A BOURGAIN-BREZIS'S DUALITY ARGUMENT FOR CONTINUOUS PRIMITIVES
UN ARGOMENTO DI DUALITÀ ALLA BOURGAIN-BREZIS PER PRIMITIVE CONTINUE

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Abstract. In this note we collect some results in $\mathbb{R}^n$ about global Poincaré inequalities for differential forms obtained in a joint research with Pierre Pansu and presented by the authors in two seminars held in Bologna respectively in 2023 and 2024. At the end of the note we comment some very new results obtained in the Heisenberg groups $\mathbb{H}^n$.

Sunto. In questa nota presentiamo alcuni risultati che riguardano disuguaglianze di Poincaré per forme differenziali. Questi risultati sono stati ottenuti in collaborazione con Pierre Pansu e presentati dagli autori in due seminari tenuti a Bologna rispettivamente nel 2023 e 2024. Alla fine della nota commenteremo alcuni risultati ottenuti nei gruppi di Heisenberg $\mathbb{H}^n$.

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1. Introduction

Let us start by recalling what we mean by a (p, q)-Poincaré inequality for differential forms in $\mathbb{R}^n$. When dealing with differential forms there is a well known topological problem, whether a given closed form is exact. Besides, for several applications to the cohomology theory for example, we can study also an analytical problem: Whether a primitive ϕ of a given exact form ω can be upgraded to one which satisfies a (p, q)-estimate of the type $\|\phi\|_q \le c\|\omega\|_p$. More precisely, if $1 \le p \le n$, we ask whether, given a closed differential h-form ω in $L^p(\mathbb{R}^n)$, there exists an $(h-1)$-form ϕ in $L^q(\mathbb{R}^n)$ for some

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$q \geq p$ such that $d\phi = \omega$ and

$$\|\phi\|_q \leq C\,\|\omega\|_p, \tag{1}$$

for $C = C(n, p, q, h)$. We refer to the above inequality as to the (p, q)-Poincaré inequality for h-forms (notice that, for $1 < p < n$, by the scale invariance, we must have $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$). Also the end-point situation $p = 1$ and $p = n$ have been considered in several papers. We refer to [9] for a long list of references. The result in $\mathbb{R}^n$ can be summarized as follows.

Theorem 1.1. *Let $h = 1, 2, \cdots n$ and let ω be a closed h-differential form in $L^p(\mathbb{R}^n)$.*

i) If $1 < p < n$ then (1) holds for any h.

ii) If $p = 1$ then (1) holds for any $h < n$.

iii) If $p = n$ then (1) holds for any $h \geq 2$.

Here we mention only a result, about the case $p = n$, covered by Bourgain and Brezis (see [15] for differential form in $\mathbb{R}^n$) and, much more recently, by Baldi, Franchi and Pansu in [8] in the setting of Heisenberg groups (and also in a more general subriemannian setting). In [14], which deals with functions, i.e. 0-forms, the statement takes the following form: *let ω be an exact n-form on the n-torus, which belongs to L^n, then there exists a bounded differential $(n-1)$-form ϕ on the torus such that $d\phi = \omega$ and*

$$\|\phi\|_{L^\infty} \leq C\|\omega\|_{L^n}.$$

Furthermore, Bourgain and Brezis show that the primitive can be taken to be *continuous*, with a similar estimate.

In this note we want to cover the endpoint limiting case $p = n$ for differential forms of any degree by showing the following global Poincaré inequality:

Theorem 1.2. *Let $2 \leq h \leq n$, and let $\omega \in L^n$ be a closed h-form, then there exists a $(h-1)$-form ϕ whose coefficients are in $C_0(\mathbb{R}^n)$ such that $d\phi = \omega$ and*

$$\|\phi\|_{C_0} \leq C\|\omega\|_{L^n}.$$

Here C_0 denotes the space of differential forms with coefficients that are continuous and vanishing at infinity, endowed with the L^∞-norm.

In the Appendix we provide a short review of the argument used by Bourgain and Brezis in [14]. Here we mention that the main idea of their proof relies on an abstract principle, i.e., that a closed unbounded operator with dense domain between two Banach spaces has a closed range if and only if its adjoint does. In order to adapting this abstract principle to our situation, first of all we see that the two Banach spaces are C_0 and L^n and the unbounded operator has to be the exterior differential d. The abstract scheme that we are going to apply is contained in the following proposition (whose proof can be found e.g. in in [9], Lemma 5.2, but basically is contained in Brezis's book, section II.7, [16]):

Proposition 1.1. *Let $A : D(A) \subset E \to F$ be a closed unbounded operator between Banach spaces. Assume that $D(A)$ is dense in E. Then the adjoint $A^* : D(A^*) \subset F^* \to E^*$ is uniquely defined, and closed. The following are equivalent:*

(1) $A(D(A)) \subset F$ *is closed.*
(2) $\exists C$, $\forall e^* \in A^*(D(A^*))$, $\exists f^* \in D(A^*)$, $A^*f^* = e^*$ *and* $\|f^*\|_{F^*} \le C\,\|e^*\|_{E^*}$.
(3) $A^*(D(A^*)) \subset E^*$ *is closed.*
(4) $\exists C$, $\forall f \in A(D(A))$, $\exists e \in D(A)$, $Ae = f$ *and* $\|e\|_E \le C\,\|f\|_F$.

In order to identify the dual of $E := C_0$ we need the notion of current. In next section we recall some basic facts about differential forms and currents.

The paper is organized as follows. In Section 2 we recall some more or less known results about differential forms and currents in $\mathbb{R}^n$. Section 3 contains the proof of our main result stated in Theorem 1.2. We add also a short Appendix containing the argument used by Bourgain and Brezis in [15]. Finally, in Section 4 we consider the Heisenberg groups $\mathbb{H}^n$ and we quickly review the main features of the so-called Rumin's complex, which replaces the de Rham complex which better fits the geometry of the group.

2. Preliminary on functions, differential forms and currents

Throughout the present note our setting will be the Euclidean space $\mathbb{R}^n$ with $n > 2$. To keep the paper self contained we recall briefly some definitions and results concerning

Euclidean currents. We refer to e.g. [20] for a detailed presentation. First of all, we start with some basic definitions and properties of functions and differential forms.

