SYMMETRY PROBLEMS FOR GAUGE BALLS IN THE HEISENBERG GROUP

PROBLEMI DI SIMMETRIA PER PALLE DELLA GAUGE NEL GRUPPO DI HEISENBERG

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ABSTRACT. In this note we focus on possible characterizations of gauge-symmetric functions in the Heisenberg group. We discuss a family of inverse problems in potential theory relating solid and surface weighted mean-value formulas, and we show a partial solution to such problems. To this aim, we review a uniqueness result for gauge balls obtained with V. Martino in [23] by means of overdetermined problems of Serrin-type. The class of competitor sets we consider enjoys partial symmetries of toric and cylindrical type.

SUNTO. In questa nota vengono discusse possibili caratterizzazioni di funzioni gauge-simmetriche nel gruppo di Heisenberg. Viene mostrata una soluzione parziale ad una famiglia di problemi inversi legati ad opportune formule di media solida e superficiale pesate per funzioni armoniche rispetto al subLaplaciano di Heisenberg. A questo scopo, viene presentato un risultato di unicità ottenuto in [23] con V. Martino per problemi sovradeterminati di tipo Serrin in questo contesto. La classe di insiemi considerata gode di proprietà di simmetria parziale di tipo torico e cilindrico.

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1. INTRODUCTION

Generalizations of the classical Gauss' theorem about mean-value properties for harmonic functions attracted a lot of attention in the literature. Solutions to linear second order partial differential operators with nonnegative characteristic form, which have

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also smooth enough coefficients and well-behaved fundamental solutions Γ, satisfy in fact mean-value formulas of solid and surface type on the (super-)level sets of Γ. If we restrict ourselves in considering the case of sub-Laplacians in Carnot groups, then the superlevel sets of Γ share self-similarity properties thanks to the left-translation and dilation invariances and they play the role of balls to all intents and purposes: it is thus possible to investigate the geometry endowed by these quasi-metric balls, which are usually called *gauge balls*. An extensive treatment to these aspects is given in [3, 2], where in particular the authors introduce and show integral representation formulas based on various average operators. If we restrict again the attention to the Heisenberg group, namely the first non-trivial example of such homogeneous groups (other than Euclidean $\mathbb{R}^N$), we have the explicit expression of the fundamental kernels and we can see in action the average operators. This is the starting point of our discussion: let us now fix the needed notations.

We identify the Heisenberg group $\mathbb{H}^n$, $n \geq 1$, with $\mathbb{R}^{2n+1}$ where we denote the generic point by $\xi = (x,t) \in \mathbb{R}^{2n} \times \mathbb{R}$ and we fix the group law

$$\xi \circ \xi' = (x,t) \circ (x',t') = (x + x', t + t' + 2\langle Jx, x'\rangle), \qquad \text{for } \xi, \xi' \in \mathbb{H}^n.$$

The $2n \times 2n$ matrix J stands for the following standard symplectic matrix

$$J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

The homogeneous dilations are given by the 1-parameter family of group homomorphisms $\{\delta_\lambda\}_{\lambda>0}$ defined as

$$\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n, \qquad \delta_\lambda(x,t) = (\lambda x, \lambda^2 t).$$

We indicate with $Q = 2n + 2$ the homogeneous dimension of the group. The canonical basis of left invariant vector fields on $\mathbb{H}^n$ is defined via

$$X_j = \frac{\partial}{\partial x_j} + 2(Jx)_j \frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}, \quad j = 1, \ldots, 2n.$$

We shall exploit the notations $D_H u$ for the so-called horizontal gradient of a given scalar valued function u and $\mathrm{div}_H(V)$ for the horizontal divergence of a vector field $(V_1, \ldots, V_{2n})$

as follows

$$D_H u = (X_1 u, \dots, X_{2n} u), \qquad \mathrm{div}_H(V) = \sum_{j=1}^{2n} X_j(V_j).$$

The Heisenberg subLaplacian is the second order partial differential operator defined by

$$\begin{aligned}\Delta_{\mathbb{H}} u &= \sum_{j=1}^{2n} X_j^2 u = \mathrm{div}_H(D_H u)\\ &= \Delta_x u + 4|x|^2 \partial_{tt}^2 u + 4\langle Jx, \nabla_x \partial_t u\rangle .\end{aligned}$$

Since the horizontal vector fields are divergence-free, left invariant, and δ_λ-homogeneous of degree 1, it is easy to check that $\Delta_{\mathbb{H}}$ is in divergence form, left-invariant, and δ_λ-homogeneous of degree 2. It is also hypoelliptic thanks to the non-commutation relation $[X_j, X_i] = 4J_{ij}T$. It is known since [8] the explicit expression for the fundamental solution of $\Delta_{\mathbb{H}}$. As a matter of fact, for any $\xi_0 \in \mathbb{H}^n$ the function

(1) $$\Gamma(\xi, \xi_0) = \frac{\beta}{\rho^{Q-2}(\xi_0^{-1} \circ \xi)} \quad \text{is the fundamental solution of } \Delta_{\mathbb{H}^n} \text{ with pole at } \xi_0,$$

where $\beta > 0$ is a renormalizing constant and the function ρ defined by

(2) $$\rho(\xi) = \left(|x|^4 + t^2\right)^{\frac{1}{4}}, \qquad \xi \in \mathbb{H}^n,$$

is the so-called gauge function. The appearance of the function ρ is connected with the underlying complex structure (or better, CR-structure) of the Heisenberg group. We refer the reader to [10] and references therein for a heat-kernel derivation of such fundamental solution. For us it is important to notice that ρ is a δ_λ-homogeneous of degree 1 norm, and we can call gauge balls the metric balls centered at $\xi_0 \in \mathbb{H}^n$ and radius $R > 0$ given by

$$B_R(\xi_0) = \left\{\xi \in \mathbb{H}^n \ : \ \rho(\xi_0^{-1} \circ \xi) < R\right\}.$$

