UNIQUENESS RESULTS FOR VARIABLE COEFFICIENT SCHRÖDINGER EQUATIONS

RISULTATI DI UNICITÀ PER EQUAZIONI DI SCHRÖDINGER A COEFFICIENTI VARIABILI

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ABSTRACT. In this note we shall present some result proved in [16] on the uniqueness of Schrödinger equations with space-variable coefficients, that is Schrödinger equations where the Laplacian is replaced by an elliptic operator with space-variable coefficients.

SUNTO. In queste note presenteremo alcuni risultati dimostrati in [16] sull'unicità di soluzioni di equazioni di Schrödinger a coefficienti variabili dipendenti dallo spazio, ovvero equazioni di Schrödinger in cui il Laplaciano viene sostituito da un operatore ellittico a coefficienti variabili che dipendono dalla variabile spaziale.

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1. Introduction

In the present note we will discuss some results proved in [16] showing that linear Schrödinger equations with variable coefficients have a unique solution whenever suitable decay assumptions are satisfied. More precisely, let $P(t, x, D_t, D_x)$ be the variable coefficient Schrödinger operator of the form

$$P = \partial_t - i(\mathcal{L} + V) \quad \text{on } [0, 1] \times \mathbb{R}^n,$$

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where $V \in L^\infty(\mathbb{R}^n, \mathbb{R})$ and

$$\mathcal{L} = \mathcal{L}(x, D_x) = \sum_{j,k=1}^{n} \partial_k(a_{kj}(x)\partial_j)$$

is a second order elliptic operator defined by a real symmetric matrix $A(x) = (a_{kj}(x))_{k,j=1,\cdots,n}$ satisfyng the boundedness and ellipticity condition

$$\exists \lambda, \Lambda > 0,\ \forall x, \xi \in \mathbb{R}^n, \quad \lambda|\xi|^2 \leq \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \leq \Lambda|\xi|^2.$$

Then the question we will answer for P is the following:

Q1: Let $P(t, x, D_t, D_x)$ be as in (2.2), and let u_1, u_2 be two solutions of the equation $Pu = 0$ in $[0, 1] \times \mathbb{R}^n$. Under which decay assumptions on u_1, u_2 we have that $u_1 \equiv u_2$?

Question Q1 – concerning the uniqueness of the solution – can be formulated in the following equivalent form.

Q1: Let $P(t, x, D_t, D_x)$ be as in (2.2), and let u_1 be a solution of $Pu = 0$ in $[0, 1] \times \mathbb{R}^n$. Under which decay assumption on u_1 we have that $u_1 \equiv 0$?

The investigation of this type of problems for general variable coefficient partial differential equations (PDEs) started after the publication of the pioneering work by Carleman in 1939 (see [4]). In [4] Carleman proved a unique continuation result for linear elliptic equations on bounded domains in $\mathbb{R}^2$ which says the following: *if u solves the elliptic equation $\Delta u = V(x)u$ in a connected open subset $\Omega \subset \mathbb{R}^2$ (where $V \in L^\infty(\Omega)$), and if $u = 0$ in an open subset $\omega \subset \Omega$, then $u \equiv 0$ in Ω.*

After Carleman's work, *local* unique continuation questions – on bounded domains or in a neighborhood of each point of the domain – were studied and answered for more general variable coefficient partial differential operators (PDOs), in other words for non-elliptic operators whose principal part may have variable coefficients. Particular attention was given to the study of unique continuation properties of PDOs (and more generally of pseudo-differential operators) across hypersurfaces (see [22, 34, 41, 27]), investigation which lead to many local results across so-called *pseudoconvex hypersurfaces*. Of course unique continuation results can be seen as uniqueness results, for that we use the two terms *unique continuation* and *uniqueness* withouth distinction.

We stress once more that the results listed above are *local.* Moreover, the initial assumption on the solution u of the homogeneus equation is a *vanishing assumption*, in that the hypothesis on u is that it vanishes in a subdomain of the domain, exactly as in Carleman's original theorem.

When dealing with evolution equations, we wish to consider unbounded domains, since very often the goal is to see what happens globally in space. For Schrödinger and other dispersive equations with constant coefficients in the principal part, uniqueness results on unbounded domains have been proved, as in the bounded domain case, under *vanishing conditions*, that is assuming that the solution vanishes somewhere (see [24, 25]) in the space-time domain.

However, for some dispersive equations, specifically for the Schrödinger, the KdV and its 2-dimensional generalization called the ZK equation, a different type of questions, that is of the form $Q1$, have been recently answered. The novelty here resides in assuming *decay conditions* on the solution of the PDE – instead of *vanishing conditions*– which allow to conclude that the solution must be identically zero on the whole (unbounded) space-time domain.