If f is a real function defined in $\mathbb{R}^n$, we denote by ${}^{\mathrm{v}}f$ the function defined by ${}^{\mathrm{v}}f(p) := f(-p)$, and, if $T \in \mathcal{D}'(\mathbb{R}^n)$, then ${}^{\mathrm{v}}T$ is the distribution defined by $\langle {}^{\mathrm{v}}T|\phi\rangle := \langle T|{}^{\mathrm{v}}\phi\rangle$ for any test function ϕ.

We recall also that the convolution $f * g$ is well defined when $f, g \in \mathcal{D}'(\mathbb{R}^n)$, provided at least one of them has compact support.

As customary, a basis of the tangent space $\bigwedge_1(\mathbb{R}^n) := \mathbb{R}^n$ is given by $(\partial_{x_1}, \dots, \partial_{x_n})$. We denote by $\langle\cdot,\cdot\rangle$ the scalar product making $(\partial_{x_1}, \dots, \partial_{x_n})$ orthonormal.

The dual space of $\bigwedge_1(\mathbb{R}^n)$ is denoted by $\bigwedge^1(\mathbb{R}^n) =: (\mathbb{R}^n)^*$. The basis of $\bigwedge^1(\mathbb{R}^n)$, dual to the basis $(\partial_{x_1}, \dots, \partial_{x_n})$, is the family of covectors $(dx_1, \dots, dx_n)$ and we again indicate as $\langle\cdot,\cdot\rangle$ the inner product in $(\mathbb{R}^n)^*$ that makes $(dx_1, \dots, dx_n)$ an orthonormal basis.

We put $\bigwedge_0 \mathbb{R}^n = \bigwedge^0 \mathbb{R}^n(\mathbb{R}^n) := \mathbb{R}$ and, for $1 \le h \le n$,

$$\bigwedge_h \mathbb{R}^n := \mathrm{span}\{\partial_{x_{i_1}} \wedge \cdots \wedge \partial_{x_{i_h}} : 1 \le i_1 < \cdots < i_h \le n\}$$

and

$$\bigwedge^h \mathbb{R}^n := \mathrm{span}\{dx_{i_1} \wedge \cdots \wedge dx_{i_h} : 1 \le i_1 < \cdots < i_h \le n\}\,.$$

If $I := (i_1, \dots, i_h)$ with $1 \le i_1 < \cdots < i_h \le n$, we set $|I| := h$ and

$$dx^I := dx_{i_1} \wedge \cdots \wedge dx_{i_h}.$$

The elements of $\bigwedge_h(\mathbb{R}^n)$ and $\bigwedge^h(\mathbb{R}^n)$ are called *h-vectors* and *h-covectors* respectively. The scalar products in the spaces of 1-vectors and 1-covectors can be canonically extended to $\bigwedge_h(\mathbb{R}^n)$ and $\bigwedge^h(\mathbb{R}^n)$ respectively.

The *Hodge star operator* is a linear operator

$$* : \bigwedge^h \mathbb{R}^n \to \bigwedge^{n-h} \mathbb{R}^n$$

defined by $\xi \wedge \eta = \langle \xi, *\eta\rangle$ for any $\eta \in \bigwedge^{n-h}(\mathbb{R}^n)$.

If $v \in \bigwedge_h(\mathbb{R}^n)$ and $\xi \in \bigwedge^h(\mathbb{R}^n)$, $|v|$ and $|\xi|$ denote as costumary their Euclidean norm. We recall now the definition of the comass norm of a covector (see [20], Chapter 2, Section 2.1).

Definition 2.1. *We denote by $\|\xi\|$ the comass norm of a covector $\xi \in \bigwedge^h(\mathbb{R}^n)$ defined by*

$$\|\xi\| = \sup\left\{ \langle \xi | v \rangle \;\Big|\; v \in \bigwedge_h(\mathbb{R}^n),\ |v| \le 1,\ v \text{ simple} \right\}.$$

By formula (13) of [20], Chapter 1, Section 2.2, there exists a geometric constant $c_1 > 0$ such that

$$c_1^{-1}|\xi| \le \|\xi\| \le |\xi| \qquad \text{for all } \xi \in \bigwedge^h(\mathbb{R}^n). \tag{2}$$

By translation, $\bigwedge^h(\mathbb{R}^n)$ defines a fibre bundle over $\mathbb{R}^n$, still denoted by $\bigwedge^h(\mathbb{R}^n)$. A differential form on $\mathbb{R}^n$ is a section of this fibre bundle.

Through this note, if $0 \le h \le n$ and $\mathcal{U} \subset \mathbb{R}^n$ is an open set, we denote by $\Omega^h(\mathcal{U})$ the space of smooth differential h-forms on $\mathcal{U}$, and by $d : \Omega^h(\mathcal{U}) \to \Omega^{h+1}(\mathcal{U})$ the exterior differential. Thus $(\Omega^\bullet(\mathcal{U}), d)$ is the de Rham complex in $\mathcal{U}$ and any $u \in \Omega^h$ can be written as $u = \sum_{|I|=h} u_I dx^I$. Finally we denote by d^* the L^2 (formal) adjoint of d. We remind the reader that, up to a sign which depends on the degree of the differential form that we consider, $d^* = \pm * d *$.

If $\mathcal{U}$ is an open set in $\mathbb{R}^n$ and ω is an h-form, then we write $\omega \in D^h(\mathcal{U})$ if its components with respect to a fixed basis belong to $D(\mathcal{U})$. Analogously, we write $\omega \in L^p(\mathcal{U})$ if its components with respect to a fixed basis are in $L^p(\mathcal{U})$, endowed with its natural norm. Clearly, these definitions are independent of the choice of the basis itself.