From the expression in (2) we can compute the following δ_λ-homogeneous of degree 0 function

(3) $$|D_H \rho(\xi)| = \frac{|x|}{\rho(\xi)}, \qquad \text{for any } \xi \in \mathbb{H}^n,\ \xi \neq 0,$$

which appears in the mean-value formulas for $\Delta_{\mathbb{H}}$. More precisely, for $r > 0$ and for continuous functions h, we denote

$$\mathrm{M}_{B_r(0)}(h) = \frac{Q(Q-2)\beta}{r^Q}\int_{B_r(0)} h(\xi)|D_H\rho(\xi)|^2 d\xi,$$

$$\mathcal{M}_{\partial B_r(0)}(h) = \frac{(Q-2)\beta}{r^{Q-1}}\int_{\partial B_r(0)} h(\xi)\frac{|D_H\rho(\xi)|^2}{|D\rho(\xi)|}d\sigma(\xi).$$

Here, and in what follows, we have indicated by $d\xi$ the Lebesgue measure, by $d\sigma(\xi)$ the surface measure for smooth hypersurfaces, by D the Euclidean gradient, and by β the constant in (1). We notice that the weight $|D_H\rho(\xi)|^2$ defined in (3) is an L^∞-function with a discontinuity at $\xi = 0$, it is bounded above by 1, and it vanishes exactly on the vertical axis (i.e. the t-axis)

$$L_v = \{(0,t) \in \mathbb{H}^n \,:\, t \in \mathbb{R}\}.$$

As we hinted at the beginning, it is known (see [12, Théorème 3] and [2, Theorems 5.5.4 and 5.6.1]) that for every solution h to $\Delta_{\mathbb{H}} h = 0$ in $B_R(0)$ (which is continuous up to the closure of $B_R(0)$) the following holds true for any $0 < r \leq R$

$$h(0) = \mathrm{M}_{B_r(0)}(h) = \mathcal{M}_{\partial B_r(0)}(h). \tag{4}$$

The reference point 0 can be substituted by any given point thanks to the underlying left-invariance properties. We recall, see e.g. the discussion in [2], that the validity of either one of the equalities in (4) at every point of a domain characterizes the $\Delta_{\mathbb{H}}$-harmonicity of h in that domain. Since the function $h \equiv 1$ is $\Delta_{\mathbb{H}}$-harmonic, the operators $\mathrm{M}_{B_r(0)}(\cdot)$ and $\mathcal{M}_{\partial B_r(0)}(\cdot)$ are actual averaging operators as $\mathrm{M}_{B_r(0)}(1) = 1 = \mathcal{M}_{\partial B_r(0)}(1)$. In particular, one can define

$$\mathrm{A}(\Omega) = \int_\Omega |D_H\rho(\xi)|^2 d\xi \quad \text{and} \quad \mathrm{M}_\Omega(h) = \frac{1}{\mathrm{A}(\Omega)}\int_\Omega h(\xi)|D_H\rho(\xi)|^2 d\xi \tag{5}$$

for any bounded open set Ω and any function h which is integrable in Ω. The first result concerning an inverse problem, with respect to the domain, for the mean-value properties displayed in (4) was established by Lanconelli in [18]: he showed that, if Ω is a bounded open set containing 0 and if $h(0) = \mathrm{M}_\Omega(h)$ for any $\Delta_{\mathbb{H}}$-harmonic function h which is integrable in Ω, then Ω is a gauge ball (centered at 0 and with radius uniquely determined by $\mathrm{A}(\Omega)$). Lanconelli's result generalizes a classical result by Kuran about the inverse problem for the Gauss' solid mean-value formula of harmonic functions, and it holds true remarkably in any Carnot group (the proof does not use the explicit expression of the weight $|D_H\rho(\xi)|^2$ and it exploits comparison principles for the equilibrium potentials related to Ω and $B_r(0)$). We refer to [6, 17] for insights and extensions concerning this issue. In this note we are interested in the inverse problem with respect to the equality case between the solid and the surface average operators in (4).

To this aim, in [23, Section 1.1] we observed that it is convenient to rewrite

$$\mathcal{M}_{\partial B_r(0)}(h) = \frac{(Q-2)\beta}{r^{Q-1}} \int_{\partial B_r(0)} h(\xi)|D_H\rho(\xi)|d\sigma_H(\xi),$$

where $d\sigma_H$ stands for the horizontal surface measure, see also the discussion in Section 2 below. The notion of horizontal surface measure is well established in the literature and it has been used by many authors: in the class of smooth hypersurfaces it is absolutely continuous with respect to the standard surface measure, see e.g. [9, Section 2.3]. We then propose to define, for any bounded open set Ω such that $\partial\Omega$ is smooth and $0 \notin \partial\Omega$, the following weighted perimeter

$$\mathcal{A}(\partial\Omega) = \int_{\partial\Omega} |D_H\rho(\xi)|d\sigma_H(\xi) \tag{6}$$

and, for continuous functions h defined on $\partial\Omega$, the relative surface average operator

$$\mathcal{M}_{\partial\Omega}(h) = \frac{1}{\mathcal{A}(\partial\Omega)} \int_{\partial\Omega} h(\xi)|D_H\rho(\xi)|d\sigma_H(\xi). \tag{7}$$

