COARSENING PHENOMENA IN THE NETWORK FLOW

FENOMENI DI TIPO COARSENING NEL FLUSSO DI NETWORK

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Abstract. In this short note we summarize recent results on the asymptotic behaviour of the network flow and we give indications of an expected coarsening–type behaviour for the network flow past singularities. The paper is complemented with a discussion on critical points and local minimizers of the length functional.

Sunto. In questa breve nota riassumiamo alcuni risultati sul comportamento asintotico del moto per curvatura di network, focalizzandoci sugli indizi di comportamenti di tipo coarsening. La nota contiene anche una discussione sui punti critici e minimi locali del funzionale lunghezza.

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1. Introduction

A network $\mathcal{N}$ is a 1–dimensional connected and planar set composed of a finite number of smooth, regular and embedded curves $\{\gamma^i\}_{i=1}^N$ that meet only at their end-points in junctions. We are interested in the so–called *network flow*, a geometric flow that can be understood as the gradient flow of the length functional

$$L(\mathcal{N}) := \sum_i \int_0^1 |\partial_x \gamma^i(x)| \, \mathrm{d}x = \sum_i \int_{\gamma^i} 1 \, \mathrm{d}s \,. \tag{1}$$

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Formally, we derive the motion equations computing the first variation of L. Each curve moves with normal velocity equal to its curvature

$$v^{\perp}(t,x) = \vec{k}(x)\,, \tag{2}$$

or, equivalently $\langle \partial_t \gamma(t,x), \nu(t,x)\rangle = \kappa(t,x)$. Moreover, to interpret the curvature as the gradient of the length, we shall put to zero the contribution of the boundary, obtaining that during the evolution the junctions are "balanced", in the sense that at the junctions the unit tangent vectors of the concurring curves sum up to zero.

The network flow, the 1–dimensional case of the *multi-phase mean curvature flow*, brings the study of the *mean curvature flow* a step further, allowing the evolution of a specific class of singular objects (regular networks, see Definition 2.2) instead of immersions of a single smooth manifold. Recently the research on this topic has been particularly flourishing and numerous results have been obtained both for weak [10, 11, 21, 5, 4] and strong solutions [9, 3, 15, 14, 19].

Even though the flow has become fashionable among researchers in geometric analysis, the origin of this evolution is definitely more applied: the flow has been indeed proposed as a model of the growth of polycrystals in metals [16].

One of the motivations of the study of this flow is the tentative formalisation of a "coarsening–type behaviour" of the flow, that ultimately would indicate how good as a model of grain growth this flow is. Generically a network flow with a highly complicated initial datum (with hundreds of loops, for instance) is expected to converge, as time goes to infinity, to a critical point of the length with a much simpler structure than the initial network. This hypothetical behaviour is evident from numerical simulations (see for instance experiment posted on the webpage of Selim Esedoglu: dept.math.lsa.umich.edu/esedoglu and of Ken Brakke: kenbrakke.com and Figure 1).

In this note we describe some arguments supporting this expectation and summarize some of the tools developed till now to get a very accurate description of the evolution.

We will adopt a classical PDE approach, and all the results will be presented in a informal and very accessible way.

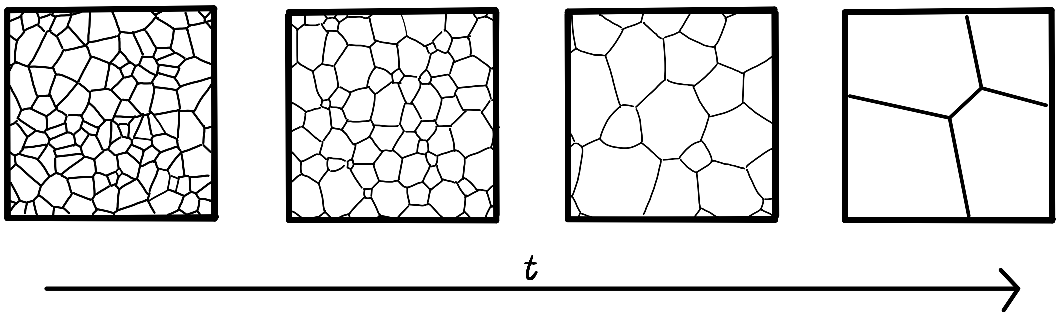

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FIGURE 1. Expected evolution of a complicated network

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2. Network flow

Consider a smooth planar curve $\gamma : [0,1] \to \mathbb{R}^2$. We say that γ is *regular* if for every $x \in [0,1]$ we have $\partial_x \gamma(x) \neq 0$. For a regular curve γ, define

$$\tau := \frac{\partial_x \gamma}{|\partial_x \gamma|}\,, \quad \nu := \mathrm{R}(\tau)\,, \tag{3}$$

the tangent and the normal vector, respectively, where R denotes the anticlockwise rotation centred in the origin of $\mathbb{R}^2$ of angle $\frac{\pi}{2}$. As usual we define $\mathrm{d}s := |\partial_x \gamma|\,\mathrm{d}x$ the arclength element and $\partial_s := |\partial_x \gamma|^{-1}\partial_x$ the arclength derivative. The curvature vector $\boldsymbol{k}$ of γ is

$$\boldsymbol{k} := \partial_s^2 \gamma = \frac{\partial_x^2 \gamma}{|\partial_x \gamma|^2} - \frac{\partial_x \gamma \, \langle \partial_x^2 \gamma, \partial_x \gamma \rangle}{|\partial_x \gamma|^4}\,, \tag{4}$$

and in the plane we have the relation $\boldsymbol{k} = \kappa \nu$ where the scalar κ is the oriented curvature.