This new approach to the uniqueness of solutions to dispersive equations, approach based on the decay of the solution, was introduced in the works [13, 11, 10, 31] for Schrödinger equations with potentials. As the authors of the listed references clearly pointed out, the new type of assumption – an L^2-exponential decay of Gaussian-type of the solution – was inspired by the so called *Hardy uncertainty Principle* (see [21]), due to the relation between the solution to the free (constant coefficient) Schrödinger equation and its Fourier transform when these two are considered at two different times. A brief discussion of Hardy uncertainty principles and of its L^2-variants is given below in Subsection 2.1, and we refer the interested reader to that part for more details on the argument.

For Schrodinger equations with and without potential, uniqueness results under an exponential decay condition – also called *Hardy uncertainty principles* – have been proved by Escauriaza, Kenig, Ponce and Vega in [13, 11, 10], while results with gradient terms were proved by Dong and Staubach in [9]. For the KdV and the ZK equation, uniqueness results can be found in [6, 12, 26, 30, 37, 42]. Other dispersive models including discrete Schrödinger equations have been treted by Bertolin and Vega in [3] and by Jaming, Kyubarski, Malinikova and Perfekt in [28], whereas higher order Schrödinger equations – where the Laplacian is replaced by a constant coefficient higher order elliptic operator – have been considered by Huang, Huang and Zheng in [23], and by Lee and Yu in [33]. Finally, Hardy uncertainty principles for magnetic Schrödinger equations can be found in the work by Cassano and Fanelli [5] (see also [2] and references therein).

Now that the picture about known *global* results, menaing on unbounded domains, for dispersive equations has been clarified, we wish to explain the novelty of the results in [16] that we are going to analyze in detail in the present paper. The difference between the operator in (2.2) and those analyzed in the aforementioned papers, is that we deal with variable coefficient Schrödinger operators in which the Laplacian is replaced by an elliptic operator with *space-variable coefficients*. Hence, the variable coefficients in our Schrödinger equation, are not introduced by the potential or by first order terms with variable coefficients as in the papers cited above, but they (also) appear in the principal symbol of the operator. This forces a new formulation of the problem, that must be based not only on the decay assumption on the solution, but also on suitable decay assumptions on the coefficients.

Let us say that the study of Schrödinger operators with variable coefficients is connected to different physical phenomena, especially in nonlinear optics. For example, the case where the Laplacian is replaced by an elliptic operator with time-variable coefficients analyzed in [17, 19], is also used in physics to understand soliton's behaviour. In particular, one can manipulate soliton's behaviour by changing the time-dependent operator's coefficients (see [43] and references therein). As for operators with space-variable coefficients, they have been studied, for instance, in [18, 8, 29, 36, 39, 38] (see also reference therein) from the mathematical point of view. Their use in physics, as in the time-variable coefficients case, is related to soliton control, and we refer to [20] and references therein for more details on the topic.

We conclude this introduction by explaining the structure of this note.

Section 2 will be composed of two subsections. In Subsection 2.1 we recall the original version of Hardy uncertainty principle and of its generalizations, and give the equivalent statements for free Schrödinger equations with constant coefficients. Afterwards, in Subsections 2.2, we state the main results of [16] holding for variable coefficient Schrödinger operators. This order of presentation will allow us to make a comparison between the results in Subsection 2.1 and 2.2 for constant and for variable coefficient operators, respectively.

In Section 3 we will delve into the strategy of the proof of the uniqueness results stated in Subsection 2.2. Here we will show some complementary results derived in [16] needed for the proof.

Finally, the note will end with Section 4 containing some final remarks and connected open problems.

2. Results for variable coefficient Schrödinger operators: the general and the structural case

This section will be devoted to the statement of the main uniqueness results obtained in [16]. These will be given in Theorem 2.4 and Theorem 2.5 in Subsection 2.2. To explain the connection with Hardy uncertainty principles, and especially to compare the variable coefficient results with those previously established for constant cefficient operators, we devote Subsection 2.1 to some Hardy uncertainty principles including those for Schrödinger equations.

2.1. Hardy uncertainty principles for Schrödinger equations.

Hardy uncertainty principle – indeed proved by Hardy in 1933 – was originally given in dimension one in the following form.

Theorem 2.1 (Hardy uncertainty Principle 1933 [21]). *For any function $f : \mathbb{R} \to \mathbb{C}$, if f itself and its Fourier transform $\widehat{f}$ satisfy*

$$f(x) = \mathcal{O}(e^{-Ax^2}) \quad \textit{and} \quad \widehat{f}(\xi) = \mathcal{O}(e^{-4B\xi^2}),$$

with $A, B > 0$ and $AB > 1/16$, then $f \equiv 0$. Moreover, if $A = B = 1/4$, then $f(x) = Ce^{-x^2/4}$.