2.1. **Currents.**

Definition 2.2. *If $\mathcal{U} \subset \mathbb{R}^n$ is an open set and $0 \le h \le n$, we say that T is a h-current on $\mathcal{U}$ if T is a continuous linear functional on smooth compactly supported differential h-forms endowed with the usual topology and we denote by $\varphi \to \langle T|\varphi\rangle$ its action on $D^h(\mathcal{U})$.*

The space of h-dimensional currents in $\mathcal{U}$ is denoted by $D_h(\mathcal{U})$. If u is an $(n-h)$-differential form in $L^1_{\rm loc}(\mathcal{U})$, then u can be identified canonically with an h-current T_u through the formula

$$\langle T_u|\varphi\rangle := \int_{\mathcal{U}} u\wedge\varphi = \int_{\mathcal{U}} \langle *u,\varphi\rangle\, dx$$

for any smooth compactly supported h-form φ on $\mathcal{U}$.

Suppose now u be a sufficiently smooth h-form (take for instance $u\in C^\infty(\mathbb{R}^n)$). If $\phi\in\mathcal{D}(\mathbb{R}^n)$ is an $(n-h+1)$-form, then, by Stokes formula,

$$\int_{\mathbb{R}^n} du\wedge\phi\, dx = (-1)^h\int_{\mathbb{R}^n} u\wedge d\phi\, dx.$$

Thus, if $T\in D_h(\mathbb{R}^n)$ it is natural to set

$$\langle \partial T|\phi\rangle = \langle T|d\phi\rangle,$$

for any $(h-1)$-form $\phi\in\mathcal{D}(\mathbb{R}^n)$ and we call the $(h-1)$-current ∂T the *boundary of T*.

Definition 2.3. *Let $\mathcal{U}$ be open set. if $T,T_j\in D_h(\mathcal{U})$, we say that the sequence $\{T_j\}$ converges in the sense of currents to T as $j\to\infty$, and we write $T_j\to T$ as $j\to\infty$ in the sense of currents, if $\langle T_j|\alpha\rangle\to\langle T|\alpha\rangle$ as $j\to\infty$ for any h-form $\alpha\in\mathcal{D}(\mathcal{U})$.*

As for distributions, the *support of a current* $T\in D_h(\mathcal{U})$ is defined as

$$\begin{aligned}\operatorname{supp} T = \bigcap\{K\subset\mathcal{U}\,|\, K \text{ relatively closed in } \mathcal{U},\ \langle T|\alpha\rangle = 0\\ \text{for all } \alpha\in D^h(\mathcal{U}) \text{ with } \operatorname{supp}\alpha\subset\mathcal{U}\setminus K\}.\end{aligned}$$

Following [20] Section 2.3 and keeping in mind Definition 2.1, we introduce also the notion of mass of a current.

Definition 2.4. *Let $\mathcal{U},\mathcal{V}$ be open sets and $\mathcal{V}\subset\mathcal{U}$. Let $T\in D_h(\mathcal{U})$. We set*

$$M_{\mathcal{V}}(T) := \sup\,\left\{\langle T|\alpha\rangle\,|\,\alpha\in\mathcal{D}^h(\mathcal{U}),\ \operatorname{supp}\alpha\subset\mathcal{V},\ \|\alpha\|\le 1\ \forall x\in\mathcal{U}\right\}\,,$$

and we say that T is of finite mass if $M_{\mathcal{V}}(T)$ is finite. If $\mathcal{V}=\mathcal{U}$ we shall simply write $M(T)$ instead of $M_{\mathcal{V}}(T)$.

The mass of currents is lower semicontinuous with respect to the previous convergence.

Remark 2.1. *If $1 \le h \le n$, $T, T_k \in \mathcal{D}_h(\mathbb{R}^n)$ for $k \in \mathbb{N}$ and $T_k \to T$ in $\mathcal{D}_h(\mathbb{R}^n)$, then*

$$\mathcal{M}(T) \le \liminf_k \mathcal{M}(T_k).$$

In addition we recall that if $\alpha \in L^1(\mathbb{R}^n)$, then

$$\mathcal{M}(T_\alpha) = \|\alpha\|_{L^1(\mathbb{R}^n)}. \tag{3}$$

Moreover, if $\alpha \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathcal{M}(T_\alpha) < \infty$, then $\alpha \in L^1(\mathbb{R}^n)$ and (3) holds.

Remark 2.2. *If $\alpha \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is an $(n-h)$-differential form, we have $\partial T_\alpha = 0$ if and only if $d\alpha = 0$ in the sense of distributions. Indeed, if $\phi \in \mathcal{D}^{h-1}(\mathbb{R}^n)$ it holds that*

$$\begin{aligned}\langle \partial T_\alpha | \phi \rangle = \langle T_\alpha | d\phi \rangle &= \int_{\mathbb{R}^n} \alpha \wedge d\phi = \int_{\mathbb{R}^n} \langle *\alpha, d\phi \rangle \, dV \\ &= \int_{\mathbb{R}^n} \langle *\alpha, ** \, d\phi \rangle \, dV = \int_{\mathbb{R}^n} \langle \alpha, *d\phi \rangle \, dV = \int_{\mathbb{R}^n} \langle \alpha, *d**\phi \rangle \, dV \\ &= \pm \int_{\mathbb{R}^n} \langle \alpha, d^* * \phi \rangle \, dV \,.\end{aligned}$$

With the previous definitions in mind, the following regularization-type result for currents holds (see [2], Theorem 3.1 and also Proposition 6.16 in [9]).

Theorem 2.1. *Let T be an h-current of finite mass $M(T)$. Then for any $\epsilon > 0$ there exists $\omega_\epsilon \in \mathcal{E}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ h-form such that, if we set $T_\epsilon := T_{*\omega_\epsilon}$, for $\epsilon \to 0$, we have:*

i) *$T_\epsilon \to T$ in the sense of currents;*
ii) *$\|\omega_\epsilon\|_{L^1} = M(T_\epsilon) \to M(T)$;*
iii) *if $T = \partial S$ with $S \in \mathcal{D}_{h+1}(\mathbb{R}^n)$, then the forms $*\omega_\epsilon$ are closed.*

The following global Poincaré inequality for currents follows by the previous result and the Poincaré inequality for differential forms in the case $p = 1$ proved in [5].