The main result we want to describe in the present note can then be read as follows

Theorem 1.1. *Let $\Omega \subset \mathbb{H}^n$ be a competitor set. Assume that one of the following conditions holds:*

- *$n = 2$ and Ω is toric symmetric;*
- *$n \geq 3$ and Ω is cylindrically symmetric.*

Suppose that

$$\mathrm{M}_\Omega(h) = \mathcal{M}_{\partial\Omega}(h)$$

for every $\Delta_{\mathbb{H}}$-harmonic function h which is suitably smooth up to the boundary of Ω. Then

$$\Omega = B_R(0) \qquad \textit{with } R = \frac{Q\mathrm{A}(\Omega)}{\mathcal{A}(\partial\Omega)}.$$

We refer the reader to Section 2 for the precise notions involved in the statement, as well as for the precise definition of the partial symmetry assumptions. We mention here that the cylindrical symmetry is meant with respect to the t-axis, and such requirement is more restrictive than the one of toric symmetry. Theorem 1.1 will follow as a particular case of Theorems 2.1-2.2 below.

Our interest for the problem stated in Theorem 1.1 comes from the known relationship (which is in fact an equivalence) between the inverse problem with respect to the domain for the equality case of solid and surface mean value formulas and Serrin's symmetry problem for the overdetermined system with respect to the torsion function [29]. For an account of such an equivalence we refer to the works [7, 26, 30] which treat the classical case of the Laplace operator[1]. In our setting, the overdetermined boundary value problem underlying Theorem 1.1 is the following

$$\begin{cases} \Delta_{\mathbb{H}} u = Q\,|D_H\rho|^2 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |D_H u| = c\,|D_H\rho| & \text{on } \partial\Omega, \end{cases} \tag{8}$$

for positive constants c, and we stress the appearance of a weighted torsion function with non-constant 0-homogeneous weight $|D_H\rho|^2$. The overdermined system (8) has been introduced for the first time in [23]. In [23] we proved that, under the partial symmetry assumptions present in Theorem 1.1, the only competitor set Ω where (8) admits a solution is the gauge ball $B_c(0)$ and in such case the solution is

$$u = \frac{\rho^2(\xi) - c^2}{2}.$$

Symmetry and rigidity questions for PDEs in CR-settings, and in particular in the Heisenberg group, are known to be a delicate issue since the lack of symmetrization techniques

[1] I thank Giovanni Cupini and Ermanno Lanconelli for having pointed out to me the reference [7] that I was not aware of.

and the obstructions in running moving-plane methods. In this direction, the most important and complete symmetry result is due to Jerison and Lee in [16] where they proved that the Aubin-Talenti type functions

$$\frac{1}{\left((|x|^2+1)^2+t^2\right)^{\frac{Q-2}{4}}}$$

are (up to translations and dilations) the unique positive solutions in the relevant variational space to the CR-Yamabe equations in the whole $\mathbb{H}^n$. Their proof follows, in a highly non-trivial way, the so-called Obata method and it relies on remarkable differential identities of Bochner type. Such identities have been proven successful also in recent important developments such as the non-existence and symmetry results established in [19, 4]. We notice that these Aubin-Talenti functions enjoy a (almost) gauge-like symmetry, and we refer to [11] for a heat-kernel derivation of such functions. We also stress that this class of symmetry results à la Jerison-Lee addresses problems which are global in nature, and which are strongly based on the underlying conformal invariances. On the other hand, much less is known about symmetry results in bounded domains of $\mathbb{H}^n$ (apart from cmc-type classifications in $\mathbb{H}^1$ which are of different nature, see [28]). In this direction, several partial results have appeared in the literature under a priori cylindrically symmetric assumption as the one in Theorem 1.1. Among these, the first was probably the initial [15, Theorem 7.8], and we mention here the moving-plane approach to infer monotonicity in the t-variable for semilinear equations in [1], the ODE approach for the classification of cmc-hypersurfaces in [27], and the weighted rearrangement approach for the isoperimetric problem in [24]. Instead, the approach we decided to follow in [23] is inspired by the elegant proof by Weinberger in [33] for the Serrin overdetermined problem with the so-called P-function method via integral and differential identities. The main novelty in [23] is a new weighted Bochner-type identity for functions with toric invariances which is suitable for showing the $\Delta_{\mathbb{H}}$-subharmonicity of the relevant P-function attached to (8), and it works as a replacement of the classical identity $\Delta(|\nabla u|^2) = 2|D^2u(x)|^2 + 2\langle \nabla u, \nabla(\Delta u)\rangle$.

In Section 2 of this note we describe a one-parameter family of results generalizing Theorem 1.1, and we recall the precise statement of the main results in [23]. At the end we briefly outline the proof contained in [23] with the role of our P-function (20), and we highlight the connection with the joint work [13] with C. Guidi and V. Martino where we established a rigidity result for gauge spheres under a suitable curvature prescription.