Hardy's result was later generalized in any dimension by Sitaram, Sundary and Tangavelu [40] in 1995, hence assuming the corresponding higher dimensional condition

$$f(x) = \mathcal{O}(e^{-A|x|^2}) \quad \text{and} \quad \widehat{f}(\xi) = \mathcal{O}(e^{-4B|\xi|^2}).$$

A further generalization of the previous principle is due to Cowling and Price [7], and it includes the following L^2-variant of Hardy's uncertainty principle.

Theorem 2.2 (L^2-Hardy uncertainty Principle). *If $e^{A|x|^2} f \in L^2(\mathbb{R}^n)$ and $e^{4B|x|^2}\widehat{f} \in L^2(\mathbb{R}^n)$, with $A, B > 0$ and $AB \geq 1/16$, then $f \equiv 0$.*

Next, consider the solution of the free Schrödinger equation

$$i\partial_t u + \Delta u = 0, \quad u(0,x) = u_0(x), \quad (t,x) \in (0,1] \times \mathbb{R}^n,$$

that is

$$u(t,x) = e^{it\Delta} u_0(x) := \mathcal{F}^{-1}(e^{-it|\xi|^2}\widehat{u}_0)(x) = \frac{e^{\frac{i|x|^2}{4t}}}{(2it)^{n/2}} \mathcal{F}(e^{-i|\cdot|^2} u_0)\Big(\frac{x}{2t}\Big). \tag{1}$$

Remark 2.1. *The reader may have noticed that in the L^2-variant of Hardy's principle we have $f \equiv 0$ even when $AB = 1/16$. This property indeed holds by virtue of the result in [7]. To be precise, the general statement of Cowling and Price works under the assumptions $e^{A|x|^2} f \in L^p(\mathbb{R}^n)$, $e^{4B|x|^2}\widehat{f} \in L^q(\mathbb{R}^n)$ and $1 \leq p, q \leq \infty$, and gives $f \equiv 0$ even when $AB = 1/16$ whenever $\min(p,q) < \infty$.*

An application of Theorem 2.1 and of Theorem 2.2 to u as in (1) provides two principles for solutions of free Schrödinger equations. If the free solution u has a certain behaviour at the initial time $t_0 = 0$ and at the final time T, then it has a prescribed behaviour at any intermediate time $t \in (0,T)$.

Hardy uncertainty principle for free Schrödinger equations. If $u(0,x) = \mathcal{O}(e^{-\frac{1}{\alpha^2}|x|^2})$ and $u(T,x) = e^{iT\Delta}u(0,x) = \mathcal{O}(e^{-\frac{4}{\beta^2}|x|^2})$, $T > 0$, and $\alpha\beta < 4T$, then $u(t,x) \equiv 0$ in $[0,T] \times \mathbb{R}^n$. Moreover, if $\alpha\beta = 4T$ then u is the solution with initial data $u(0,x) = \omega e^{-\left(\frac{1}{\beta^2} + \frac{i}{4T}\right)|x|^2}$, for some $\omega \in \mathbb{C}$.

L^2***- Hardy uncertainty principle for free Schrödinger equations.*** if $e^{\frac{1}{\alpha^2}|x|^2}u(0,x)$ and $e^{\frac{1}{\beta^2}|x|^2}u(T,x)$ are in $L^2(\mathbb{R}^n)$, and $\alpha\beta \leq 4T$, then $u(t,x) \equiv 0$ in $[0,T] \times \mathbb{R}^n$.

Notice that the first part of these principles can be seen as uniqueness results. In fact they show that if the solution has a certain behaviour at two different times, then it must be identically zero.

As already mentioned in the Introduction, uniqueness problems for Schrödinger equations with potentials, that is of the form

$$i\partial_t u + (\Delta + V)u = 0, \quad V \not\equiv 0,$$

were first considered by Escauriaza, Kenig, Ponce and Vega. These authors, inspired by the previous principles, obtained the following Hardy uncertanty principle for Schrödinger equations with potential.

Theorem 2.3. ***Hardy uncertainty principle for Schrödinger equations with potential.*** *Assume that* $u \in C([0,1], L^2(\mathbb{R}^n))$ *solves* $i\partial_t u + \Delta u = V(t,x)u, \quad (t,x) \in [0,1] \times \mathbb{R}^n$. *Then, under proper boundedness and decay assumptions on* V, *if for* $A, B > 0$ *with* $AB > 1/16$, *we have*

$$\left\| e^{A|x|^2} u(0,x) \right\|_{L^2(\mathbb{R}^n)} + \left\| e^{B|x|^2} u(1,x) \right\|_{L^2(\mathbb{R}^n)} < \infty, \tag{2}$$

then $u \equiv 0$.