Theorem 2.2. *Let $h = 1, \dots, n$. If $T \in \mathcal{D}_h(\mathbb{R}^n)$ is a current of finite mass of the form $T = \partial S$ with $S \in \mathcal{D}_{h+1}(\mathbb{R}^n)$, then there exists a $(2n-h)$-form $\phi \in L^{n/(n-1)}$, such that*

$$\partial T_\phi = T \qquad \textit{and} \qquad \|\phi\|_{L^{n/(n-1)}(\mathbb{R}^n)} \le C\, \mathcal{M}(T).$$

We omit the proof of this result that is similar to the one of Theorem 1.1 in [2] (in the case the domain of T is a ball). For more details, see also the corresponding result in Heisenberg groups contained in [9] (see Theorem 6.17), keeping in mind that $\mathbb{R}^n$ just requires less technicalities.

3. Main result

We denote again by $C_0(\mathbb{R}^n)$ the Banach space of continuous functions vanishing at infinity with the L^∞-norm. The same notation is used also for differential forms with coefficients in $C_0(\mathbb{R}^n)$. Sometimes we may also write $C_0(\mathbb{R}^n, h)$ if we want to stress that we are dealing with an h-differential form, and similarly for $L^n(\mathbb{R}^n, h)$ or other spaces.

In order to prove our main result, which is stated in Theorem 1.2 we are going to use the abstract scheme proposed in Proposition 1.1. Therefore, for $2 \le h \le n$, we set

$$E := C_0(\mathbb{H}^n, h-1) \quad \text{and} \quad F = L^n(\mathbb{R}^n, h)\,.$$

By Riesz representation theorem, the dual space E^* can be identified with the set of currents with finite mass and F^* can be identified with $L^{n/(n-1)}(\mathbb{R}^n, n-h)$. We set also

$$\mathcal{D}(A) := \{\psi \in E,\, d\psi \in F\} \subset E,$$

with

$$A : \mathcal{D}(A) \to F, \qquad A\psi := d\psi.$$

Notice that $\mathcal{D}(A)$ is dense since contains $\mathcal{D}(\mathbb{R}^n)$ and A is closed since is a differential operator.

We can prove that

$$A^*(\beta) = \partial T_\beta, \tag{4}$$

and we have description of the domain of A^*, directly from the definition passing through an approximation argument, and hence we can prove that

$$\mathcal{D}(A^*) = \{\beta \in F,\, \mathcal{M}(\partial T_\beta) < \infty\}.$$

See Lemma 7.3 and Proposition 7.4 in [9] for a more detailed discussion about $\mathcal{D}(A^*)$.

Proposition 3.1. *With the notation introduced above, $\mathcal{D}(A^*)$ is dense in F^* and*

$$A^*(\mathcal{D}(A^*)) \qquad \textit{is closed in } E^*. \tag{5}$$

Proof. The fact that $\mathcal{D}(A^*)$ is dense in F^* holds since F^* is reflexive (see [16], Remark 15 of Section 2.6). Let us now show the second assertion. To this end, let $\{T_k\}_{k\in\mathbb{N}}$ be a sequence of $(h-1)$-currents in $A^*(\mathcal{D}(A^*))$ that converges to an $(h-1)$-current $T \in E^*$ (i.e. in the mass norm). Hence, $\mathcal{M}(T_k) = \|T_k\|_{E^*} \le C_1$ for all $k \in \mathbb{N}$. Moreover, in particular $\langle T|\sigma\rangle = \lim_{k\to\infty}\langle T_k|\sigma\rangle$, for all $\sigma \in \mathcal{D}(\mathbb{R}^n, h-1)$ i.e. $\{T_k\}_{k\in\mathbb{N}}$ converges to T also in the sense of currents. By (3) and (4) there exists a corresponding sequence $\{S_k\}_{k\in\mathbb{N}}$ in $\mathcal{D}_h(\mathbb{R}^n)$ such that

$$T_k = \partial S_k, \quad \text{with } S_k = T_{\beta_k},\ \ \beta_k \in F^* \tag{6}$$

for any $k \in \mathbb{N}$. Since the $(h-1)$-currents ∂S_k's satisfy $\mathcal{M}(\partial S_k) = \mathcal{M}(T_k) < \infty$, by Theorem 2.2 there exists $\phi_k \in F^*$ such that $\partial T_{\phi_k} = \partial S_k = \partial T_{\beta_k}$, and

$$\|\phi_k\|_{F^*} \le C_2\mathcal{M}(\partial S_k) = C\mathcal{M}(T_k) \le C_1C_2.$$

Since $F^* = L^{n/(n-1)}$ and $n/(n-1) > 1$ we can assume that

$$\phi_k \to \phi \qquad \text{weakly in } L^{n/(n-1)}.$$

Thus $T_{\phi_k} \to T_\phi$ in the sense of currents and $T_k = \partial S_k = \partial T_{\phi_k} \to \partial T_\phi$ in the sense of currents; therefore, since also $T_k \to T$, it follows that $T = \partial T_\phi$. To prove that $T \in A^*(\mathcal{D}(A^*))$, by (3) we have only to show that $\mathcal{M}(\partial T_\phi) < \infty$. Because of the lower semicontinuity of the mass with respect to the convergence in the sense of currents (see Remark 2.1), we have

$$\mathcal{M}(\partial T_\phi) \le \liminf_{k\to\infty}\mathcal{M}(\partial S_k) = \liminf_{k\to\infty}\mathcal{M}(T_k) = \mathcal{M}(T).$$

Thus (5) is proved.

□

We are now ready to prove Theorem 1.2

Proof of Theorem 1.2. By Theorem II.18 in [16], (5) implies that

$$A(\mathcal{D}(A)) = (\ker A^*)^{\perp}. \tag{7}$$

Moreover, by Proposition 1.1, (5) implies that there exists $C > 0$ such that for all $f \in A(\mathcal{D}(A))$ there exists $e \in \mathcal{D}(A)$, satisfying

$$Ae = f \quad \text{and} \quad \|e\|_E \le C\,\|f\|_F. \tag{8}$$

We are left to show that $\{\omega \in L^n(\mathbb{R}^n),\, d\omega = 0\} \subset A(\mathcal{D}(A))$. This will be done by showing that

$$\{\omega \in F,\, d\omega = 0\} \subset (\ker A^*)^{\perp}. \tag{9}$$

Suppose for a while that (9) holds. Then combining (9) and (7),

$$\{\alpha \in L^n(\mathbb{R}^n),\, d\alpha = 0\} \subset A(\mathcal{D}(A)),$$

and hence, by (8), we have proved the theorem.