2. Overdetermined problems in $\mathbb{H}^n$

Fix $\alpha > 0$ and define

$$F_\alpha(\xi) = |D_H\rho(\xi)|^2\rho^{\alpha-2}(\xi) = |x|^2\rho^{\alpha-4}(\xi). \tag{9}$$

We notice that F_α is nonnegative, it is smooth out of $\xi = 0$, and it is also locally integrable around 0 since $Q + \alpha - 2 > 0$. Actually, the regularity of F_α around 0 depends on α in the following sense

$$F_\alpha \in \begin{cases} L^\infty_{\text{loc}} & \text{if } \alpha \geq 2, \\ L^p_{\text{loc}} & \text{for some } p > \frac{Q}{2} \text{ if } 0 < \alpha < 2. \end{cases}$$

We are then ready to define the notion of competitor set introduced in [23]: we consider open, bounded and connected sets $\Omega \subset \mathbb{H}^n$ with smooth boundaries, we assume that around characteristic points of $\partial\Omega$ the set Ω has interior and exterior tangent gauge-balls, and we ask that the unique weak solutions to

$$\begin{cases} \Delta_{\mathbb{H}} u = (Q + \alpha - 2)F_\alpha & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{10}$$

are also smooth in a neighborhood of the boundary of Ω. If $\alpha \neq 4$, we require in addition that $0 \in \Omega$ (keep in mind that F_4 is C^∞).

We recall here the well-known notion of characteristic point: a point $\xi_0 \in \partial\Omega$ is said to be characteristic for $\partial\Omega$ if the tangent space to $\partial\Omega$ at ξ_0 coincides with the vector space $\text{span}\{X_1(\xi_0), \dots, X_{2n}(\xi_0)\}$. The characteristic set has surface measure 0, but around such characteristic points the requirement for the solution u to (10) to be smooth is non-trivial, and we refer to [23, Section 2] for a discussion about this issue and for the related literature. We recall here that the exterior gauge-ball condition and the fact that $F_\alpha \in L^p(\Omega)$ for some $p > \frac{Q}{2}$ ensures that the function u is Hölder continuous in $\overline{\Omega}$. Moreover, the interior gauge-ball condition ensures the validity of the Hopf-lemma at boundary points (see [21]) which says that $|Du|$ cannot vanish on $\partial\Omega$ (notice that the points of $\partial\Omega$ are points of strict maximum for u in Ω by the strong maximum principle). To summarize, if Ω is a competitor set we have in particular that

$$u < 0 \text{ in } \Omega, \quad u \equiv 0 \text{ on } \partial\Omega, \quad \text{and} \quad |Du| > 0 \text{ on } \partial\Omega,$$

i.e. the torsion-like function u is a defining function for $\partial\Omega$. As a safe remark, the gauge balls $B_r(0)$ are competitor sets and in such case a direct computation shows that the function $u_\alpha(\xi) = \frac{1}{\alpha}(\rho^\alpha(\xi) - r^\alpha)$ is the relevant torsion function.

Working with the defining function u, we can write the horizontal surface measure $d\sigma_H$ in terms of the standard surface measure as $d\sigma_H = \frac{|D_H u|}{|Du|} d\sigma$. We can then recall the notion of horizontal outer unit normal to $\partial\Omega$ which is defined for any non-characteristic ξ as $\nu^H(\xi) = \frac{D_H u(\xi)}{|D_H u(\xi)|}$. With these notations being fixed, for vector fields $V \in C(\overline{\Omega}; \mathbb{R}^{2n})$ such that $\mathrm{div}_H(V)$ is locally integrable around 0 and smooth around $\partial\Omega$ one has

$$\int_\Omega \mathrm{div}_H(V)(\xi)\, d\xi = \int_{\partial\Omega} \left\langle V(\xi), \frac{D_H u(\xi)}{|Du(\xi)|} \right\rangle d\sigma(\xi) = \int_{\partial\Omega} \left\langle V(\xi), \nu^H(\xi) \right\rangle d\sigma_H(\xi).$$

Therefore, exploiting the solution u to (10) in a competitor set Ω and the previous form of the divergence theorem, we recognize the following: for any $\Delta_{\mathbb{H}}$-harmonic function h with $h \in C(\overline{\Omega})$ such that h is smooth up to non-characteristic boundary points and $D_H h$ is bounded, we have

$$\begin{aligned}
\int_\Omega h(\xi) F_\alpha d\xi &= \frac{1}{Q+\alpha-2} \int_\Omega h(\xi) \Delta_{\mathbb{H}} u(\xi) d\xi \\
&= \frac{1}{Q+\alpha-2} \int_\Omega \left(h(\xi)\Delta_{\mathbb{H}} u(\xi) - u(\xi)\Delta_{\mathbb{H}} h(\xi) \right) d\xi \\
&= \frac{1}{Q+\alpha-2} \int_\Omega \mathrm{div}_H \left(h D_H u - u D_H h \right)(\xi) d\xi \\
&= \frac{1}{Q+\alpha-2} \int_{\partial\Omega} h(\xi) |D_H u(\xi)| d\sigma_H(\xi).
\end{aligned} \tag{11}$$

We agree to let $\mathcal{H}_\Omega$ the set of such harmonic functions h.