Definition 2.1. *A network $\mathcal{N}$ is a connected set in the Euclidean plane, composed of finitely many regular, embedded smooth curves that meet only at their end–points in junctions.*

We distinguish between interior and exterior vertices of the network: at the former, more than one curve concur, and the latter are the end-points of the network.

We say that a network is a *tree* if it does not contain loops. We call *grain* a bounded region enclosed by one or more curve of the network.

We denote by L^i the length of the i–th curve of a network, namely

$$L^i := \int_0^1 |\partial_x \gamma^i(x)| \,\mathrm{d}x = \int_{\gamma^i} 1 \,\mathrm{d}s\,,$$

and the length of the network is nothing but the sum of the length of all its curves.

Definition 2.2. *A network whose interior vertices are only triple junctions, where the unit tangent vectors form angles of* 120 *degrees, is called* regular*. Otherwise we refer to is as* irregular*, so irregular ones can have either triple junctions where the angle condition is not satisfied or junctions where more than three curves meet. A network is said to be* minimal *if it is regular and it is composed of straight segments.*

We define now the network flow, namely the formal geometric gradient flow of the length functional. The flow is described as solution of a system of partial differential equation. We require that each curve of the network moves by curvature

$$(\partial_t \gamma^i)^{\perp} = \vec{\kappa}^i\,. \tag{5}$$

Moreover, apart from the initial time, the evolving network will be regular: for all times $t > 0$ we impose the following balancing condition at each triple junction

$$\begin{aligned} &\gamma^{i_1} = \gamma^{i_2} = \gamma^{i_3}\,, \quad \text{and} \\ &\tau^{i_1} + \tau^{i_2} + \tau^{i_3} = 0, \end{aligned} \tag{6}$$

Definition 2.3. *A time dependent family of networks $\mathcal{N}_t$, with $0 \leq t < T$, is a solution of the motion by curvature of regular networks if $\mathcal{N}_t$ converges to the initial network $\mathcal{N}_0$ as $t \searrow 0$, $\mathcal{N}_t$ is regular for all $t > 0$ and satisfies* (5)*,* (6) *for all $0 < t < T$. We say that the solution is maximal if it does not exists another solution defined on $[0, \widetilde{T})$ with $\widetilde{T} > T$ that coincide with $\mathcal{N}_t$ on $[0, T)$.*

Every solution can be extended to a maximal solution. From now on we consider only maximal solutions.

Remark 2.1. *To maintain the presentation as simple as possible, we have not specified the type of convergence towards the initial datum. One can think that the set $\mathcal{N}_t$ converges in Hausdorff distance or that the collection of maps $(\gamma_t^1, \dots, \gamma_t^N)$ describing the network converges uniformly to the collection of maps $(\gamma_0^1, \dots, \gamma_0^N)$ that describes $\mathcal{N}_0$ (some of the γ_0^i could be the constant map).*

If we suppose that the initial datum is a regular network, with linearization and a fixed point argument, one can prove that there exists a unique (up to reparametrization) maximal solution to the network flow with initial datum $\mathcal{N}_0$ in the maximal time interval $[0, T_{\max})$ (see [1, 7, 15]).

With definitely much more effort one can still prove a short-time existence result with irregular networks as initial data [8, 13].

3. Singularities

In this paper we focus our attention on the asymptotic behaviour of the flow. Ideally, one would like to prove that either the maximal time of existence T is finite and everything vanishes at T (as in the case of closed curves) or $T = +\infty$ and the evolving family of networks convergence to a critical point of the length functional. Unfortunately complications arises during the evolution in the form of "singularities". With the Definition 2.3 of the flow, one can describe the long–time behaviour as follows [15]:

Theorem 3.1. *Let $T > 0$ and let $(\mathcal{N}_t)$, with $0 \leq t < T$, be a maximal solution of the motion by curvature of regular networks in the maximal time interval $[0, T)$. If $T = +\infty$ the family of evolving networks converges (up to subsequences) to a network composed of straight segments and balanced junctions (the sum of the unit tangent vector at the junctions equals zero). If T is finite, as $t \to T$ at least one of the following happens:*

i) *the inferior limit of the length of at least one curve of the network is zero;*

ii) *the superior limit of the L^2-norm of the curvature is $+\infty$;*

and the two possibilities are not mutually exclusive.