Hence we are left to show (9).

To this end, let $\omega \in F$ be a closed form, and take $\beta \in \ker A^*$. Thus $\beta \in L^{n/(n-1)}(\mathbb{R}^n)$ and, by Remark 2.2, is a closed form. Therefore, it is possible to show that

$$\int_{\mathbb{R}^n} \omega \wedge \beta = 0. \tag{10}$$

Indeed, without loss of generality, the previous formula can be proved by assuming that β and ω are smooth. We set $\frac{1}{n''} = \frac{1}{n/n-1} - \frac{1}{n}$. Then, by Poincaré inequality stated in Theorem 1.1-i), there exists an $(h-1)$-form $\phi \in L^{n''}$ such that $d\phi = \beta$ and

$$\|\phi\|_{L^{n''}} \le C\|\beta\|_{L^{n/n-1}}. \tag{11}$$

In particular, $\omega \wedge \beta = \omega \wedge d\phi$. If $N > 0$, let now χ_N be a smooth cut-off function supported in $B(0, 2N)$, $\chi_N \equiv 1$ on $B(0, N)$, $|d\chi_N| \le 2/N$. Obviously, since $\omega \wedge \beta \in L^1$

$$\int_{\mathbb{R}^n} \chi_N \omega \wedge \beta \to \int_{\mathbb{R}^n} \omega \wedge \beta,$$

as $N \to \infty$. On the other hand, by Stokes' theorem and keeping in mind that $d\omega = 0$,

$$|\int_{\mathbb{R}^n} \chi_N \omega \wedge \beta| = |\int_{\mathbb{R}^n} \chi_N \omega \wedge d\phi| = |\int_{\mathbb{R}^n} \omega \wedge d\chi_N \wedge \phi|.$$

We have $\omega \wedge d\chi_N \wedge \phi \to 0$ pointwise since $d\chi_N \equiv 0$ on $B(0, R)$.

To finish, let us prove that $\int_{\mathbb{R}^n} \omega \wedge d\chi_N \wedge \phi \to 0$ as $N \to \infty$. We invoke (11) and, by Hölder inequality, we get

$$\begin{aligned} |\int_{\mathbb{R}^n} \omega \wedge d\chi_N \wedge \phi| &\le \|\omega\|_{L^n} \, \|d\chi_N\|_{L^n} \, \|\phi\|_{L^{n/(n-2)}} \\ &< \frac{2}{N} |B(0, 2N)|^{1/N} \, \|\omega\|_{L^n} \, \|\phi\|_{L^{n/(n-2)}}, \end{aligned}$$

since $\frac{1}{n} + \frac{1}{n} + \frac{n-2}{n} = 1$.

This proves (10) and therefore we have proved that (9) holds.

□

Appendix: the Bourgain-Brezis duality argument used in [14]

In this section we give a very rough gist of the proof underlying the result of Bourgain and Bresis for the case $p = n$. For precise definitions and statements we refer to [14].

In [14] the authors want to get continuous vector-fields solving the equation

$$\mathrm{div}(Y) = f$$

for f in L^n on the torus and with $\int_{(0,2\pi)^n} f = 0$. They prove that given a such periodic $f \in L^n$, there exists some $Y \in L^\infty \cap C^0$ solving the equation (in the sense of distribution) so that

$$\|Y\|_\infty \le C(n) \|f\|_n .$$

By the well known Sobolev-Gagliardo-Nirenberg imbedding $BV \subset L^{n/n-1}$, it holds

$$\|u\|_{L^{n/n-1}} \le C(n) \|Du\|_{\mathcal{M}} \qquad \forall u \in BV,$$

where $\mathcal{M}$ denotes the space of measures.

They consider the two Banach spaces $E := C^0$ and $F = L^N_0$, and the unbounded linear operator $A = \mathcal{D}(A) \subset E \to F$, defined by

$$\mathcal{D}(A) = \{Y \in E \, : \mathrm{div} Y \in L^n\}, \quad AY = \mathrm{div} Y,$$

so that A is densely defined and has closed graph.

The dual spaces of E and F are

$$E^* = \mathcal{M}, \quad F^* = L_0^{n/(n-1)}$$

and

$$\mathcal{D}(A^*) = F^* \cap BV, \quad A^*u = \operatorname{grad} u.$$

Therefore, the Sobolev-Gagliardo-Nirenberg inequality above reads as

$$\|u\|_{F^*} \leq C(n)\|A^*u\|_{E^*} \quad \forall u \in \mathcal{D}(A^*).$$

Using the geometric version of Hahn-Banach and Closed Range Theorems they achieve the result.

More precisely, if we consider $\|f\|_{L^n} = 1$ and take the two convex sets

$$B = \{Y \in E \, : \|Y\|_E < 2C(n)\}, \qquad L = \{Y \in E \, : \operatorname{div} Y = f\},$$

by contraddiction it is possible to prove that $B \cap L \neq \emptyset$. Indeed, suppose that that $B \cap L = \emptyset$. By the first geometric form of the Hahn-Banach theorem there exist $\mu \in E^*$, $\mu \neq 0$ and $t \in \mathbb{R}$ such that:

$$\langle \mu | Y \rangle \leq t, \quad \forall \; Y \in B, \qquad \langle \mu | Y \rangle \geq t, \quad \forall \; Y \in L\,.$$

The first of the two inequalities says that

$$\|\mu\| \leq \frac{t}{2C(n)}\,.$$

From the second condition they deduce that $\langle \mu | Y \rangle = 0$ for all $Z \in \operatorname{Ker}(A)$, and thus $\mu \in \operatorname{Ker}(A)^\perp = R(A^*)$.

Hence, there exists some $u \in F^* \cap BV$ such that $\operatorname{grad} u = \mu$. Applying the Gagliardo-Sobolev inequality they deduce that

$$\|u\|_{L^{n/n-1}} \leq C(n)\|\mu\| \leq t/2\,.$$

On the other hand, since $\langle \mu | Y \rangle \geq t$ for all $Y \in L$, then for $Y \in L$ one has

$$t \leq \langle \mu | Y \rangle = \langle \operatorname{grad} u, Y \rangle = -\int u \operatorname{div} Y = -\int uf \leq \|u\|_{L^{n/n-1}} \leq t/2\,.$$

This is impossible since $t > 0$ (because $\mu \neq 0$). The proof is therefore complete.