Definition 2.1. *Fix $\alpha > 0$, and let Ω be a competitor set. Let us define the α-weighted horizontal volume of Ω and the α-weighted horizontal perimeter of $\partial\Omega$ respectively as*

$$\mathrm{V}^{\mathbb{H}}_\alpha(\Omega) = \int_\Omega F_\alpha(\xi) d\xi \qquad \textit{and} \qquad \mathcal{P}^{\mathbb{H}}_\alpha(\partial\Omega) = \int_{\partial\Omega} F_\alpha^{\frac{1}{2}}(\xi) d\sigma_H(\xi). \tag{12}$$

Moreover, for any function $h \in C(\overline{\Omega})$ *we put*

$$\mathrm{M}_{\Omega}^{(\alpha)}(h) = \frac{1}{\mathrm{V}_{\alpha}^{\mathbb{H}}(\Omega)} \int_{\Omega} h(\xi) F_{\alpha}(\xi) d\xi \tag{13}$$

and

$$\mathcal{M}_{\partial\Omega}^{(\alpha)}(h) = \frac{1}{\mathcal{P}_{\alpha}^{\mathbb{H}}(\partial\Omega)} \int_{\partial\Omega} h(\xi) F_{\alpha}^{\frac{1}{2}}(\xi) d\sigma_H(\xi). \tag{14}$$

Remark 2.1. *In case* $\alpha = 2$, *it is clear from the definitions in* (5), (6), (9), *and* (12) *that*

$$\mathrm{V}_2^{\mathbb{H}}(\Omega) = \mathrm{A}(\Omega) \qquad \textit{and} \qquad \mathcal{P}_2^{\mathbb{H}}(\partial\Omega) = \mathcal{A}(\partial\Omega).$$

Moreover, comparing (5)-(7) *with* (13)-(14), *we have also*

$$\mathrm{M}_{\Omega}^{(2)}(h) = \mathrm{M}_{\Omega}(h) \qquad \textit{and} \qquad \mathcal{M}_{\partial\Omega}^{(2)}(h) = \mathcal{M}_{\partial\Omega}(h).$$

In the following lemma we show that, for $\Omega = B_R(0)$, the mean-value properties recalled in (4) (i.e. the case $\alpha = 2$ in our notations) hold true for any $\alpha > 0$. Such properties basically follow from the discussion in [2, Section 5.6] but we provide the details for the convenience of the reader.

Lemma 2.1. *Fix* $\alpha > 0$ *and* $R > 0$. *Let* h *be a* $\Delta_{\mathbb{H}}$*-harmonic function such that* $h \in C(\overline{B_R(0)})$. *For any* $0 < r \leq R$ *we have*

$$h(0) = \mathrm{M}_{B_r(0)}^{(\alpha)}(h) = \mathcal{M}_{\partial B_r(0)}^{(\alpha)}(h).$$

Moreover

$$\mathrm{V}_{\alpha}^{\mathbb{H}}(B_r(0)) = \frac{r^{Q+\alpha-2}}{(Q+\alpha-2)(Q-2)\beta} \quad \textit{and} \quad \mathcal{P}_{\alpha}^{\mathbb{H}}(\partial B_r(0)) = \frac{r^{Q+\frac{\alpha}{2}-2}}{(Q-2)\beta}. \tag{15}$$

Proof. Since $\rho(\xi) = r$ on $\partial B_r(0)$ and $d\sigma_H = \frac{|D_H \rho|}{|D\rho|} d\sigma$ in such case, we just rewrite

$$\begin{aligned}
\mathcal{M}_{\partial B_r(0)}(h) &= \frac{(Q-2)\beta}{r^{Q-1}} \int_{\partial B_r(0)} h(\xi) \frac{|D_H \rho(\xi)|^2}{|D\rho(\xi)|} d\sigma(\xi) \\
&= \frac{(Q-2)\beta}{r^{Q+\frac{\alpha}{2}-2}} \int_{\partial B_r(0)} h(\xi) \rho^{\frac{\alpha}{2}-1}(\xi) \frac{|D_H \rho(\xi)|^2}{|D\rho(\xi)|} d\sigma(\xi) \\
&= \frac{(Q-2)\beta}{r^{Q+\frac{\alpha}{2}-2}} \int_{\partial B_r(0)} h(\xi) F_{\alpha}^{\frac{1}{2}}(\xi) d\sigma_H(\xi) \\
&= \frac{(Q-2)\beta \mathcal{P}_{\alpha}^{\mathbb{H}}(\partial B_r(0))}{r^{Q+\frac{\alpha}{2}-2}} \mathcal{M}_{\partial B_r(0)}^{(\alpha)}(h).
\end{aligned}$$

Hence, by plugging $h \equiv 1$ in the previous identity and from (4), we have

$$\frac{(Q-2)\beta\mathcal{P}^{\mathbb{H}}_{\alpha}(\partial B_r(0))}{r^{Q+\frac{\alpha}{2}-2}} = 1 \quad \text{and} \quad h(0) = \mathcal{M}^{(\alpha)}_{\partial B_r(0)}(h).$$

In order to deduce the solid counterpart, we put

$$\varphi_r(\tau) = \begin{cases} \frac{(Q+\alpha-2)}{r^{Q+\alpha-2}}\tau^{Q+\alpha-3} & \text{if } \tau \in [0,r], \\ 0 & \text{if } \tau > r. \end{cases}$$