If $T = +\infty$, in certain cases, the result can be strengthened, as we will discuss in Section 7.1. Let us instead elaborate a bit more on the case $T < +\infty$. First of all, if

$T < +\infty$, we call T singular time, we refer to the phenomena i) and ii) as singularities and the limit network as $t \to T$ (if it exists) is a singular network.

Consider a grain of the network enclosed in a loop ℓ composed of m curves $(\gamma^1, \dots, \gamma^m)$ and let $A(t)$ be the area of a grain (see the central picture in Figure 2). Using Gauss–Bonnet theorem we get that the time-derivative of the area is given by

$$A'(t) = -\sum_{i=1}^{m} \int_{\gamma^i} \left\langle \partial_t \gamma^i, \nu^i \right\rangle = -\sum_{i=1}^{m} \int_{\gamma^i} \kappa^i = -\left(2 - \frac{m}{3}\right)\pi \,. \tag{7}$$

Thus $A(t)$ increases linearly in time if $m > 6$, remains constant if $m = 6$, and decreases linearly in time if $m < 6$. In this last case, in particular, the area is zero at

$$\hat{T} = \frac{A_0}{(2 - {}^{m}\!/_{3})\,\pi} \tag{8}$$

where A_0 is the initial area enclosed by the loop. It is possible to construct examples of flows where the maximal time of existence T coincides with $\hat{T}$ and both the area enclosed by a loop ℓ and the length of ℓ go to zero as $t \to \hat{T}$. In particular, these are examples of singularity in which an entire region enclosed by several curves vanishes in finite time [2, 18, 15]. To do so, we have to consider as initial data networks with a loop of $m < 6$ curves that satisfy suitable symmetries. When the length of a loop goes to zero, the L^2–norm of the curvature blows up: indeed by Hölder inequality we get

$$\left(2 - \frac{m}{3}\right)\pi \leq \int_\ell |\vec{\kappa}| \leq \left(\int_\ell \kappa^2\right)^{1/2} \sqrt{\mathrm{L}(\ell)}\,, \tag{9}$$

that is

$$\int_\ell \kappa^2 \,\mathrm{d}s \geq \frac{C}{\mathrm{L}(\ell)} \quad \text{with}\, C > 0 \text{ for cells with } m \text{ edges, } m < 6\,. \tag{10}$$

We thus have examples of singularities in which both *i)* and *ii)* simultaneously happens. There are also explicit examples of evolution in which as $t \to T$ a single curve disappear [15, 19] as depicted in Figure 2 (left and right). Whenever two triple junctions coalesce without the disappearance of a region, the curvature of the networks remains bounded [14]. It is instead widely believed that there are no singularities where the length of each curve is uniformly bounded away from zero and the curvature is unbounded.

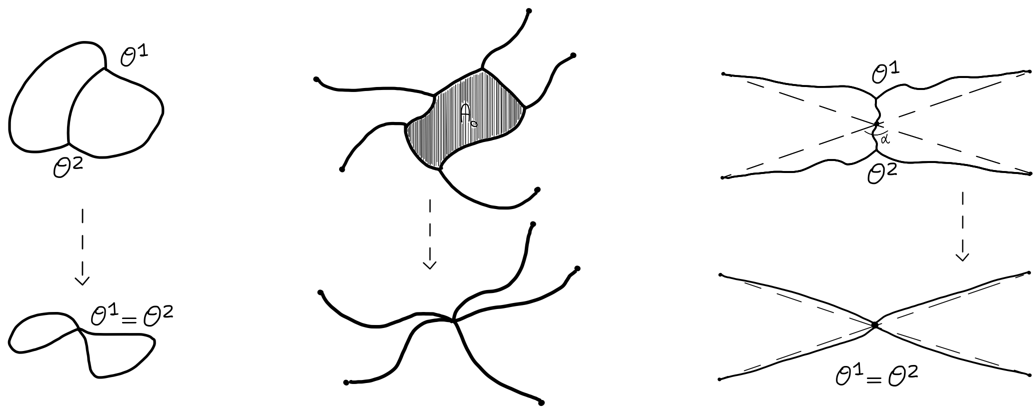

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FIGURE 2. Examples of singularity.

The take-home message of this section is that singularities actually happen and thus we need to introduce a notion of flow past singularities.

4. Flow past singularities

We now give a different definition of the network flow, it can still be describe by smooth solutions of a system of partial differential equation and it is still a motion by curvature of regular networks apart from a finite set of singular times $\{a_1, \dots, a_\ell\}$.

Definition 4.1. *A time dependent family of networks $\mathcal{N}_t$, $0 \le t < T$, is a solution of the network flow if $\mathcal{N}_t$ converges to the initial network $\mathcal{N}_0$ as $t \searrow 0$, and if $[0, T)$ decomposes as a finite union of subintervals $[0, t_1) \cup [t_1, t_2) \cup \dots, [t_\ell, T)$ so that $\mathcal{N}_t$ is regular for all t except possibly $t_1, \dots, t_\ell$. On each open interval (t_j, t_{j+1}) the family $\mathcal{N}_t$ satisfies* (5), (6) *and it is continuous across each t_j. The times t_j are the singular times. The solution is assumed to be maximal.*

Note that it is implicit in the definition that there exists the limit as $t \nearrow t_j^-$ (again we do not specify here the type of convergence as said in Remark 2.1) but this fact is far from being trivial.