We end this section by recalling that, apart from [9], an abstract scheme similar to the previous one has been applied also by L. Moonens and T. Picon [22], and T. De Pauw and M. Torres [17]. In addition, a special instance in general Carnot groups for top degree Rumin forms, is considered by A. Baldi and F. Montefalcone [1].

4. The case of Heisenberg groups

In [9], the previous results are generalized to differential forms of Rumin's complex in Heisenberg groups. Let us give a gist of the notion of Rumin complex. When dealing with differential forms in $\mathbb{H}^n$, the de Rham complex lacks scale invariance under anisotropic dilations (see (17)). M. Rumin, in [24] has defined a substitute of the de Rham's complex for arbitrary contact manifolds, that recovers scale invariance under the family of anisotropic dilations of $\mathbb{H}^n$. In the present section, we shall merely list a few properties of Rumin's complex that we used in [9]. We send a reader, interested to understand better Rumin's complex, to the Appendix of [8] for a quick review, or to [24] and [10], [7] for more details of the construction.

We denote by $\mathbb{H}^n$ the n-dimensional Heisenberg group, identified with $\mathbb{R}^{2n+1}$ through exponential coordinates. A point $p \in \mathbb{H}^n$ is denoted by $p = (x, y, t)$, with both $x, y \in \mathbb{R}^n$ and $t \in \mathbb{R}$. If p and $p' \in \mathbb{H}^n$, the group operation is defined by

$$p \cdot p' = (x + x', y + y', t + t' + \frac{1}{2} \sum_{j=1}^{n} (x_j y_j' - y_j x_j')).$$

The unit element of $\mathbb{H}^n$ is the origin, that will be denoted by e. For any $q \in \mathbb{H}^n$, the *(left) translation* $\tau_q : \mathbb{H}^n \to \mathbb{H}^n$ is defined as

$$p \mapsto \tau_q p := q \cdot p.$$

The Lebesgue measure in $\mathbb{R}^{2n+1}$ is a Haar measure in $\mathbb{H}^n$ (i.e., a bi-invariant measure on the group). It is denoted by $\mathcal{L}^{2n+1}$, and when we need to stress the integration variable p, will be denoted also by dp.

For a general review on Heisenberg groups and their properties, we refer to [25], [21], [13], and to [26]. We limit ourselves to fix some notation, following [18].

The Heisenberg group $\mathbb{H}^n$ can be endowed with the homogeneous norm (Cygan-Korányi norm)

$$\varrho(p) = \left(|p'|^4 + 16\, p_{2n+1}^2\right)^{1/4}, \tag{12}$$

and we define the gauge distance (a true distance, see [25], p. 638), that is left invariant i.e. $d(\tau_q p, \tau_q p') = d(p,p')$ for all $p, p' \in \mathbb{H}^n$ as

$$d(p,q) := \varrho(p^{-1}\cdot q). \tag{13}$$

Finally, the balls for the metric d are the so-called Korányi balls

$$B(p,r) := \{q \in \mathbb{H}^n;\ d(p,q) < r\}. \tag{14}$$

Notice that Korányi balls are convex smooth sets.

A straightforward computation shows that there exists $c_0 > 1$ such that

$$c_0^{-2}|p| \le \rho(p) \le |p|^{1/2}, \tag{15}$$

provided p is close to e. In particular, for $r > 0$ small, if we denote by $B_{\mathrm{Euc}}(e,r)$ the Euclidean ball centred at e of radius r,

$$B_{\mathrm{Euc}}(e,r^2) \subset B(e,r) \subset B_{\mathrm{Euc}}(e,c_0^2 r). \tag{16}$$

We denote by $\mathfrak{h}$ the Lie algebra of the left invariant vector fields of $\mathbb{H}^n$. The standard basis of $\mathfrak{h}$ is given, for $i = 1,\dots,n$, by

$$X_i := \partial_{x_i} - \frac{1}{2}y_i\partial_t, \quad Y_i := \partial_{y_i} + \frac{1}{2}x_i\partial_t, \quad T := \partial_t.$$

The only non-trivial commutation relations are $[X_i, Y_i] = T$, for $i = 1,\dots,n$. The *horizontal subspace* $\mathfrak{h}_1$ is the subspace of $\mathfrak{h}$ spanned by $X_1,\dots,X_n$ and $Y_1,\dots,Y_n$:

$$\mathfrak{h}_1 := \text{span}\,\{X_1,\dots,X_n,Y_1,\dots,Y_n\}\ .$$

Coherently, from now on, we refer to $X_1,\dots,X_n,Y_1,\dots,Y_n$ (identified with first order differential operators) as to the *horizontal derivatives*. Denoting by $\mathfrak{h}_2$ the linear span of T, the 2-step stratification of $\mathfrak{h}$ is expressed by

$$\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2.$$

The stratification of the Lie algebra $\mathfrak{h}$ induces a family of non-isotropic dilations $\delta_\lambda : \mathbb{H}^n \to \mathbb{H}^n$, $\lambda > 0$ as follows: if $p = (x, y, t) \in \mathbb{H}^n$, then

$$\delta_\lambda(x, y, t) = (\lambda x, \lambda y, \lambda^2 t). \tag{17}$$

Notice that the gauge norm (12) is positively δ_λ-homogenous, so that the Lebesgue measure of the ball $B(x, r)$ is r^{2n+2} up to a geometric constant (the Lebesgue measure of $B(e, 1)$).

The constant

$$Q := 2n + 2$$

is said the *homogeneous dimension* of $\mathbb{H}^n$ with respect to δ_λ, $\lambda > 0$. It is well known that the topological dimension of $\mathbb{H}^n$ is $2n + 1$, since as a smooth manifold it coincides with $\mathbb{R}^{2n+1}$, whereas the Hausdorff dimension of $(\mathbb{H}^n, d)$ is Q.