Since $\int_0^\infty \varphi_r(\tau)d\tau = 1$, from (4) and coarea formula we deduce that

$$\begin{aligned} h(0) &= \int_0^\infty \varphi_r(\tau)h(0)d\tau = \int_0^\infty \varphi_r(\tau)\mathcal{M}_{\partial B_\tau(0)}(h)d\tau \\ &= \frac{(Q+\alpha-2)(Q-2)\beta}{r^{Q+\alpha-2}}\int_0^r \int_{\partial B_\tau(0)} \frac{\tau^{Q+\alpha-3}}{\tau^{Q-1}} h(\xi)\frac{|D_H\rho(\xi)|^2}{|D\rho(\xi)|}d\sigma(\xi)d\tau \\ &= \frac{(Q+\alpha-2)(Q-2)\beta}{r^{Q+\alpha-2}}\int_{B_r(0)} h(\xi)|D_H\rho(\xi)|^2\rho^{\alpha-2}(\xi)d\xi \\ &= \frac{(Q+\alpha-2)(Q-2)\beta}{r^{Q+\alpha-2}}\int_{B_r(0)} h(\xi)F_\alpha(\xi)d\xi \\ &= \frac{(Q+\alpha-2)(Q-2)\beta \mathrm{V}^{\mathbb{H}}_{\alpha}(B_r(0))}{r^{Q+\alpha-2}}\mathrm{M}^{(\alpha)}_{B_r(0)}(h). \end{aligned}$$

Again, by plugging $h \equiv 1$ in the previous identity, we have

$$\frac{(Q+\alpha-2)(Q-2)\beta \mathrm{V}^{\mathbb{H}}_{\alpha}(B_r(0))}{r^{Q+\alpha-2}} = 1 \quad \text{and} \quad h(0) = \mathrm{M}^{(\alpha)}_{B_r(0)}(h).$$

□

W say that a bounded open set $\Omega \subset \mathbb{H}^n$ is cylindrically symmetric (with respect to the t-axis) if

$$(x,t) \in \Omega \quad \Longrightarrow \quad (x',t) \in \Omega \quad \text{for every } x' \in \mathbb{R}^{2n} \text{ with } |x'| = |x|.$$

Under this assumption we have the following

Theorem 2.1. *Fix $n \geq 1$, and $\alpha \in (\alpha_n, 4]$ where $\alpha_n = \frac{3}{4}$ for $n \geq 2$ and $\alpha_1 = 2$. Let $\Omega \subset \mathbb{H}^n$ be a competitor set, and assume that Ω is cylindrically symmetric. Suppose that*

$$\mathrm{M}^{(\alpha)}_{\Omega}(h) = \mathcal{M}^{(\alpha)}_{\partial\Omega}(h) \quad \textit{for any } h \in \mathcal{H}_\Omega.$$

Then Ω is a gauge ball of radius R given by

$$R^{\frac{\alpha}{2}} = \frac{(Q+\alpha-2)\mathrm{V}_\alpha^{\mathbb{H}}(\Omega)}{\mathcal{P}_\alpha^{\mathbb{H}}(\partial\Omega)}.$$

The geometric meaning of the cylindrical symmetry assumption is that the t-sections of Ω are required to be balls in $\mathbb{R}^{2n}$ centered on the vertical axis L_v. Clearly, this assumption is more and more restrictive when we raise the dimension n. In case $n=2$ we can assume a sharper condition (i.e. a weaker requirement), which is to require that the t-sections of Ω are Reinhardt domains with respect to the complex structure inherited by J. More precisely, we say that a bounded open set $\Omega \subset \mathbb{H}^n$ is toric symmetric (with respect to the symplectic matrix J and the t-axis) if

$$(x,t)\in\Omega \quad \Longrightarrow \quad (x',t)\in\Omega \quad \text{for every } x'\in\mathbb{R}^{2n} \text{ such that}$$
$$(x')_k^2 + (x')_{n+k}^2 = x_k^2 + x_{n+k}^2 \text{ with } k\in\{1,\dots,n\}.$$

Theorem 2.2. *Fix $n=2$, and $\alpha\in(\frac{3}{4},4]$. Let $\Omega\subset\mathbb{H}^2$ be a competitor set, and assume that Ω is toric symmetric. Suppose that*

$$\mathrm{M}_\Omega^{(\alpha)}(h) = \mathcal{M}_{\partial\Omega}^{(\alpha)}(h) \quad \textit{for any } h\in\mathcal{H}_\Omega.$$

Then Ω is a gauge ball of radius R given by

$$R^{\frac{\alpha}{2}} = \frac{(Q+\alpha-2)\mathrm{V}_\alpha^{\mathbb{H}}(\Omega)}{\mathcal{P}_\alpha^{\mathbb{H}}(\partial\Omega)}.$$

.

In view of Remark 2.1 it is clear that the case $\alpha=2$ of Theorem 2.1 and Theorem 2.2 yields Theorem 1.1. In the following remark we display the connection with the overdetermined problems studied in [23].

Remark 2.2. *In any competitor set Ω, we can consider the unique solution u to* (10). *Suppose we know that u also satisfies the (overdetermined) condition*

$$|D_H u(\xi)| = c\, F_\alpha^{\frac{1}{2}}(\xi) \quad \textit{for } \xi\in\partial\Omega \tag{16}$$

for some positive constant c. Then, for any $h \in \mathcal{H}_\Omega$, we obtain from (11) *that*

$$\int_\Omega h(\xi) F_\alpha d\xi = \frac{1}{Q+\alpha-2} \int_{\partial\Omega} h(\xi)|D_H u(\xi)| d\sigma_H(\xi)$$
$$= \frac{c}{Q+\alpha-2} \int_{\partial\Omega} h(\xi) F_\alpha^{\frac{1}{2}}(\xi) d\sigma_H(\xi).$$

By keeping in mind the definitions in (13) *and* (14)*, the previous identity can be written as*

$$\mathrm{M}_\Omega^{(\alpha)}(h) = \frac{c\mathcal{P}_\alpha^{\mathbb{H}}(\partial\Omega)}{(Q+\alpha-2)\mathrm{V}_\alpha^{\mathbb{H}}(\Omega)} \mathcal{M}_{\partial\Omega}^{(\alpha)}(h). \tag{17}$$