For every $j \in \{1, \dots, \ell\}$, as $t \nearrow t_j$ the length of some arcs of $\mathcal{N}_t$ tends to zero, while for every $t_j < t < t_{j+1}$ the flow $\mathcal{N}_t$ has a collection of new arcs emanating from all vertices in $\mathcal{N}_{t_j}$ with order greater than 3.

At singular times t_j irregular networks appears. In many situations, we are able to "restart" the flow after such singularities [8, 13].

We give now an idea of the construction of a solution past singularities. The notion of expanding solitons is crucial for the construction. Expanding solitons are solutions that self-similarly dilates during the evolution, each evolving curve has the form $\gamma(t,x) = \lambda(t)\eta(x)$ where the expanding factor $\lambda(t)$ equals $\sqrt{2t}$.

Let the irregular junction coincides with the origin, and let $\gamma^{i_1}, \dots \gamma^{i_k}$ be the concurring curves, with unit tangent vectors $\tau^{i_1}, \dots, \tau^{i_k}$. Consider k halflines from the origin, whose direction coincides with $\tau^{i_1}, \dots, \tau^{i_k}$ and a small disk centred at O. Replace the part of the network inside the disk with a miniatures of a expanding soliton. The expanding soliton must have k non-compact branches whose directions at infinity coincide with the k halflines. Connect the soliton with the remaining part of the network nicely and let it flow. We get our evolution past singularity.

Note that the irregular junction is somehow locally replaced by a regular network whose combinatorics/topology is the same as one of the expanding solitons of the flow. We stress the fact that the number of solutions past singularities coincides with the number of expanding solutions compatible with the irregular junction. Moreover, a key feature of the solution past singularity is that a single irregular junction gives birth to a cluster of triple junctions.

Now, to discuss the asymptotic behaviour of the flow, we should know that the singular times are finite.

Figure 3 shows an hypothetical pathological behaviour: the flow oscillates infinitely many times between two different topologies.

In principle, the singular times could not only be infinite, but even "accumulate". We have neither analytical examples nor indications of such a behaviour, however at the moment we are not able to exclude it (for a partial result in this direction see [17]). From

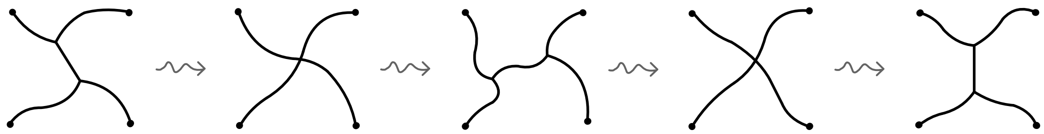

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Figure 3. The family of evolving networks switches between two different topologies.

now on, we simply suppose to be always able to restart the flow after a singularity and we exclude any "pathological" behaviour.

5. Simplification of the topology through singularities

If to restart the flow after a singularity we consider only tree-like solitons without loops, then, we can easily show also that the number of grains, of curves and of triple junctions is non–increasing during the evolution.

To be precise, when a singularity occurs with no vanishing of regions, the number of grains, of curves and of triple junctions is preserved. On the other hand, when a bounded region disappears and we desingularize the irregular junctions by gluing–in a tree like soliton, we do not add grains to the network, the number of grains is non–increasing, the total number of curves decreases at least by three and the total number of triple junctions decreases at least by two.

One may wonder if the alternative of consider solitons with loops is relevant to the problem. If an expanding soliton contains a grain, then by (7) it is bounded by a loop composed of at least seven curves. Thus to desingularise a junctions where at most five curves concur we can use only trees. The eventuality of considering solitons with loops in the restarting procedure seems not relevant because of the following:

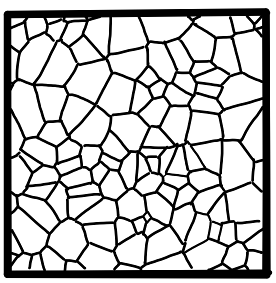
Conjecture [T. Ilmanen] *Let $\mathcal{N}_t$ be a solution of the network flow in $[0,T)$, let $\tilde{T} > 0$ be a singular time of the evolution and let $\mathcal{O}$ be an irregular junction of $\mathcal{N}_{\tilde{T}}$. Then, at most 5 curves concur at $\mathcal{O}$.*

If the statement were true, then extra grains can appear only in the desingularisation of the initial datum and not at a later singular time.

6. Average growth of the area of the grains

Till now we have described numerous aspects of the network flow summarising the known results. In this section instead we change perspective: we present an argument to support the simplification of evolving network through time. The computations are formal and at the moment we do not have a precise mathematical proof of the stated facts.

For simplicity we set the evolution in the flat torus $\mathbb{T}^2$.



