$

and

$$S_2 f_k \rightharpoonup S_2 f_0 \tag{16}$$

in $C([0,T];H)$.

Let us show that

$$\begin{aligned}&\int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0 + g(q_k))\Big)(s)\ ds \\ &\qquad \rightharpoonup \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0 + g(q))\Big)(s)\ ds\end{aligned} \tag{17}$$

for every $t \in [0,T]$. Consider $\phi \in H$ and $t \in [0,T]$ and define the linear functional $\widetilde{\phi} : H \to \mathbb{R}$ as

$$\widetilde{\phi}(w) = \left(\phi, \int_0^t S(t-s)B\widetilde{G}^{-1}(w)(s)\ ds\right),\ w \in H.$$

By the boundedness of the operators B, $\widetilde{G}^{-1}$ and $S(t)$, for every $t \geq 0$, and by Cauchy-Schwarz inequality in $L^2(0,T)$, we have

$$\begin{aligned}|\widetilde{\phi}(w)| &\leq \|\phi\| \left\|\int_0^t S(t-s)B\widetilde{G}^{-1}(w)(s)\ ds\right\| \leq \|\phi\| \int_0^t \left\|S(t-s)B\widetilde{G}^{-1}(w)(s)\right\|\ ds \\ &\leq \|\phi\| C \|B\|_{\mathcal{L}(U,H)} \int_0^t \left\|\widetilde{G}^{-1}(w)(s)\right\|_U\ ds \leq \|\phi\| C \|B\|_{\mathcal{L}(U,H)} \|\widetilde{G}^{-1}(w)\|_{L^1([0,T];U)} \\ &\leq \|\phi\| C \|B\|_{\mathcal{L}(U,H)} \sqrt{T} \|\widetilde{G}^{-1}(w)\|_{L^2([0,T];U)} \\ &\leq \|\phi\| C \|B\|_{\mathcal{L}(U,H)} \sqrt{T} \|\widetilde{G}^{-1}\|_{\mathcal{L}(H, L^2([0,T];U))} \|w\|\end{aligned} \tag{18}$$

for every $w \in H$, where $C := \sup_{t\in[0,T]} \|S(t)\|$. Therefore $\widetilde{\phi}$ is bounded and, by (13), we obtain that

$$\begin{aligned}\widetilde{\phi}(g_k) &= \left(\phi, \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0 + g(q_k))\Big)(s)\ ds\right) \\ &\to \left(\phi, \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0 + g(q))\Big)(s)\ ds\right) = \widetilde{\phi}(g_0)\end{aligned} \tag{19}$$

for every $t \in [0,T]$. Then (17) follows.

Thus, by (12), (15), (16) and (17) we have

(20)

$$\begin{aligned} y_k(t) &= S(t)(y_0 + g(q_k)) + S_1 f_k(t) \\ &\quad + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0 + g(q_k))\Big)(s)\ ds + S_2 f_k(t) \\ &\quad \rightharpoonup S(t)(y_0 + g(q)) + S_1 f_0(t) + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0 + g(q))\Big)(s)\ ds \\ &\quad + S_2 f_0(t) =: l(t) \quad \forall\, t \in [0,T]. \end{aligned}$$

For the uniqueness of the weak limit in H, we obtain that $l(t) = y(t)$ for all $t \in [0,T]$. To conclude, we have only to prove that $f_0(t) \in F(t, q(t))$ for a.e. $t \in [0,T]$. By Mazur's convexity Theorem (see, e.g. [11]) we have that there exists a sequence

$$\widetilde{f}_k = \sum_{i=0}^{m_k} \lambda_{ki} f_{k+i}, \quad \lambda_{ki} \geq 0,\ \sum_{i=0}^{m_k} \lambda_{ki} = 1 \tag{21}$$