The vector space $\mathfrak{h}$ can be endowed with an inner product, indicated by $\langle \cdot, \cdot \rangle$, making $X_1, \dots, X_n$, $Y_1, \dots, Y_n$ and T orthonormal.

Throughout this note, we write also

$$W_i := X_i, \quad W_{i+n} := Y_i \quad \text{and} \quad W_{2n+1} := T, \quad \text{for } i = 1, \dots, n. \tag{18}$$

Let $1 \le p \le \infty$ and $m \in \mathbb{N}$, $W^{m,p}_{\mathrm{Euc}}(U)$ denotes the usual Sobolev space.

The dual space of $\mathfrak{h}$ is denoted by $\bigwedge^1 \mathfrak{h}$. The basis of $\bigwedge^1 \mathfrak{h}$, dual to the basis $\{X_1, \dots, Y_n, T\}$, is the family of covectors $\{dx_1, \dots, dx_n, dy_1, \dots, dy_n, \theta\}$ where

$$\theta := dt - \frac{1}{2} \sum_{j=1}^{n} (x_j dy_j - y_j dx_j)$$

is the *contact form* in $\mathbb{H}^n$. We denote by $\langle \cdot, \cdot \rangle$ the inner product in $\bigwedge^1 \mathfrak{h}$ that makes $(dx_1, \dots, dy_n, \theta)$ an orthonormal basis and by dV the associated volume form

$$dV := dx_1 \wedge \cdots \wedge dx_n \wedge dy_1 \wedge \cdots \wedge dy_n \wedge \theta.$$

Throughout this paper, $\bigwedge^h \mathfrak{h}$ denotes the h-th exterior power of the Lie algebra $\mathfrak{h}$. Keeping in mind that the Lie algebra $\mathfrak{h}$ can be identified with the tangent space to $\mathbb{H}^n$ at $x = e$ (see, e.g. [19], Proposition 1.72), starting from $\bigwedge^h \mathfrak{h}$ we can define by left translation a fiber bundle over $\mathbb{H}^n$ that we can still denote by $\bigwedge^h \mathfrak{h} \simeq \bigwedge^h T^*\mathbb{H}^n$. Moreover, a scalar product in $\mathfrak{h}$ induces a scalar product and a norm on $\bigwedge^h \mathfrak{h}$.

We can think of h-forms as sections of $\bigwedge^h \mathfrak{h}$ and we denote by Ω^h the vector space of all smooth h-forms.

- For $h = 0, \dots, 2n+1$, the space of Rumin h-forms, E_0^h, is the space of smooth sections of a left-invariant subbundle of $\bigwedge^h \mathfrak{h}$ (that we still denote by E_0^h). Hence it inherits the inner product and the norm of $\bigwedge^h \mathfrak{h}$.
- If we denote by $\star$ the Hodge duality operator associated with the inner product in $E_0^\bullet$ and the volume form dV, then $\star E_0^h = E_0^{2n+1-h}$.

  In particular we have

  Remark 4.1. *If $\alpha \in E_0^h$, then $\star\star\alpha = (-1)^{(2n+1-h)h}\alpha = \alpha$. Thus*

  $$\alpha \wedge \phi = \phi \wedge (\star\star\alpha) = \langle \star\alpha, \phi \rangle \, dV.$$

  Moreover, if $\beta \in E_0^h$

  $$\begin{aligned} \langle \star\alpha, \star\beta \rangle \, dV &= \alpha \wedge \star\beta = (-1)^{h(2n+1-h)} \star\beta \wedge \alpha \\ &= \langle \star\star\beta, \alpha \rangle \, dV = \langle \beta, \alpha \rangle \, dV = \langle \alpha, \beta \rangle \, dV. \end{aligned}$$

- A differential operator $d_c : E_0^h \to E_0^{h+1}$ is defined. It is left-invariant, homogeneous with respect to group dilations. It is a first order homogeneous operator in the horizontal derivatives in degree $\neq n$, whereas *it is a second order homogeneous horizontal operator in degree n.*
- Altogether, operators d_c form a complex: $d_c \circ d_c = 0$.
- This complex is homotopic to de Rham's complex $(\Omega^\bullet, d)$. More precisely there exist a sub-complex (E, d) of the de Rham complex and a suitable "projection" $\Pi_E : \Omega^\bullet \to E^\bullet$ such that Π_E is a differential operator of order ≤ 1 in the horizontal derivatives.
- Π_E is a chain map, i.e.

  $$d\Pi_E = \Pi_E d.$$

- Let Π_{E_0} be the orthogonal projection on $E_0^\bullet$. Then
$$\Pi_{E_0}\Pi_E\Pi_{E_0} = \Pi_{E_0} \qquad \text{and} \qquad \Pi_E\Pi_{E_0}\Pi_E = \Pi_E$$
(we stress that Π_{E_0} is an algebraic operator).
- The exterior differential d_c can be written as
$$d_c = \Pi_{E_0} d \Pi_E \Pi_{E_0}.$$
- The L^2-formal adjoint d_c^* of d_c on E_0^h satisfies

$$d_c^* = (-1)^h \star d_c \star . \tag{19}$$

When d_c is second order (when acting on forms of degree n), $(E_0^\bullet, d_c)$ stops behaving like a differential module. This is the source of many complications. In particular, the classical Leibniz formula for the de Rham complex $d(\alpha\wedge\beta) = d\alpha\wedge\beta \pm \alpha\wedge d\beta$ is true in Rumin's complex only in special degrees, as shown in [11], Proposition A.1 and [23], Proposition 4.1. However, in general, the Leibniz formula fails to hold (see [11], Proposition A.7). This causes several technical difficulties when we want to localize our estimates by means of cut-off functions.

In fact, the main difficulty of our proof is hidden in the following Leibniz' formula for Rumin's differential forms.