We suppose that the initial network $\mathcal{N}_0$ is composed by a large number of curves and triple junctions, let's say that it has N^2 grains.

Then the average diameter of grains is of order $1/N$, the average area of order $1/N^2$ and the global length of the network is of order N.

We have shown that grains bounded by less than six curve should disappear during the evolution. We argue that the average area of the (surviving) grains grows linearly.

By formula (10) we have that along each loop ℓ there holds

$$\int_\ell k^2 \,\mathrm{d}s \geq \frac{C}{L(\ell)} \gtrsim N \,,$$

with $C > 0$ for cells with m edges, $m < 6$. Till the percentage of non–hexagonal grains is sufficiently high (the number of non–hexagonal grains is of order N^2), we can pass from an estimate on a single loop to an integral estimate on the whole network:

$$\int_{\mathcal{N}} k^2 \,\mathrm{d}s \gtrsim N \,\sharp\,(\text{non–hexagonal grains}) \approx N^3 \,. \tag{11}$$

Computing the evolution of the total length of $\mathcal{N}$, from the gradient flow structure of the problem we get

$$\frac{d}{dt} L(\mathcal{N}) = -\int_{\mathcal{N}} k^2 \,\mathrm{d}s \,. \tag{12}$$

Putting together (11) and (12) we derive the following differential inequality:

$$\frac{d}{dt} N(t) \lesssim -N^3(t) \,, \tag{13}$$

from which we get

$$\frac{1}{N(t)^2} \gtrsim 2t + c_0\,, \quad \text{that is} \quad N(t) \lesssim \frac{1}{\sqrt{2t+c_0}}\,, \tag{14}$$

with c_0 a constant encoding the number of initial grains. From this computation we obtain that the average area of the grains grows at least linearly in time: $\frac{1}{N(t)^2} \geq 2Ct$. However, there are two main limitations: we basically supposed that all grains are very similar to each other and we supposed that we see a certain fixed amount of non–hexagonal cell during the whole evolution.

7. Stability

The last topic we consider in the note is the analysis the flow as $t \to +\infty$. The asymptotic analysis of solutions $t \to +\infty$ is a very relevant subject for parabolic problems in general and in our specific case it could shed light on the question of coarsening.

A "soft" statement (part of Theorem 3.1) reads as follow: if $\mathcal{N}_t$ is a solution to the network flow in $[0, +\infty)$, then as $t \to +\infty$, the evolving networks $\mathcal{N}_t$ converge, up to subsequences, in $C^{1,\alpha} \cap W^{2,2}$, for every $\alpha \in (0, 1/2)$, to a critical point of the length functional. We stress the fact that the limit is not necessarily a regular network, but it is merely composed of straight segments and balanced junctions (sum of the unit tangent vectors equals zero) and it is not necessarily a global minimizer of the length functional [19].

At this point three questions naturally arise.

- Does the full sequence $\mathcal{N}_t$ converges to a limit network as $t \to +\infty$?
- Under which hypothesis on the initial datum are we able to ensure global existence?
- When is it possible to prove that there exists a time $\widetilde{T}$ such that $\mathcal{N}_t$ with $t \in [\widetilde{T}, +\infty)$ and the limit network $\mathcal{N}_\infty$ have the same topology?

It turns out that these three questions are intimately related, as shown in [19]:

Theorem 7.1. *Let $\mathcal{N}_*$ be a minimal network. Then, there exists $\varepsilon = \varepsilon(\mathcal{N}_*) > 0$ such that the network flow starting from any regular network $\mathcal{N}_0$ with*

$$\|\mathcal{N}_* - \mathcal{N}_0\|_{H^2} < \varepsilon \tag{15}$$

exists for all times and converges to a network $\mathcal{N}_\infty$ *with the same topology and same length of* $\mathcal{N}_*$.

In the statement the norm H^2 of the difference of two network is a shortcut to write the sum of the Sobolev H^2 norm of the difference of the parametrizations of the curve of the two networks.

We point out two facts.

- In general, it is not true that $\mathcal{N}_\infty$ coincide with $\mathcal{N}_*$. However, it is easy to prove that if $\mathcal{N}_*$ is a tree, then $\mathcal{N}_\infty$ must coincide with $\mathcal{N}_*$.
- Being H^2–close is a very strong condition. One would like to replace the Sobolev H^2 norm with a weaker one, for example the Hausdorff distance between the two networks.

Apparently Theorem 7.1 goes against the supposed coarsening–type behaviour of the evolution, under the hypothesis the evolving network does not simplify, its topology is preserved. However, we expect the cases in which one has stability to be scarce. The length functional lacks deeply convexity in the class of networks with fixed end-points, and the number of critical points is bigger and bigger as the number of end-points increases. We thus expect the basin of attraction of a critical point to become smaller and smaller.

We would like to have a bound from above on the thickness of the basin of attraction of critical points. As a first step in the estimate of the basin of attraction of minimal networks for the flow, we now present a result on local minimizers of the length functional. We give a quantitative bound from below on their local minimality. One should hope that the order of the bound from above would coincide with the order of the bound from below.