Since (17) *holds true for all $h \in \mathcal{H}_\Omega$, we can use the relation for $h \equiv 1$ to obtain*

$$\frac{c\mathcal{P}_\alpha^{\mathbb{H}}(\partial\Omega)}{(Q+\alpha-2)\mathrm{V}_\alpha^{\mathbb{H}}(\Omega)} = 1.$$

Substituting the previous information back in (17) *we deduce $\mathrm{M}_\Omega^{(\alpha)}(h) = \mathcal{M}_{\partial\Omega}^{(\alpha)}(h)$ for any $h \in \mathcal{H}_\Omega$. Viceversa, if we assume*

$$\mathrm{M}_\Omega^{(\alpha)}(h) = \mathcal{M}_{\partial\Omega}^{(\alpha)}(h) \quad \textit{for any } h \in \mathcal{H}_\Omega,$$

then (11) *yields*

$$\frac{\mathrm{V}_\alpha^{\mathbb{H}}(\Omega)}{\mathcal{P}_\alpha^{\mathbb{H}}(\partial\Omega)} \int_{\partial\Omega} h(\xi) F_\alpha^{\frac{1}{2}}(\xi) d\sigma_H(\xi) = \mathrm{V}_\alpha^{\mathbb{H}}(\Omega) \mathcal{M}_{\partial\Omega}^{(\alpha)}(h)$$
$$= \mathrm{V}_\alpha^{\mathbb{H}}(\Omega) \mathrm{M}_\Omega^{(\alpha)}(h) = \int_\Omega h(\xi) F_\alpha d\xi$$
$$= \frac{1}{Q+\alpha-2} \int_{\partial\Omega} h(\xi)|D_H u(\xi)| d\sigma_H(\xi)$$

which says that

$$\int_{\partial\Omega} h(\xi) \left(|D_H u(\xi)| - \frac{(Q+\alpha-2)\mathrm{V}_\alpha^{\mathbb{H}}(\Omega)}{\mathcal{P}_\alpha^{\mathbb{H}}(\partial\Omega)} F_\alpha^{\frac{1}{2}}(\xi) \right) d\sigma_H(\xi) = 0 \quad \textit{for any } h \in \mathcal{H}_\Omega. \tag{18}$$

We can then consider the $\Delta_{\mathbb{H}}$-harmonic function $\bar{h}$ solving the Dirichlet problem

$$\begin{cases} \Delta_{\mathbb{H}} \bar{h} = 0 & \textit{in } \Omega, \\ \bar{h} = |D_H u| - \frac{(Q+\alpha-2)\mathrm{V}_\alpha^{\mathbb{H}}(\Omega)}{\mathcal{P}_\alpha^{\mathbb{H}}(\partial\Omega)} F_\alpha^{\frac{1}{2}} & \textit{on } \partial\Omega. \end{cases}$$

Thanks to the smoothness of $\partial\Omega$ and the exterior gauge condition, such Dirichlet problem has a unique classical solution, and we refer e.g. to [32] *for gradient bounds. Hence, we can plug $h = \bar{h}$ into* (18) *and we obtain*

$$\int_{\partial\Omega} \left(|D_H u(\xi)| - \frac{(Q+\alpha-2)\mathrm{V}_\alpha^{\mathbb{H}}(\Omega)}{\mathcal{P}_\alpha^{\mathbb{H}}(\partial\Omega)} F_\alpha^{\frac{1}{2}}(\xi) \right)^2 d\sigma_H(\xi) = 0.$$

From the vanishing of the previous integrand function, we realize that u solves in fact the overdetermined system

$$\begin{cases} \Delta_{\mathbb{H}} u = (Q+\alpha-2) F_\alpha & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ |D_H u| = c\, F_\alpha^{\frac{1}{2}} & \text{on } \partial\Omega \end{cases} \tag{19}$$

with

$$c = \frac{(Q+\alpha-2)\mathrm{V}_\alpha^{\mathbb{H}}(\Omega)}{\mathcal{P}_\alpha^{\mathbb{H}}(\partial\Omega)}.$$

Under the same hypotheses of Theorem 2.1 and Theorem 2.2, we showed in [23, Theorem 1.1 and Theorem 1.2] that the unique domain where (19) admits a solution is in fact a gauge ball. We summarize these results from [23] in the following

Theorem 2.3. *Fix $n \geq 1$, and $\alpha \in (\alpha_n, 4]$ where $\alpha_n = \frac{3}{4}$ for $n \geq 2$ and $\alpha_1 = 2$. Fix also $c > 0$. Let $\Omega \subset \mathbb{H}^n$ be a competitor set. Suppose that Ω is cylindrically symmetric, or even toric symmetric if $n = 2$. If there exists a solution u to* (19)*, then Ω is a gauge ball of radius $R = c^{\frac{2}{\alpha}}$.*

Thanks to the discussion in Remark 2.2, the previous theorem implies Theorems 2.1 and 2.2 once we recognize from (15) that

$$c = \frac{(Q+\alpha-2)\mathrm{V}_\alpha^{\mathbb{H}}(B_R(0))}{\mathcal{P}_\alpha^{\mathbb{H}}(\partial B_R(0))} = R^{\frac{\alpha}{2}}.$$

We stress that, whenever $\alpha < 4$, all the previous theorems characterize the gauge balls centered at the null element of the group 0. On the other hand, in the particular case $\alpha = 4$, the system (19) inherits an extra degree of freedom as the function $F_4(\xi) = |x|^2$ is independent of the t-variable: this fact is responsible for the existence of gauge-symmetric solutions in $\Omega = B_R((0, t_0))$ for any $t_0 \in \mathbb{R}$ and $R > 0$ which are given by

$u_4(\xi) = \frac{1}{4}(|x|^4 + (t-t_0)^2 - R^4)$. Hence, the characterizations in the previous theorems for $\alpha = 4$ ensure the existence of $t_0 \in \mathbb{R}$ such that $\Omega = B_R((0,t_0))$ (see [23, Theorem 4.2 and Theorem 4.4] for the precise statements).