This theorem, as the aformentioned principles, gives information about the solution u at any time of existence assuming a certain decay of the solution at the initial and the final time. The information given by the theorem is very precise, that is, under the above L^2-decay condition, the solution of the equation must be identically zero. This fact translates in a uniqueness result for solutions to Schrödinger equations with potential.

Let us finally stress that the decay assumption in order to have a unique solution in presence of potentials is an L^2-Gaussian decay of $u(0,x)$ and $u(T,x)$, which is the L^2-decay required when $V \equiv 0$, hence the sharp one. This is important to keep in mind since this decay rate will be compared with the one needed in the variable coefficients case.

2.2. **Variable coefficients Schrödinger equations: results.** We recall that the *variable coefficient* Schrödinger equations we are interested in are those of the form

$$Pu = \partial_t u - i(\mathcal{L} + V)\, u = 0, \quad (t,x) \in [0,1] \times \mathbb{R}^n,$$

where

$$\mathcal{L} = \mathcal{L}(x) = \sum_{j,k=1}^{n} \partial_k (a_{kj}(x)\partial_j),$$

$V \in L^\infty(\mathbb{R}^n, \mathbb{R})$, and $A(x) = (a_{kj}(x))_{k,j=1,\cdots,n}$ is a real symmetric matrix satisfyng the boundedness and ellipticity condition

$$\exists \lambda, \Lambda > 0, \ \forall x, \xi \in \mathbb{R}^n, \quad \lambda|\xi|^2 \le \sum_{j,k=1}^{n} a_{jk}(x)\xi_j\xi_k \le \Lambda|\xi|^2.$$

Here the coefficients are said to be variable not because of the presencee of the potential V (which can be assumed identically zero), but due to the presence of a second order elliptic operator $\mathcal{L}$ having, in general, variable coefficients (of course the constant coefficients case is covered as well).

To summarize the previous discussion, the question we will answer concerning these operators is the following.

Question: Assume that u solves $Pu = 0$ in $[0,1] \times \mathbb{R}^n$. Under which

(1) conditions on $A(x)$,
(2) decay assumption on $u(0,x), u(1,x)$,

we have that $\mathbf{u} \equiv \mathbf{0}$?

This question was answered in [16] in two different cases, called, respectively, the *general* and the *structural* case. We state below the uniqueness results we have in the two cases, and comment on them right afterwards. We specifiy that the notation $A \in C^k(\mathbb{R}^n)$ used from now one stands for $a_{jk} \in C^k(\mathbb{R}^n)$ for all $i, j = 1, \ldots, n$.

Theorem 2.4 (Uniqueness in the general case)**.** *Let $u \in C([0,1], L^2(\mathbb{R}^n))$ be a solution to $Pu = 0$ with*

$$A \in C^3(\mathbb{R}^n),\ V = V(x) \in L^\infty(\mathbb{R}^n, \mathbb{R}). \tag{3}$$

Then there exists a small positive number $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda)$, such that if

$$\sup_{\mathbb{R}^n} |x||\nabla A| \le \varepsilon_0 \tag{4}$$

and

$$e^{k|x|^3}|u(0,x)|,\ e^{k|x|^3}|u(1,x)| \in L^2_x(\mathbb{R}^n), \quad \forall k > 0,$$

then $u \equiv 0$.

Now we assume the following transversally anisotropic type condition on $A(x)$: under a suitable choice of coordinates, for $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$,

$$A = A(x_1, x') = \begin{pmatrix} a_{11}(x_1) & 0 \\ 0 & \widetilde{A}(x') \end{pmatrix}, \quad \text{where } \widetilde{A} \text{ is symmetric.}$$

Theorem 2.5 (Uniqueness in the structural case)**.** *Let* $u \in C([0,1], L^2(\mathbb{R}^n))$ *be a solution to* $Pu = 0$*, let* A *with the structure above be s.t.*

$$a_{11} \in C^2(\mathbb{R}), \quad \widetilde{A} \in C^3(\mathbb{R}^{n-1}), \text{ and } V = V(x) \in L^\infty(\mathbb{R}^n, \mathbb{R}). \tag{5}$$

Then there exists a small positive number $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda)$ *and a large number* $k = k(n, \lambda, \Lambda, \|a_{11}\|_{C^2}, \|\tilde{A}\|_{C^3})$*, such that if*

$$\sup_{\mathbb{R}^{n-1}} |x'||\nabla_{x'} \widetilde{A}| \leq \varepsilon_0, \tag{6}$$

and

$$e^{k|x|^2}|u(0,x)|, \; e^{k|x|^2}|u(1,x)| \in L^2_x(\mathbb{R}^n),$$

then $u \equiv 0$.