Lemma 4.1 (see also [6], Lemma 4.1)**.** *If ζ is a smooth real function, then the following formulae hold in the sense of distributions:*

i) *if $h \neq n$, then on E_0^h we have*
$$[d_c, \zeta] = P_0^h(W\zeta),$$
where $P_0^h(W\zeta) : E_0^h \to E_0^{h+1}$ is a linear homogeneous differential operator of order zero with coefficients depending only on the horizontal derivatives of ζ. If $h \neq n+1$, an analogous statement holds if we replace d_c in degree h with d_c^* in degree $h+1$;

ii) *if* $h = n$, *then on* E_0^n *we have*

$$[d_c, \zeta] = P_1^n(W\zeta) + P_0^n(W^2\zeta),$$

where $P_1^n(W\zeta) : E_0^n \to E_0^{n+1}$ *is a linear homogeneous differential operator of order 1 (and therefore horizontal) with coefficients depending only on the horizontal derivatives of* ζ, *and where* $P_0^h(W^2\zeta) : E_0^n \to E_0^{n+1}$ *is a linear homogeneous differential operator in the horizontal derivatives of order 0 with coefficients depending only on second order horizontal derivatives of* ζ. *If* $h = n+1$, *an analogous statement holds if we replace* d_c *in degree* n *with* d_c^* *in degree* $n+1$.

The result we present here within Rumin's complex is a part of along-standing project initiated by the authors in collaboration with Pierre Pansu starting from [4] (see also [3] and [12].)

As for the de Rham complex, we can associate with Rumin's complex a class of currents. Their main properties can be found, e.g., in Section 6 of [9]. In the setting of Rumin's complex, Theorem 2.2 reads as follows.

Theorem 4.1. *The following global Poincaré inequalities hold.*

i) *if* $2 \le h \le 2n+1$, $h \ne n+1$, *then a* d_c*-exact Rumin* h*-form* $\omega \in L^{2n+2}(\mathbb{H}^n, E_0^h)$ *admits a primitive* $\phi \in C_0(\mathbb{H}^n, E_0^{h-1})$ *such that*

$$\|\phi\|_{C_0(\mathbb{H}^n, E_0^{h-1})} \le C\|\omega\|_{L^{2n+2}(\mathbb{H}^n, E_0^h)};$$

ii) *a* d_c*-exact Rumin* $(n+1)$*-form* $\omega \in L^{n+1}(\mathbb{H}^n, E_0^{n+1})$ *admits a primitive* $\phi \in C_0(\mathbb{H}^n, E_0^n)$ *such that*

$$\|\phi\|_{C_0(\mathbb{H}^n, E_0^n)} \le C\|\omega\|_{L^{n+1}(\mathbb{H}^n, E_0^{n+1})}.$$

The proof of statement i) it is not very far from that of Theorem 2.2, whereas the proof of statement ii) requires an utterly different functional setting, because of the structure of Leibniz' formula in degree n. More precisely, we have to use the so-called Beppo Levi-Sobolev spaces that allow us to control extra derivatives in Leibniz formula when d_c is a second order operator. Below we give the basic definitions and properties, and we refer to Section 8 in [9] for definitions and proofs.

Definition 4.1. *If $1 < p < Q$, we denote by $BL^{1,p}(\mathbb{H}^n)$ the homogeneous Sobolev space (called also Beppo Levi space) defined as the completion of $\mathcal{D}(\mathbb{H}^n)$ with respect to the norm*

$$\|u\|_{BL^{1,p}(\mathbb{H}^n)} := \sum_{j=1}^{2n} \|W_j u\|_{L^p(\mathbb{H}^n)}.$$

Remark 4.2. *Since $BL^{1,p}(\mathbb{H}^n)$ is reflexive, it can be identified with its bidual via the canonical isomorphism $\tau(u)(f) = f(u)$ for all $BL^{1,p}(\mathbb{H}^n)$ and $f \in (BL^{1,p}(\mathbb{H}^n))^*$.*

Proposition 4.1. *If $1 < p < Q$, then*

$$(BL^{1,p}(\mathbb{H}^n))^* = \{T \in \mathcal{D}'(\mathbb{H}^n)\,;\, T = \sum_j W_j f_j\,,\; f_j \in L^{p'}(\mathbb{H}^n)\}.$$

If $F = (f_1, \dots, f_{2n})$ is a horizontal vector field, then we set

$$\operatorname{div}_{\mathbb{H}} F := \sum_j W_j f_j.$$

Definition 4.2. *If $1 \le h \le 2n+1$, then a form α belongs to $BL^{1,p}(\mathbb{H}^n, E_0^h)$ if and only if all its components with respect to a fixed left invariant basis of E_0^Q*

$$\Xi_0^h = \{\xi_1^h, \dots \xi_{N_h}^h\} \tag{20}$$

belong to $BL^{1,p}(\mathbb{H}^n)$.

Proposition 4.2. *The dual space $(BL^{1,p}(\mathbb{H}^n, E_0^h))^*$ can be identified with a family of currents $T \in \mathcal{D}'(\mathbb{H}^n, E_0^h)$ such that, with the notation of* (20)

$$T = \sum_j T_j (\xi_j^h)^*,$$

with $T_j \in \mathcal{D}'(\mathbb{H}^n)$, $j = 1, \dots, N_h$ of the form

$$T_j = \operatorname{div}_{\mathbb{H}} F_j,$$

with $F_j \in (L^{p'})^{2n}$, $j = 1, \dots, N_h$.

More precisely, by the density of $\mathcal{D}(\mathbb{H}^n, E_0^h)$ in $BL^{1,p}(\mathbb{H}^n, E_0^h)$, an element of $(BL^{1,p}(\mathbb{H}^n, E_0^h))^*$ is fully identified by its restriction to $\mathcal{D}(\mathbb{H}^n, E_0^h)$.

We can now define a new functional setting where Brezis and Bourgain's abstract theory applies.

Definition 4.3. *We set*

$$E := C_0(\mathbb{H}^n, E_0^n), \quad and \quad F := (BL^{1,Q/(Q-1)}(\mathbb{H}^n, E_0^n))^*,$$

$$\mathcal{D}(A) := \{\psi \in E, \partial_c T_\psi \in F\}, \quad and \quad A\psi := \partial_c T_\psi,$$

where, according to Proposition 4.2, we use the identification of $(BL^{1,p}(\mathbb{H}^n, E_0^n))^*$ *with a space of currents.*

Again, as for de Rham currents, the dual space of E *can be identified with the set of currents with finite mass.*

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