7.1. **Critical points of the length functional.** To consider the grains of the network as a partition of an open subset of $\mathbb{R}^2$ could be particularly convenient. We briefly summarise the relevant jargon relative to Cacciopoli partitions.

Let $\Omega \subset \mathbb{R}^2$ be open. A partition $\mathbf{E} = (E_1, \dots, E_n)$ of Ω is a collection of finite perimeter sets $E_i \subset \Omega$ such that $|E_i \cap E_j| = 0$ for $i \neq j$ and $|\Omega \setminus \cup_{i=1}^n E_i| = 0$. We define the perimeter of the partition of Ω as

$$P(\mathbf{E},\Omega) = \frac{1}{2}\sum_{i=1}^{N} P(E_i,\Omega) \tag{16}$$

where $P(E_i,\Omega) := |D\chi_{E_i}|(\Omega)$ is the (relative) perimeter of E_i in Ω.

We denote by $\Sigma_{ij}^{\mathbf{E}} := \partial^* E_i \cap \partial^* E_j$ and by $\nu_{ij} = \nu_i = -\nu_j$ the unit normal to Σ_{ij}, where ν_i is the generalized outer unit normal to the set E_i. In particular we can think of ν_{ij} as a normal pointing from E_i into E_j. We informally refer to $\bigcup_{i=1}^{N}(\partial^* E_i \cap \Omega) = \bigcup_{i<j=1}^{N}\left(\Sigma_{ij}^{\mathbf{E}} \cap \Omega\right)$ as the *boundary* of the partition $\mathbf{E}$.

Given a network $\mathcal{N}$ we denote by d be the minimum between the minimal distance of any two external vertices of the network and the length of the shortest edge of $\mathcal{N}$.

We say that a compact and connected set $\mathcal{N}$ disconnects two points x, y of $\partial\Omega$ if any continuous path σ in $\overline{\Omega}$ from x to y intersects $\mathcal{N}$.

Let Ω be a open subset of the plane. Suppose that a network $\mathcal{N}^* \subset \overline{\Omega}$ has all its end-points on $\partial\Omega$. We call $\mathcal{A}(\mathcal{N}^*,\Omega)$ the class of all networks with the following property: if $\mathcal{N}^*$ disconnects two points x, y of $\partial\Omega$ then also $\mathcal{N}$ disconnects x and y.

Theorem 7.2. *Let $\mathcal{N}^*$ be a minimal network. Then, there exists a δ–neighbourhood Ω of $\mathcal{N}^*$ with $0 < \delta \le \frac{\sqrt{3}}{8}d$ such that $\mathcal{N}^*$ is a minimizer of the length among all networks $\mathcal{N}$ in $\mathcal{A}(\mathcal{N}^*,\Omega)$.*

Remark 7.1. *For the sake of clarity in the statement we refer to Ω as a δ–neighbourhood of $\mathcal{N}^*$. To be precise, Ω is truncated as in Figure 4*

Idea of the proof

The result is a direct consequence of [20, Theorem 3.9]. Instead of explaining only how our current statement fits in the framework of Theorem 3.9, we prefer, in addition, to summarise its proof here.

Construct Ω as in Figure 4 Since locally $\mathcal{N}^*$ is an hexagonal lattice, one shows that $\mathcal{N}^*$ can be interpreted as the boundary of a suitable partition $\mathbf{E} = (E_1, E_2, E_3)$ of Ω.

Now to prove that $\mathbf{E}$ is perimeter minimizing among all partitions $\mathbf{F}$ of Ω with the same trace on the boundary of Ω of $\mathbf{E}$ we construct an explicit *calibration* of $\mathbf{E}$ in Ω.

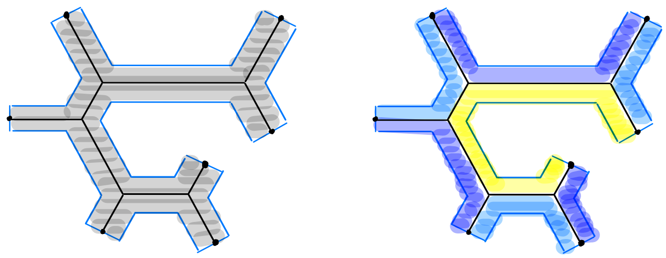


FIGURE 4. Left: A minimal network and the truncated neighborhood Ω. Right: the associated partition of three phases $\mathbf{E} = (E_1, E_2, E_3)$.