Remark 2.3. *To understand the role of the partial symmetry assumptions, one should have in mind that such requirement introduces a sort of additional boundary condition on the axis of symmetry. For example, in the case of cylindrical symmetry, the function* $W(|x|^2, t) = u(x,t)$ *is forced to satisfy a homogeneous Neumann-like condition at the vertical axis* L_v. *Furthermore, in the interior of the relevant domain the pde that* W *satisfies is the following*

$$W_{\sigma\sigma} + \frac{n}{\sigma} W_\sigma + W_{tt} = \frac{2n+2}{4}\left(\sigma^2 + t^2\right)^{\alpha-4}.$$

It is then evident that the case $\alpha = 4$ *with cylindrical symmetry is the easiest one to be treated as the weight disappears in the cylindrical coordinates. We refer to* [25] *for partially overdetermined problems of Serrin-type where homogeneous Neumann conditions are prescribed on the boundary of fixed cones.*

As we mentioned in the Introduction, the proof of Theorem 2.3 follows the well-known P-function approach devised by Weinberger in [33]. We briefly highlight here the key steps. In [23, Section 3] we introduced the following function

$$v = \frac{1}{F_\alpha}|D_H u|^2 - \alpha u \tag{20}$$

associated with the solution u to our overdetermined system (19) in a competitor set. We stress that this function is smoothly defined only in $\overline{\Omega} \smallsetminus L_v$. Because of the peculiar Dirichlet and Neumann conditions for u, we know that v approaches the constant c^2 at $\partial\Omega$ (at least at the points of the boundary which are not on the vertical axis L_v). It is shown in [23, formula (29)] that

$$\int_\Omega v\, F_\alpha\, d\xi = c^2 \int_\Omega F_\alpha\, d\xi$$

for any $\alpha > 0$. This is saying that, with respect to the weight F_α, the function v is in average with its boundary datum c^2. In order to claim that v is the Weinberger-type P-function suitable for the problem (19) and to exploit the needed maximum principle argument, sub-harmonicity properties for the function v are essential. At this point the partial symmetry assumptions for Ω and the restrictions for the parameter α come into play. In [23, Corollary 3.1] we proved that, for any $0 < \alpha \leq 4$, the function v is $\Delta_{\mathbb{H}}$-subharmonic in $\Omega \setminus L_v$ under the geometric assumptions of cylindrical and toric symmetries stated in the theorems. The proof of this fact is highly non-trivial and it depends on a new differential identity for functions with toric symmetry established in [23, Lemma 3.1]. With this in hands, we adopted two different techniques in low dimensions and higher dimensions to conclude the maximum principle argument and this last step required the further restrictions on α: concerning the cylindrical case for $n \geq 2$ we dealt with the singularity via the local summability properties of $\Delta_{\mathbb{H}^n} v$, concerning the case $n = 1$ and the toric symmetric case in $n = 2$ we exploited the polarity of isolated points.

As a closing comment, we touch upon parallelisms with Alexandrov type problems. We studied the problem of characterizing gauge spheres via the prescription of the horizontal mean curvature (which can be defined as the horizontal divergence of the horizontal unit normal) in a companion project, and also in such case one can see the appearance of the relevant weights. In [13] we established a rigidity result for gauge spheres by assuming the horizontal mean curvature to be proportional to $F_4^{\frac{1}{2}}$ under partial symmetry assumptions of cylindrical type. Differently from the equation displayed in Remark 2.3 for the case $\alpha = 4$, the ode underlying such curvature problem cannot be treated via classical methods. Interestingly, this ode shows some analogies with the one analyzed in [14] for the constant Levi curvature characterization of spheres in $\mathbb{C}^2$ in the class of Reinhardt domains. We also refer to [20, 31, 22] for similar studies. In particular, in [22] we devised a pde-approach to this uniqueness result which works for both Reinhardt domains and starshaped circular domains in $\mathbb{C}^2$: the proof ultimately hinges on a smooth auxiliary function which enjoys subharmonicity properties under the assumption of constant Levi curvature. Having this in mind, in [13] we did not study the underlying ode for cylindrically symmetric hypersurfaces but we decided instead to follow the strategy of the classical Darboux' theorem for the characterization of umbilical surfaces: using the notion of horizontal umbilicality defined in [5] we were able to build the key auxiliary functions whose constancy provides the desired gauge symmetry. These auxiliary functions exhibit singularities on the characteristic set, see in this respect [13, formulas (3.2) and (3.11)]. It is worth noticing that also the P-function (20) has singularities exactly on the characteristic set of $\partial\Omega$, since the defining function u satisfies the constraint (16) which vanishes exactly on L_v. We plan to explore further these analogies in future studies.

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