satisfying $\widetilde{f}_k \to f_0$ in $L^1([0,T];H)$ and, up to subsequence, there is $N_0 \subset [0,T]$ with Lebesgue measure zero such that $\widetilde{f}_k(t) \to f_0(t)$ for all $t \in [0,T] \setminus N_0$. With no loss of generality we can also assume that $F(t,\cdot) : H^\omega \multimap H^\omega$ is weakly sequentially closed and $\sup_{\|x\| \leq n} \|F(t,x)\| \leq v_n(t)$ for every $t \notin N_0$. Fix $t_0 \notin N_0$ and assume, by contradiction, that $f_0(t_0) \notin F(t_0, q(t_0))$. By the reflexivity of the space H and (F_3), the restriction $F_{nB}(t_0,\cdot)$ of the multimap $F(t_0,\cdot)$ on the set nB is weakly compact. Moreover, by assumption (F_2'), $F_{nB}(t_0,\cdot)$ is weakly sequentially closed and thus, by Corollary 2.1, we have that $F_{nB}(t_0,\cdot)$ is a weakly closed multimap and, as a result, it is weakly u.s.c. (see Proposition 2.2). Since $\|q(t_0)\| \leq n$ and since $F_{nB}(t_0, q(t_0))$ is closed and convex, from the Hahn Banach Theorem there is a weakly open convex set $V \supset F_{nB}(t_0, q(t_0))$ satisfying $f_0(t_0) \notin \overline{V}^\omega$. Since $F_{nB}(t_0,\cdot)$ is weakly u.s.c., we can also find a weak neighborhood V_1 of $q(t_0)$ such that $F_{nB}(t_0, x) \subset V$ for all $x \in V_1$ with $\|x\| \leq n$. Moreover, $\|q_k(t_0)\| \leq n$ for all $k \in \mathbb{N}$ and $q_k(t_0) \rightharpoonup q(t_0)$ as $k \to \infty$. Then, there exists $k_0 \in \mathbb{N}$ such that $q_k(t_0) \in V_1$ for all $k > k_0$. Therefore $f_k(t_0) \in F_{nB}(t_0, q_k(t_0)) \subset V$ for all $k > k_0$. Since V is convex, we have $\widetilde{f}_k(t_0) \in V$ for all $k > k_0$ and, since $t_0 \notin N_0$, we also have $\widetilde{f}_k(t_0) \to f_0(t_0)$. Then, we obtain the contradictory conclusion that $f_0(t_0) \in \overline{V}^\omega$. We can conclude that $f_0(t) \in F(t, q(t))$ for a.e. $t \in [0,T]$. □

Proposition 3.4. *The multioperator $\mathcal{T}_n : Q_n \multimap C([0,T];H)$ is weakly compact.*

Proof. We first prove that $\mathcal{T}_n(Q_n)$ is weakly relatively sequentially compact.
Let $\{q_k\}_{k\in\mathbb{N}} \subset Q_n$ and let $\{y_k\}_{k\in\mathbb{N}} \subset C([0,T];H)$ satisfying $y_k \in \mathcal{T}_n(q_k)$ for all $k \in \mathbb{N}$. By the definition of the multioperator $\mathcal{T}_n$, there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$, $f_k \in \Sigma_{q_k}$, such that

$$\begin{aligned} y_k(t) =& S(t)(y_0 + g(q_k)) + S_1 f_k(t) \\ &+ \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0+g(q_k))\Big)(s)\ ds + S_2 f_k(t) \end{aligned} \tag{22}$$

for every $t \in [0,T]$.
By the boundedness of the sequence $\{q_k\}_{k\in\mathbb{N}}$, since by assumption (g') the operator g maps bounded sets into bounded sets and by the reflexivity of the space H, we have that, up to subsequence, there exists $\overline{y} \in H$ such that $g(q_k) \rightharpoonup \overline{y}$. Moreover, reasoning as in Proposition 3.3, we have that there exists a subsequence, still denoted as the sequence, and a function f_0 such that $f_k \rightharpoonup f_0$ in $L^1([0,T];H)$. Therefore,

$$\begin{aligned} y_k(t) \rightharpoonup & S(t)(y_0 + \overline{y}) + S_1 f_0(t) \\ &+ \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0+\overline{y})\Big)(s)\ ds + S_2 f_0(t) = y(t) \end{aligned} \tag{23}$$

for all $t \in [0,T]$.
Furthermore, by the boundedness of the operators B, $\widetilde{G}^{-1}$, $S(t)$, for every $t \geq 0$, S_1 and S_2, it follows that