We shall refer to (3) and (4) as the *regularity* and *smallness* conditions when dealing with the general case. Similarly, we shall refer to (5) and (6) as the *regularity* and *smallness* conditions when dealing with the structural case.

Let us now give a few comments on the above statements. Theorem 2.4 is for operators in general form, that is with the minimum regularity requirement on the coefficients of $\mathcal{L}$. Nevertheless, compared to Theorem 2.3 we need to ask higher that L^2-Gaussian decay for $u(0,x)$ and $u(T,x)$, indeed due to the generality of the considered framework. On the other hand, by adding the structural condition in Theorem 2.5, we can get a sharp result in terms of decay requirement, since by requiring L^2-Gaussian type decay of $u(0,x)$ and $u(T,x)$ – the same as in the constant coefficients case – we reach the desired conclusion.

3. Complementary results and strategy of the proof

The strategy to prove Theorem 2.4 and Theorem 2.5 is based on three key ingredients: a log-convexity result, a Carleman estimate, and a lower bound for a suitable localized norm of the solution. For completeness we state these complementary results – which, per se, show other properties of the solution – below.

3.1. log-**convexity.** The log-convexity result in Theorem 3.1 should be interpreted in the following way: if the solution u of the variable coefficient Schrödinger equation $Pu = 0$ has a certain L^2-exponential decay at two different times, then the same rate of decay is possessed by u at any intermediate time. The result is a *log*-convexity result because it amounts to the *log*-convexity of $H(t) := \|e^{\beta|x|^2}u(t,x)\|^2_{L^2_x}$. Moreover, it gives the boundedness of a weighted $L^2_tH^1_x$-norm of u, an estimate which is crucial to get the proof of our main results.

Theorem 3.1 (log-convexity)**.** *Let $u \in C([0,1], L^2(\mathbb{R}^n))$ be a solution to $Pu = 0$ with $A \in C^3(\mathbb{R}^n), M_1 := \|V\|_{L^\infty} < \infty$. Then there exist a small enough $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda) > 0$ and a large enough $\beta_0 = \beta_0(n, \lambda, \Lambda, \|A\|_{C^3})$, s.t. if*

$$\sup_{\mathbb{R}^n} |x||\nabla A| \le \varepsilon_0, \ \text{and } e^{\beta|x|^2}u(0,x),\ e^{\beta|x|^2}u(1,x) \in L^2(\mathbb{R}^n), \quad \text{with } \beta > \beta_0,$$

then for all $t \in (0,1)$, we have

$$\|e^{\beta|x|^2}u(t,x)\|^2_{L^2_x} \le Ce^{M_1^2}\left(\|e^{\beta|x|^2}u(0,x)\|^2_{L^2_x}\right)^{1-t}\left(\|e^{\beta|x|^2}u(1,x)\|^2_{L^2_x}\right)^t,$$

and

$$\begin{aligned}&\beta\|\sqrt{t(1-t)}e^{\beta|x|^2}|\nabla u|\|^2_{L^2_{t,x}} + \beta^3\|\sqrt{t(1-t)}e^{\beta|x|^2}|xu|\|^2_{L^2_{t,x}}\\ &\le Ce^{M_1^2}(\|e^{\beta|x|^2}u(0,x)\|^2_{L^2_x} + \|e^{\beta|x|^2}u(1,x)\|^2_{L^2_x}).\end{aligned}$$

Here C is an absolute constant and $L^2_{t,x} := L^2([0,T] \times \mathbb{R}^n)$.

The subsequent corollary shows that the same *log*-convexity result as above holds with higher order exponential weights if the solution enjoys additional decay. This, in particular, is needed to prove Theorem 2.4.

Corollary 3.1. *Under the assumptions of Theorem 3.1, $\forall\alpha > 1$, $\exists\kappa_0 = \kappa_0(\beta_0, \alpha)$, s.t. if we further assume that for some $\kappa > \kappa_0$*

$$e^{\kappa|x|^{2\alpha}}u(0,x),\ e^{\kappa|x|^{2\alpha}}u(1,x) \in L^2(\mathbb{R}^n),$$

then for any $t \in (0,1)$, we also have

$$e^{\kappa|x|^{2\alpha}}u(t,x) \in L^2(\mathbb{R}^n),$$

$$\sqrt{t(1-t)}e^{\kappa|x|^{2\alpha}}\nabla u(t,x),\ \sqrt{t(1-t)}e^{\kappa|x|^{2\alpha}}xu(t,x) \in L^2([0,1], L^2(\mathbb{R}^n)),$$

with estimates

$$\|e^{\kappa|x|^{2\alpha}}u(t,x)\|^2_{L^2_x} \leq Ce^{M_1^2}\left(\|e^{\kappa|x|^{2\alpha}}u(0,x)\|^2_{L^2_x}\right)^{1-t}\left(\|e^{\kappa|x|^{2\alpha}}u(1,x)\|^2_{L^2_x}\right)^t,$$