A calibration for $\mathbf{E}$ is a collection of three (sufficiently regular) vector fields (Φ_1, Φ_2, Φ_3), $\Phi_i : \overline{\Omega} \to \mathbb{R}^2$ with (distributional) divergence equal to zero fulfilling the following properties:

$$|\Phi_i - \Phi_j| \leq 1 \quad \mathcal{H}^1 - \text{a.e. in } \Omega\,, \text{for } i,j = 1,2,3\,,\,, i \neq j\,, \tag{17}$$

$$(\Phi_i - \Phi_j) \cdot \nu_{ij} = 1 \quad \mathcal{H}^1 - \text{a.e. in } \Sigma^{\mathbf{E}}_{ij}\,, \text{for } i,j = 1,2,3\,, i \neq j\,. \tag{18}$$

We then have

(19)

$$\mathcal{P}(\mathbf{E}) = \int_{\Sigma^{\mathbf{E}}_{12}\cap\Omega} (\Phi_1 - \Phi_2)\cdot\nu_{12}\,\mathrm{d}\mathcal{H}^1 + \int_{\Sigma^{\mathbf{E}}_{23}\cap\Omega} (\Phi_2 - \Phi_3)\cdot\nu_{23}\,\mathrm{d}\mathcal{H}^1 + \int_{\Sigma^{\mathbf{E}}_{31}\cap\Omega} (\Phi_3 - \Phi_1)\cdot\nu_{31}\,\mathrm{d}\mathcal{H}^1$$

(20)

$$= \sum_{i=1}^{3} \int_{\Omega} \Phi_i \cdot D\chi_{E_i} = \sum_{i=1}^{3} \int_{\Omega} \Phi_i \cdot D\chi_{F_i}$$

(21)

$$\leq \int_{\Sigma^{\mathbf{F}}_{12}\cap\Omega} |\Phi_1 - \Phi_2|\,\mathrm{d}\mathcal{H}^1 + \int_{\Sigma^{\mathbf{F}}_{23}\cap\Omega} |\Phi_2 - \Phi_3|\,\mathrm{d}\mathcal{H}^1 + \int_{\Sigma^{\mathbf{F}}_{31}\cap\Omega} |\Phi_3 - \Phi_1|\,\mathrm{d}\mathcal{H}^1 = \mathcal{P}(\mathbf{F})\,.$$

for every partition $\mathbf{F}$ that have the same trace of the boundary of Ω of $\mathbf{E}$.

Note that the differences $\Phi_i - \Phi_j$ play a crucial role, we can actually focus directly on the differences. Indeed, any time we are able to find three divergence free vector fields $\Psi_{12}, \Psi_{23}, \Psi_{31} : \overline{\Omega} \to \mathbb{R}^2$ such that

- $|\Psi_{12}|, |\Psi_{23}|, |\Psi_{31}| \leq 1$ $\mathcal{H}^1$-a.e. in Ω,
- $\Psi_{ij} \cdot \nu_{ij} = 1$ $\mathcal{H}^1$-a.e. in Σ_{ij}, for $i,j = 1,2,3$ such that Ψ_{ij} is defined,

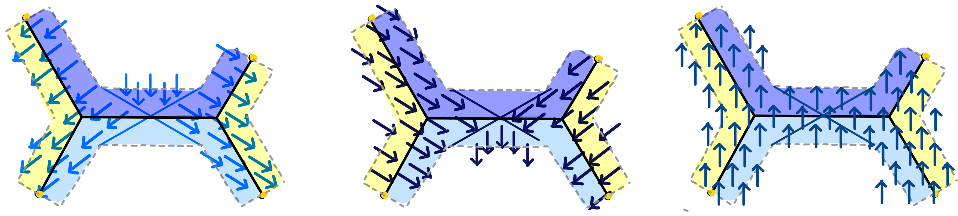


FIGURE 5. A calibration of a minimal network

- $\Psi_{12} + \Psi_{23} + \Psi_{31} = 0$ $\mathcal{H}^1$-a.e. in Ω.

We are then able to exhibit a calibration (Φ_1, Φ_2, Φ_3). Indeed, we can fix for example, $\Phi_1(x, y) := (0, 0)|_{\overline{\Omega}}$ and set $\Phi_2 := \Phi_1 - \Psi_{12}$ and $\Phi_3 := \Psi_{31} + \Phi_1$. The three vector fields $\Psi_{12}, \Psi_{23}, \Psi_{31}$ that do the job are the unitary vector fields depicted in Figure 5.

To conclude it remains to prove that to any network $\mathcal{N}$ satisfying the hypothesis of the theorem we can associate a partition (F_1, F_2, F_3) with $\mathrm{tr}_D \chi_{F_i} = \mathrm{tr}_D \chi_{E_i}$ whose boundary is contained in $\mathcal{N}$.

We associate to $\mathcal{N}$ a partition $(F_1, \ldots, F_n)$. Then each connected component of $\Omega \setminus \mathcal{N}$ corresponds to one of the F_i and there exists a unique $j \in \{1, 2, 3\}$ such that $F_i \cap \partial\Omega$ coincides with $E_j \cap \partial\Omega$. It is enough to rename F_i as F_j with $j \in \{1, 2, 3\}$. □

By checking the proof, one realises that the theorem can be immediately improved to the following stronger but less direct/transparent statement:

Corollary 7.1. *Let $\mathcal{N}^*$ be a minimal network. Then there exists Ω as in Theorem 7.2 and there exists a partition $\mathbf{E} = (E_1, E_2, E_3)$ of Ω whose boundary coincides with $\mathcal{N}^*$ such that $\mathcal{N}^*$ is a minimizer of the length functional in Ω among all networks $\mathcal{N}$ inducing a partition $\mathbf{F} = (F_1, F_2, F_3)$ of Ω with $\mathrm{tr}_D \chi_{F_i} = \mathrm{tr}_D \chi_{E_i}$.*

As one can see in Figure 6, there are networks that are competitors in the sense of the corollary but not in the sense of the theorem.