$$\begin{aligned} \|y_k(t)\| = & \Big\| S(t)(y_0 + g(q_k)) + S_1 f_k(t) \\ &+ \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0+g(q_k))\Big)(s)\ ds + S_2 f_k(t)\Big\| \\ \leq & C(\|y_0\| + \|g(q_k)\|) + C\|f_k\|_{L^1([0,T];H)} \\ &+ C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}(\|y_1\| + C(\|y_0\| + \|g(q_k)\|)) \\ &+ C^2\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}\|f_k\|_{L^1([0,T];H)} \leq N \end{aligned} \tag{24}$$

for all $k \in \mathbb{N}$ and for all $t \in [0,T]$ and for some $N > 0$.
By (23) and (24) and applying Theorem 2.1, we have that $y_k \rightharpoonup y$ in $C([0,T];H)$. Thus

$\mathcal{T}_n(Q_n)$ is weakly relatively sequentially compact. Applying Theorem 2.4, we obtain that $\mathcal{T}_n(Q_n)$ is weakly relatively compact. □

Proposition 3.5. *The multioperator* $\mathcal{T}_n : Q_n \multimap C([0,T];H)$ *has convex and weakly compact values.*

Proof. Fix $q \in Q_n$. Since the multimap F is convex valued, the set $\mathcal{T}_n(q)$ is convex from the linearity of the integral operator and of the operators B, $\widetilde{G}^{-1}$, g and $S(t)$, for every $t \geq 0$. The weak compactness of $\mathcal{T}_n(q)$ follows by Propositions 3.3 and 3.4. □

We are able now to state the main result of this section.

Theorem 3.1. *Let conditions* (A), (F_0), (F_1), (F_2'), (B), (G) *and* (g') *hold. In addition suppose that, for every* $n \in \mathbb{N}$, *there exists a function* $\phi_n \in L^1([0,T];\mathbb{R}_+)$ *such that*

$$(F_3') \begin{cases} \sup\limits_{\|c\|\leq n} \|F(t,c)\| \leq \phi_n(t) \text{ for a.e. } t \in [0,T], \\ \liminf\limits_{n\to\infty} \dfrac{1}{n} \displaystyle\int_0^T \phi_n(s)\, ds = 0 \end{cases} \tag{25}$$

and

$$(g_1) \quad \lim_{\|u\|_0 \to +\infty} \frac{\|g(u)\|}{\|u\|_0} = 0, \tag{26}$$

then the problem $(\mathcal{P})$ *is controllable.*

Proof. We want to apply Theorem 2.3 in order to prove the existence of at least one fixed point y of $\mathcal{T}_n$, that is a mild solution of the controllability problem $(\mathcal{P})$ such that $y(T) = y_1$. It remains to prove that there exists $n \in \mathbb{N}$ such that the multioperator $\mathcal{T}_n : Q_n \multimap C([0,T];H)$ maps the ball Q_n into itself. By contradiction, assume that there exist two sequences $\{q_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ such that $q_n \in Q_n$, $y_n \in \mathcal{T}_n(q_n)$ and $y_n \notin Q_n$, for all $n \in \mathbb{N}$. By the definition of the multioperator $\mathcal{T}_n$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^1([0,T];H)$, $f_n(t) \in F(t,q_n(t))$ for all $n \in \mathbb{N}$ and a.e. $t \in [0,T]$, such that

$$\begin{aligned} y_n(t) =& S(t)(y_0 + g(q_n)) + S_1 f_n(t) \\ &+ \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0 + g(q_n))\Big)(s)\, ds + S_2 f_n(t) \end{aligned} \tag{27}$$

for all $t \in [0,T]$. Since $q_n \in Q_n$ for all $n \in \mathbb{N}$, condition (F_3') implies that there exists $\phi_n \in L^1([0,T];\mathbb{R}_+)$ such that $\|f_n(t)\| \leq \phi_n(t)$ for a.e. $t \in [0,T]$. Since $y_n \notin Q_n$ for all $n \in \mathbb{N}$, we have