and

$$\beta\|\sqrt{t(1-t)}e^{\kappa|x|^{2\alpha}}|\nabla u|\|^2_{L^2_{t,x}} + \beta^3\|\sqrt{t(1-t)}e^{\kappa|x|^{2\alpha}}|xu|\|^2_{L^2_{t,x}}$$
$$\leq Ce^{M_1^2}(\|e^{\kappa|x|^{2\alpha}}u(0,x)\|^2_{L^2_x} + \|e^{\kappa|x|^{2\alpha}}u(1,x)\|^2_{L^2_x}).$$

3.2. **Carleman estimates.** Next, we discuss the second crucial ingredient in the proof of Theorems 2.4 and 2.5, i.e. Carleman estimates. In the two cases, general and structural, this estimate is slightly different. However, this small difference becomes essential to get a sharp lower bound estimate for a certain norm of the solution (see Subsection 3.3), which, in the end, translates into a sharp uniqueness in terms of decay assumptions.

It will be clear from the following statements that Carleman estimates are weighted estimates of the form

$$\|e^{\phi}Pu\|_X \geq C\|e^{\phi}u\|_Y, \quad \forall u \in C_0^\infty(\Omega)$$

for some X, Y functional spaces, usually Lebesgue or Sobolev spaces or a weighted version of them, and for some open set $\Omega \subseteq \mathbb{R}^n$. The most challenging part is to find a *weight function* ϕ allowing for an estimate of this kind and being suitable for a cutting-off procedure employed in the proof of the uniqueness theorems.

Theorem 3.2 (Carleman estimate in the general case). *Assume $A \in C_b^3(\mathbb{R}^n)$, $\varphi = \varphi(t) \in C_c^\infty(\mathbb{R})$, and $r_0 > 0$. There exists an $\varepsilon_0 = \varepsilon_0(n, \lambda, \Lambda)$, such that if*

$$\sup_{x \in \mathbb{R}^n} |x||\nabla A| < \varepsilon_0,$$

then for any function

$$f \in C_c^\infty(\mathbb{R} \times B_{r_0}^{\mathsf{c}})$$

and β, R satisfying

$$\beta \geq \beta_1 := \max\{\lambda^{-1}\|\varphi''\|_{L^\infty}^{1/2} r_0^{-1} R^3, C_1(1 + r_0^{-1})R^2\}, \quad R \geq 1$$

with $C_1 = C_1(n, \lambda, \Lambda, \|A\|_{C^3})$, we have

$$\begin{aligned}\beta R^{-2} \int_{\mathbb{R}} \int_{\mathbb{R}^n} |\nabla f|^2 \, dx dt &+ \beta^3 R^{-6} \int_{\mathbb{R}} \int_{\mathbb{R}^n} |x|^2 |f|^2 \, dx dt \\ &\leq \lambda^{-2} \int_{\mathbb{R}} \int_{\mathbb{R}^n} \left| e^{\beta(|x/R|^2 + \varphi(t))} (i\partial_t + \mathcal{L}) e^{-\beta(|x/R|^2 + \varphi(t))} f \right|^2 dx dt.\end{aligned}$$

If we set $f = e^{\beta(|x/R|^2 + \varphi(t))} u$ and $\phi(t, x) = \beta(|x/R|^2 + \varphi(t))$, we have

$$\begin{aligned}\beta R^{-2} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{2\phi} |\nabla u|^2 \, dx dt &+ \beta^3 R^{-6} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{2\phi} |x|^2 |u|^2 \, dx dt \\ &\lesssim \lambda^{-2} \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{2\phi} \left| (i\partial_t + \mathcal{L}) u \right|^2 dx dt.\end{aligned}$$

Theorem 3.3 (Carleman estimate in the structural case). *Let $R > 1$ and $\varphi = \varphi(t) \in C_c^\infty(\mathbb{R})$. Then there exists a large constant $c_0 = c_0(n, \lambda, \Lambda, \|A\|_{C^3}, \|\varphi'\|_{L^\infty}, \|\varphi''\|_{L^\infty}) > 0$, such that if the smallness and regularity conditions are satisfied, then for any $R \geq 1$, $f \in C_c^\infty(\mathbb{R} \times \mathbb{R}^n)$ with*