$$\begin{aligned} n &< \|y_n\|_0 \\ &\leq C(\|y_0\| + \|g(q_n)\|) + C\left(\int_0^T \|f_n(\eta)\|\, d\eta\right) \\ &\quad + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}(\|y_1\| + C(\|y_0\| + \|g(q_n)\|)) \\ &\quad + C^2\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}\left(\int_0^T \|f_n(\eta)\|\, d\eta\right) \\ &\leq C\|y_0\| + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}(\|y_1\| + C\|y_0\|) \\ &\quad + C(1 + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T})\|g(q_n)\| \\ &\quad + C(1 + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T})\left(\int_0^T \phi_n(\eta)\, d\eta\right) \\ &= C_1 + C_2\left(\int_0^T \phi_n(\eta)\, d\eta\right) + C_2\|g(q_n)\| \end{aligned} \tag{28}$$

for all $n \in \mathbb{N}$, with

$$C_1 = C\|y_0\| + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}(\|y_1\| + C\|y_0\|) \tag{29}$$

and

$$C_2 = C(1 + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}). \tag{30}$$

Now, dividing by $n \in \mathbb{N}$ the terms of the previous inequality we have

$$1 < \frac{\|y_n\|_0}{n} \leq \frac{C_1}{n} + \frac{C_2\left(\int_0^T \phi_n(\eta)\, d\eta\right)}{n} + \frac{C_2\|g(q_n)\|}{n}. \tag{31}$$

Notice that, if $\|q_n\|_0 \leq H < +\infty$ for any $n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{\|g(q_n)\|}{n} = 0$, since g maps bounded sets into bounded sets. If $\limsup_{n\to\infty}\|q_n\|_0 = +\infty$, then, by condition (g_1), we have

$$\limsup_{n\to+\infty} \frac{\|g(q_n)\|}{n} \leq \limsup_{n\to+\infty} \frac{\|g(q_n)\|}{\|q_n\|_0} \leq \limsup_{\|u\|_0\to+\infty} \frac{\|g(u)\|}{\|u\|_0} = 0. \tag{32}$$

In both cases, passing to the limit for $n \to \infty$ and using (25), we obtain the contradiction

$$1 \leq \frac{C_1}{n} + \frac{C_2\left(\displaystyle\int_0^T \phi_n(\eta)\, d\eta\right)}{n} + \frac{C_2\|g(q_n)\|}{n} \to 0. \tag{33}$$

Now, fix $n \in \mathbb{N}$ such that $\mathcal{T}_n(Q_n) \subseteq Q_n$. By Proposition 3.4 the set $V_n = \overline{\mathcal{T}_n(Q_n)}^{\omega}$ is weakly compact. Let now $W_n = \overline{co}(V_n)$, where $\overline{co}(V_n)$ denotes the closed convex hull of V_n. By Theorem 2.5 W_n is a weakly compact set. Moreover, since $\mathcal{T}_n(Q_n) \subset Q_n$ and Q_n is a convex closed set, we have that $W_n \subset Q_n$ and hence

$$\mathcal{T}_n(W_n) = \mathcal{T}_n(\overline{co}(\mathcal{T}_n(Q_n))) \subseteq \mathcal{T}_n(Q_n) \subseteq \overline{\mathcal{T}_n(Q_n)}^{\omega} = V_n \subset W_n. \tag{34}$$

Therefore, from Proposition 3.3 and from Corollary 2.1, we obtain that the restriction of the multimap $\mathcal{T}_n$ on W_n has a weakly closed graph, hence, by Proposition 3.4 and Proposition 3.5, applying Proposition 2.2, it is weakly u.s.c.. Finally, applying Theorem 2.3, we obtain the existence of a fixed point y of $\mathcal{T}_n$, which is a solution to $(\mathcal{P})$. □

It is possible to prove the controllability result also under less restrictive growth assumptions.

Theorem 3.2. *Let conditions (A), (F_0), (F_1), (F_2'), (B), (G) and (g') hold. In addition suppose that there exists a function $\alpha \in L^1([0,T];\mathbb{R}_+)$ such that*

$$(F_3'')\ \|F(t,c)\| \leq \alpha(t)(1+\|c\|) \text{ for a.e. } t \in [0,T] \text{ and for all } c \in H \tag{35}$$

and there exists $\beta > 0$ such that

$$(g_2)\ \|g(u)\| \leq \beta(1+\|u\|_0) \text{ for all } u \in C([0,T];H). \tag{36}$$

Moreover, suppose that

$$C(1 + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T})(\|\alpha\|_{L^1([0,T];\mathbb{R}_+)} + \beta) < 1, \tag{37}$$

then the problem $(\mathcal{P})$ is controllable.

Proof. Reasoning as in Theorem 3.1, we have to prove that there exists $n \in \mathbb{N}$ such that the multioperator $\mathcal{T}_n : Q_n \multimap C([0,T];H)$ maps the ball Q_n into itself. By contradiction, we assume that there exist two sequences $\{q_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ such that $q_n \in Q_n$, $y_n \in \mathcal{T}_n(q_n)$

and $y_n \notin Q_n$, for all $n \in \mathbb{N}$. By the definition of the multioperator $\mathcal{T}_n$, there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^1([0,T];H)$, $f_n(t) \in F(t,q_n(t))$ for all $n \in \mathbb{N}$ and a.e. $t \in [0,T]$, such that

(38)
$$y_n(t) = S(t)(y_0 + g(q_n)) + S_1 f_n(t) + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0 + g(q_n))\Big)(s)\, ds + S_2 f_n(t)$$

for all $t \in [0,T]$. Since $q_n \in Q_n$ for all $n \in \mathbb{N}$, by condition (F_3''), there exists $\alpha \in L^1([0,T];\mathbb{R}_+)$ such that $\|f_n(t)\| \le \alpha(t)(1+n)$ for a.e. $t \in [0,T]$. Since $y_n \notin Q_n$ for all $n \in \mathbb{N}$, by condition (g_2), we have

$$\begin{aligned}
n &< \|y_n\|_0 \\
&\le C(\|y_0\| + \|g(q_n)\|) \\
&\quad + C(1 + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T})\left(\int_0^T \|f_n(\eta)\|\, d\eta\right) \\
&\quad + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}(\|y_1\| + C(\|y_0\| + \|g(q_n)\|)) \\
&\le C\|y_0\| + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}(\|y_1\| + C\|y_0\|) \\
&\quad + C(1 + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T})\beta(1+n) \\
&\quad + C(1 + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T})(1+n)\|\alpha\|_{L^1([0,T];\mathbb{R}_+)} \\
&= C_1 + C_2(1+n)\|\alpha\|_{L^1([0,T];\mathbb{R}_+)} + C_2(1+n)\beta
\end{aligned} \tag{39}$$

for all $n \in \mathbb{N}$, with constants C_1 and C_2 defined respectively in (29) and (30).
Now, dividing by $n \in \mathbb{N}$ the first and the last terms of the previous inequality, we have

$$1 < \frac{C_1}{n} + C_2\left(\frac{1}{n} + 1\right)\|\alpha\|_{L^1([0,T];\mathbb{R}_+)} + C_2\left(\frac{1}{n} + 1\right)\beta \tag{40}$$

and, passing to the limit for $n \to \infty$, we get

$$1 \le C_2\|\alpha\|_{L^1([0,T];\mathbb{R}_+)} + C_2\beta. \tag{41}$$

By (37) and by the definition of C_2, it follows the contradiction

$$\begin{aligned}
1 &\le C_2\|\alpha\|_{L^1([0,T];\mathbb{R}_+)} + C_2\beta \\
&= C(1 + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T})(\|\alpha\|_{L^1([0,T];\mathbb{R}_+)} + \beta) < 1.
\end{aligned} \tag{42}$$

The conclusion then follows by Theorem 2.3, as in Theorem 3.1. □

3.2. **Existence with a Gelfand triple type method.** We consider the Hilbert space $(H, \|\cdot\|)$ compactly embedded in a Banach space $(E, \|\cdot\|_E)$, therefore there exists a costant $\lambda > 0$ such that $\|\cdot\|_E \leq \lambda \|\cdot\|$, and we consider the problem $(\mathcal{P})$ under the following assumptions:

- (A') $A : H \to H$ is a bounded linear operator;
- (F_2'') (a) the multimap $F(t, \cdot) : H \multimap H^\omega$ is closed for a.e. $t \in [0, T]$;
  - (b) the multimap $F(t, \cdot) : H \multimap H$ is $E - E$ u.s.c. for a.e. $t \in [0, T]$ in the following sense: for each $w \in H$ and $\varepsilon > 0$ there exists $\delta > 0$ such that from $w' \in B_E(w, \delta)$ it follows $F(t, w') \subset F(t, w) + B_E(0, \varepsilon)$;
- (g'') $g : C([0, T]; H) \to H$ is a bounded linear operator.