$$\operatorname{supp}(f) \subset \{|x/R + \varphi(t)\vec{e}_1| \geq 1\},$$

and

$$\beta \geq \beta_3 := c_0 R^2,$$

we have

$$\begin{aligned}\frac{\beta}{R^2} \|\nabla_x f\|_{L^2_{t,x}}^2 + &\frac{\beta^3}{R^6} \||x/R + \varphi \vec{e}_1| f\|_{L^2_{t,x}}^2 \\ &\leq C \|e^{\beta|x/R + \varphi\vec{e}_1|^2} (i\partial_t + \mathcal{L}) e^{-\beta|x/R + \varphi\vec{e}_1|^2} f\|_{L^2_{t,x}}^2,\end{aligned}$$

where $\vec{e}_1$ *is the unit vector* $(1,0,\dots,0)$. *If we set* $f = e^{|x/R+\varphi(t)\vec{e}_1|}u$ *and* $\phi(t,x) = \beta|x/R + \varphi(t)\vec{e}_1|$, *we have*

$$\begin{aligned} &\frac{\beta}{R^2}\|e^{\phi}\nabla_x f\|^2_{L^2_{t,x}} + \frac{\beta^3}{R^6}\||x/R+\varphi\vec{e}_1|e^{\phi}u\|^2_{L^2_{t,x}} \\ &\le C\|e^{\phi}(i\partial_t + \mathcal{L})u\|^2_{L^2_{t,x}}, \end{aligned} \tag{7}$$

Remark 3.1. *Observe that in Theorems 3.2 and 3.3 the weight function* ϕ *in the weighted norms is different, respectively* $\phi(t,x) = \beta(|x/R|^2 + \varphi(t))$ *and* $\phi(t,x) = \beta|x/R + \varphi(t)\vec{e}_1|$. *In addition, the functions on which the two theorems apply are supported in different sets. This is due to the choice of the weight function, which, in turn, is dictated by the form of the operator.*

3.3. **Lower bounds for nontrivial solutions and proof of the main results.** We can finally state the last fundamental ingredient to prove the uniqueness theorems. We will provide two lower bounds for nontrivial solutions valid in the general and in the structural case. These bounds are stated in Theorem 3.4 and Theorem 3.5 respectively. Let us also say that the proof of the subsequent theorems relies on the use of the previous Carleman estimates.

Theorem 3.4 (Lower bound in the general case)**.** *Let* $u \in L^\infty([0,1], L^2(\mathbb{R}^n)) \cap L^2_{loc}((0,1), H^1(\mathbb{R}^n))$ *solves* $Pu = 0$ *where* $A = A(x)$ *satisfy the smallness condition* (4) *in Theorem 2.4 and* $V \in L^\infty(\mathbb{R}\times\mathbb{R}^n, \mathbb{C})$. *Denote by* $M_1 := \|V\|_{L^\infty}$.

Furthermore, let $E_1, E_2, R_0, \varepsilon$ *be the numbers such that*

$$\int_{1/8}^{7/8}\int_{\mathbb{R}^n} (|u(t,x)|^2 + |\nabla u(t,x)|^2)\, dtdx \le E_1^2 < \infty$$

and

$$\int_{1/4}^{3/4}\int_{B_{R_0}\setminus B_{2\varepsilon}} |u(t,x)|^2\, dtdx \ge E_2^2,$$

then there exist positive constants $R_1 = R_1(n,\lambda,\Lambda,\|A\|_{C^3},\varepsilon,M_1,E_1,E_2,R_0)$, $C_0 = C_0(\lambda,\varepsilon)$, *and* $C = C(n,\lambda,\Lambda,\|A\|_{C^3},\varepsilon,M_1,E_1,E_2,R_0)$, *such that, for any* $R > R_1$,

$$\delta(R) := \int_{1/8}^{7/8}\int_{B_R\setminus B_{R-1}} (|u(t,x)|^2 + |\nabla u(t,x)|^2)\, dxdt \ge C\, e^{-C_0 R^3}.$$

Theorem 3.5 (Lower bound in the structural case). *Let $u \in L^\infty L^2 \cap L^2 H^1$ be a solution of $Pu = 0$ with A satisfying the smallness and regularity assumption* (6) *in Theorem 2.5. Let also $V \in L^\infty([0,1] \times \mathbb{R}^n, \mathbb{R})$ and $M_1 := \|V\|_{L^\infty}$.*

Furthermore, let E_1, E_2, R_0 be the numbers such that

$$\int_{1/8}^{7/8} \int_{\mathbb{R}^n} \left(|u(t,x)|^2 + |\nabla u(t,x)|^2\right) dxdt \le E_1^2 < \infty$$

and

$$\int_{1/4}^{3/4} \int_{B_{R_0}} |u(t,x)|^2 \, dxdt \ge E_2^2 > 0,$$

then there exist some positive constants $R_1 = R_1(n, \lambda, \Lambda, \left\|\tilde{A}\right\|_{C^3}, M_1, E_1, E_2, R_0)$, $C_0 = C_0(n, \lambda, \Lambda, \|\tilde{A}\|_{C^3})$, and $C_1 = C_1(n, \lambda, \Lambda, \left\|\tilde{A}\right\|_{C^3}, M_1, E_1, E_2, R_0)$, such that, for any $R > R_1$,

$$\delta(R) := \int_{1/8}^{7/8} \int_{B_R \setminus B_{R-1}} (|u(t,x)|^2 + |\nabla u(t,x)|^2) \, dxdt \ge C_1 \, e^{-C_0 R^2}.$$