As we recalled in the preliminary section, $A : H \to H$ generates a uniformly continuous semigroup $\{S(t)\}_{t \geq 0}$, that in particular is a C_0-semigroup.

Remark 3.2. Notice that, since condition $(F_2'')(a)$ implies (F_2), by Proposition 3.2 the set Σ_q, defined in (6), is nonempty for every $q \in C([0, T]; H)$ also in this setting.

Now, we describe the properties of the multioperator $\mathcal{T}_n : Q_n \multimap C([0, T]; H)$ needeed to apply the Gliksberg - Fan fixed point Theorem (see Theorem 2.3). We recall that in this subsection we require assumptions (A'), (F_0), (F_1), (F_2''), (F_3), (B), (G) and (g'').

Proposition 3.6. *The multioperator $\mathcal{T}_n : Q_n \multimap C([0, T]; H)$ has a closed graph.*

Proof. We prove in the following that $\mathcal{T}_n$ has a sequentially closed graph. Then, let $\{q_k\}_{k \in \mathbb{N}} \subset Q_n$ a sequence such that $q_k \to q$ in $C([0, T]; H)$. Let $\{y_k\}_{k \in \mathbb{N}} \subset C([0, T]; H)$ satisfying $y_k \in \mathcal{T}_n(q_k)$ for all $k \in \mathbb{N}$ and such that $y_k \to y$ in $C([0, T]; H)$. We shall prove that $y \in \mathcal{T}_n(q)$. Since $y_k \in \mathcal{T}_n(q_k)$, it follows that there exists a sequence $\{f_k\}_{k \in \mathbb{N}}$, $f_k \in \Sigma_{q_k}$, such that

$$
\begin{aligned}
y_k(t) =& S(t)(y_0 + g(q_k)) + S_1 f_k(t) \\
& + \int_0^t S(t-s) B \widetilde{G}^{-1}\Big(y_1 - S(T)(y_0 + g(q_k))\Big)(s)\, ds + S_2 f_k(t)
\end{aligned}
\tag{43}
$$

for every $t \in [0, T]$. Reasoning as in Proposition 3.3, it is possible to prove that there exists a subsequence, still denoted as the sequence, and a function f_0 such that $f_k \rightharpoonup f_0$ in

$L^1([0,T];H)$. Then, by the linearity and continuity of the operators S_1 and S_2, we have

$$S_1 f_k \rightharpoonup S_1 f_0 \tag{44}$$

and

$$S_2 f_k \rightharpoonup S_2 f_0 \tag{45}$$

in $C([0,T];H)$. Since $q_k \to q$ in $C([0,T];H)$, by assumption (g''), we have

$$g(q_k) \xrightarrow{H} g(q) \tag{46}$$

and, by the linearity and continuity of the operator $S(t)$, for every $t \geq 0$, we obtain

$$S(t)g(q_k) \xrightarrow{H} S(t)g(q) \tag{47}$$

for every $t \in [0,T]$. In particular,

$$S(t)(y_0 + g(q_k)) \xrightarrow{H} S(t)(y_0 + g(q)) \tag{48}$$

for every $t \in [0,T]$. Moreover, by (46) and by the boundedness of the operators B, $\widetilde{G}^{-1}$ and $S(t)$, for every $t \geq 0$, we have

(49)
$$\begin{aligned}
&\left\| \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0 + g(q_k))\Big)(s)\, ds - \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0 + g(q))\Big)(s)\, ds \right\| \\
&\quad \leq \int_0^t \left\| S(t-s)B\widetilde{G}^{-1}\Big(-S(T)(g(q_k) - g(q))\Big)(s)\right\| ds \\
&\quad \leq C\|B\|_{\mathcal{L}(U,H)} \int_0^t \left\| \widetilde{G}^{-1}\Big(-S(T)(g(q_k) - g(q))\Big)(s)\right\|_U ds \\
&\quad \leq C\|B\|_{\mathcal{L}(U,H)} \left\| \widetilde{G}^{-1}\Big(-S(T)(g(q_k) - g(q))\Big)\right\|_{L^1([0,T];U)} \\
&\quad \leq C\|B\|_{\mathcal{L}(U,H)}\sqrt{T} \left\| \widetilde{G}^{-1}\Big(-S(T)(g(q_k) - g(q))\Big)\right\|_{L^2([0,T];U)} \\
&\quad \leq C^2\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}\|g(q_k) - g(q)\| \to 0.
\end{aligned}$$

Then (44), (45) and (48) imply that

(50)
$$
\begin{aligned}
y_k(t) &= S(t)(y_0+g(q_k)) + S_1 f_k(t) + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0+g(q_k))\Big)(s)\,ds + S_2 f_k(t)\\
&\overset{H}{\rightharpoonup} S(t)(y_0+g(q)) + S_1 f_0(t) + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0+g(q))\Big)(s)\,ds + S_2 f_0(t)\\
&= l(t)
\end{aligned}
$$

for every $t \in [0,T]$. By the uniqueness of the weak limit in H, we obtain that $y(t) = l(t)$ for every $t \in [0,T]$. Reasoning as at the end of the proof of Proposition 3.3, we get that $f_0(t) \in F(t, q(t))$ for a.e. $t \in [0,T]$, thus the claimed result. □

Proposition 3.7. *The multioperator $\mathcal{T}_n : Q_n \multimap C([0,T];H)$ is compact.*

Proof. We prove that $\mathcal{T}_n(Q_n)$ is relatively sequentially compact in $C([0,T];H)$. Then, let $\{q_k\}_{k\in\mathbb{N}} \subset Q_n$ and let $\{y_k\}_{k\in\mathbb{N}} \subset C([0,T];H)$ satisfying $y_k \in \mathcal{T}_n(q_k)$ for all $k \in \mathbb{N}$. By the definition of the multioperator $\mathcal{T}_n$, there exists a sequence $\{f_k\}_{k\in\mathbb{N}}$, $f_k \in \Sigma_{q_k}$, such that

(51)
$$
y_k(t) = S(t)(y_0+g(q_k)) + S_1 f_k(t) + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0+g(q_k))\Big)(s)\,ds + S_2 f_k(t)
$$

for every $t \in [0,T]$.

By the boundedness of the sequence $\{q_k\}_{k\in\mathbb{N}}$, since by assumption (g'') the operator g maps bounded sets into bounded sets and by the reflexivity of the space H, we have that, up to subsequence, there exists $\overline{y} \in H$ such that $g(q_k) \xrightarrow{H} \overline{y}$. Moreover, reasoning as in Proposition 3.3, we have that there exists a subsequence, still denoted as the sequence, and a function f_0 such that $f_k \rightharpoonup f_0$ in $L^1([0,T];H)$. Therefore,

(52)
$$
y_k(t) \overset{H}{\rightharpoonup} S(t)(y_0+\overline{y}) + S_1 f_0(t) + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0+\overline{y})\Big)(s)\,ds + S_2 f_0(t) = l(t)
$$

for all $t \in [0,T]$. From the compact embedding it follows

$$
y_k(t) \xrightarrow{E} l(t) \text{ for every } t \in [0,T]. \tag{53}
$$

Now, fix $q \in Q_n$ and $t \in [0,T]$ and consider $y(t) \in \mathcal{T}_n(q)(t)$. It follows that there exists a selection $f \in \Sigma_q$, such that

(54)
$$y(t) = S(t)(y_0 + g(q)) + S_1 f(t) + \int_0^t S(t-s)B\widetilde{G}^{-1}\Big(y_1 - S(T)(y_0+g(q))\Big)(s)\ ds + S_2 f(t)$$
$$= S(t)(y_0 + g(q)) + \int_0^t S(t-s)f(s)\ ds + \int_0^t S(t-s)B\widetilde{G}^{-1}(p_q)(s)\ ds,$$

where $p_q = y_1 - S(T)(y_0 + g(q)) - \int_0^T S(T-s)f(s)\ ds$. Recall that the map $y : [0,T] \to H$ is a strong solution of the problem

$$\begin{cases} y'(t) \in Ay(t) + F(t, q(t)) + B\widetilde{G}^{-1}(p_q)(t), \text{ a.e. } t \in [0,T] \\ y(0) = y_0 + g(q), \end{cases} \tag{55}$$