Remark 3.2. *In Theorem 3.4 the lower bound estimate in the general case involves an exponential function with exponent $-C_0 R^3$, while the estimate in the structural case exhibits a quadratic exponent on the right hand side, that is $-C_0 R^2$. This is the origin of the decay conditions in Theorems 2.4 and 2.5.*

Let us now sketch the proof of Theorems 2.4 and 2.5 by using the results presented so far in this section.

proof of Theorems 2.4 and 2.5. Assume that the hypotheses of Theorem 2.4 are satisfied when dealing with the general case, and that those of Theorem 2.5 hold in the structural case.

We proceed by contradiction assuming that $u \not\equiv 0$. Then (due to the form of the solution when V is real) there exist $R_0 > 0$ sufficiently large and $\varepsilon > 0$ sufficiently small such that

$$\int_{1/4}^{3/4} \int_{B_{R_0} \setminus B_{2\varepsilon}} |u(t,x)|^2 \, dxdt \in (0, \infty).$$

Now the last part of Theorem 3.1 yields

$$\int_{1/8}^{7/8}\int_{\mathbb{R}^n}(|u(t,x)|^2+|\nabla u(t,x)|^2)\,dxdt<\infty,$$

which shows that the hypotheses in Theorem 3.4 and in Theorem 3.5 are satisfied in the general and in the structural case respectively. An application of these theorems gives

(8)

$$\int_{1/8}^{7/8}\int_{B_{R+1}\setminus B_R}(|u(t,x)|^2+|\nabla u(t,x)|^2)\,dxdt\geq\begin{cases}C_1e^{-C_0R^3}, & \text{in the general case}\\ C_1e^{-C_0R^2}, & \text{in the structural case.}\end{cases}$$

Next, by Theorem 3.1 and Corollary 3.1, for $\kappa>\kappa_0$, we have

$$\int_0^1\int_{\mathbb{R}^n}e^{\kappa|x|^\alpha}\left(|u(t,x)|^2+t(1-t)|\nabla u(t,x)|^2\right)dxdt<\infty,$$

implying that

$$e^{\kappa R^\alpha}\left(\int_{1/8}^{7/8}\int_{B_{R+1}\setminus B_R}(|u(t,x)|^2+|\nabla u(t,x)|^2)\,dxdt\right)=0$$

with $\alpha=3$ in the general case, and with $\alpha=2$ in the structural case. Therefore, choosing κ large enough and $\alpha=3$, we reach a contradiction with the lower bound (8) in the general case. By the same argument with $\alpha=2$ we reach a contradiction with the lower bound (8) in the structural case. This concludes the proof. □

4. Final remarks

In this final section we wish to make some comments on the role of the assumptions used in the uniqueness theorems, i.e. the smallness and the structural condition, and share some open problems close to that studied in this note.

The smallness conditions (4) and (6) are quite different, in the sense that the second one is not exactly a translation of the first one, but a new suitable requirement. Moreover, smallness assumptions of this kind – and even more restrictive ones – are used to show the validity of smoothing and Strichartz estimates for variable coefficient Schrödinger operators as in (2.2) (see, for instance, [39, 38, 36]). The reason behind the smallness condition is that it is related with the behaviour of the bicharacteristics of the operator $\mathcal{L}$, or, in other words, with the so called *non-trapping condition.*

The structural condition in Theorem 2.5 is not surprising. A similar structure – giving a translation invariance in one direction – is also required in [5] to get a sharp uniqueness result for magnetic Schrödinger equations, that is to say for operators involving an elliptic operator with constant coefficient leading part and variable coefficient lower order terms. This raises the question of whether it is possible to remove the structural condition and get the sharp result (in terms of decay) in the general case.

Uniqueness results for other dispersive equations have been investigated for KdV and ZK equations, i.e. involving constant coefficients operators. It is an open problem to understand the validity of Hardy-type uniqueness results as the preceding ones, that is based on decay assumptions, for variable coefficient operators of this type. A first step in this direction is the proof of a suitable Carleman estimate. Such an estimate has been recently derived in [14] for some KdV-type operators with variable coefficients in a local framework. This estimate is not ready to be used in a uniqueness argument, however it can give an insight on the weighted estimate one can hope for.

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