(see [14, Theorem 4.1.3]). By the boundedness of the operators B, $\widetilde{G}^{-1}$ and $S(t)$, for every $t \geq 0$, we have

$$\begin{aligned} \|y(t)\| \leq\ & C(\|y_0\| + \|g(q)\|) + C\|f\|_{L^1([0,T];H)} \\ & + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}(\|y_1\| + C(\|y_0\| + \|g(q)\|)) \\ & + C^2\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}\|f\|_{L^1([0,T];H)} \end{aligned}$$

and, since $q \in Q_n$, according to (F_3), there exists a function $v_n \in L^1([0,T];\mathbb{R}_+)$ such that

$$\|f(t)\| \leq v_n(t)$$

for a.e. $t \in [0,T]$, then it follows

$$\begin{aligned} \|y(t)\| \leq\ & C(\|y_0\| + \|g(q)\|) + C\|v_n\|_{L^1([0,T];H)} \\ & + C\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}(\|y_1\| + C(\|y_0\| + \|g(q)\|)) \\ & + C^2\|B\|_{\mathcal{L}(U,H)}\|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))}\sqrt{T}\|v_n\|_{L^1([0,T];H)} =: D_1 \end{aligned} \tag{56}$$

and

$$\begin{aligned} \|p_q\| & \leq \|y_1\| + C(\|y_0\| + \|g(q)\|) + C\int_0^T \|f(s)\|\ ds \\ & \leq \|y_1\| + C(\|y_0\| + \|g(q)\|) + C\int_0^T v_n(s)\ ds \\ & = \|y_1\| + C(\|y_0\| + \|g(q)\|) + C\|v_n\|_{L^1([0,T];H)} =: D_2. \end{aligned} \tag{57}$$

Then by (56) and (57), for every $0 \leq t_1 < t_2 \leq T$, we have

(58)

$$\|y(t_2) - y(t_1)\|_E \leq \lambda \|y(t_2) - y(t_1)\| = \lambda \left\| \int_0^{t_2} y'(s)\ ds - \int_0^{t_1} y'(s)\ ds \right\| = \lambda \left\| \int_{t_1}^{t_2} y'(s)\ ds \right\|$$
$$\leq \lambda \int_{t_1}^{t_2} \|y'(s)\|\ ds = \lambda \int_{t_1}^{t_2} \|Ay(s) + f(s) + B\widetilde{G}^{-1}(p_q)(s)\|\ ds$$
$$\leq \lambda \Big(\|A\|_{\mathcal{L}(H)} D_1(t_2 - t_1) + \|v_n\|_{L^1([0,T];\mathbb{R}_+)}$$
$$+ \sqrt{t_2 - t_1} \|B\|_{\mathcal{L}(U,H)} \|\widetilde{G}^{-1}\|_{\mathcal{L}(H,L^2([0,T];U))} D_2 \Big).$$

Then the functions in $\mathcal{T}_n(Q_n)$ are equicontinuous in $C([0,T];E)$. Applying the Ascoli-Arzelà Theorem (see [23, Theorem A.2.1.]), we can conclude that the multioperator $\mathcal{T}_n$ is compact in $C([0,T];E)$. □

Reasoning as in Proposition 3.5 it is possible to prove the following.

Proposition 3.8. *The multioperator* $\mathcal{T}_n : Q_n \multimap C([0,T];H)$ *has convex and compact values.*

From Propositions 3.6 and 3.7, again applying Proposition 2.2, we obtain that $\mathcal{T}_n$ is a compact u.s.c. multimap.

Now, reasoning as in the proofs of Theorems 3.1 and 3.2 we can apply Theorem 2.3 to obtain the following existence results.

Theorem 3.3. *Let conditions* (A'), (F_0), (F_1), (F_2''), (F_3'), (B), (G), (g'') *and* (g_1) *hold. Then the problem* $(\mathcal{P})$ *is controllable.*

Theorem 3.4. *Let conditions* (A'), (F_0), (F_1), (F_2''), (F_3''), (B), (G), (g''), (g_2) *and* (37) *hold. Then the problem* $(\mathcal{P})$ *is controllable.*

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