We have repeatedly use the fact that a network in Ω induces a partition of Ω. On the contrary, the boundary of a partition can be understood as a network in the plane. Based

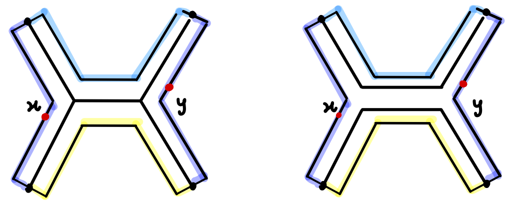


FIGURE 6. Left: a minimal network. Right: a competitor.

on this observation it is not hard to imagine that Theorem 7.2 can be stated directly in the language of partitions.

Corollary 7.2. *Let $\Omega \subset \mathbb{R}^2$ be open. Let $\mathbf{E} = (E_1, \dots, E_n)$ be a partition of Ω whose boundary is a minimal network and let d be the minimum of the distance between any two end-points and the shortest edge of $\mathcal{N}$. Then there exists a δ–neighbourhood D of $\mathcal{N}$ with $0 < \delta \leq \frac{\sqrt{3}}{8} d$ such that $\mathbf{E}$ is a minimizer in D for the perimeter among all partitions $\mathbf{F} = (F_1, \dots, F_n)$ with $\mathrm{tr}_D \chi_{F_i} = \mathrm{tr}_D \chi_{E_i}$.*

Proof. Let $\mathbf{E}$ be a partition as in the statement and $\mathbf{F}$ a competitor. To apply [20, Theorem 3.9] it is enough to canonically associate in D a partition of three sets $\widetilde{\mathbf{E}}$ to $\mathbf{E}$ and $\widetilde{\mathbf{F}}$ to $\mathbf{F}$, so that $P(\widetilde{\mathbf{E}}) = P(\mathbf{E})$ and $P(\widetilde{\mathbf{F}}) \leq P(\mathbf{F})$. In a δ–neighbourhood of $\mathcal{N}$ it is always possible to associate a partition of three sets $\widetilde{\mathbf{E}} = (\widetilde{E}_1, \widetilde{E}_2, \widetilde{E}_3)$ to $\mathbf{E} = (E_1, \dots, E_n)$ (see [20, Theorem 3.9] for details). Consider now a competitor $\mathbf{F}$. Since $\mathrm{tr}_D \chi_{F_i} = \mathrm{tr}_D \chi_{E_i}$, we can associate to each F_i such that $F_i \cap \partial D \neq \emptyset$ one of the three $\widetilde{F}_i$ in such a way that $\mathrm{tr}_D \chi_{\widetilde{F}_i} = \mathrm{tr}_D \chi_{\widetilde{E}_i}$. To conclude we associate to all the remaining F_i the set $\widetilde{F}_1$. $\square$

The corollary establishes the local minimality of the partition among all partitions that are close to the minimal candidate in a L^∞ sense. With extra work, one can obtain a statement of the same flavour but with the L^1 distance in place of the L^∞ distance [6].

8. Conclusions

In the paper we listed the properties of the flow that indicate that the structure/topology of the networks should simplify during the evolution.

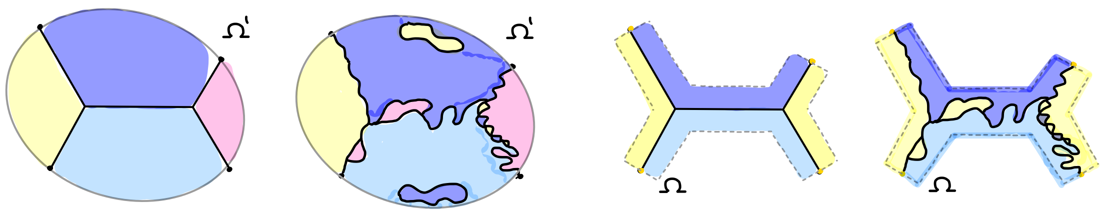


Figure 7. Left: two partition of Ω', the boundary of the first is a minimal network. Right: construction of Ω and relabelling of the two partitions inside of Ω.

- When at most five curves concur at an irregular junction, locally all flowouts are without loops. Hence, the number of grains, of curves and of triple junctions is non–increasing during the evolution. In particular, when a region enclosed by a loop vanishes, the total number of curves decreases at least by three and the total number of triple junctions decreases at least by two.
- If we suppose that all grains are very similar to each other and that percentage of non–hexagonal grains is sufficiently high during the evolution, then we proved that grains bound by less than six curve should disappear during the evolution and the average area of the (surviving) grains grows linearly.
- It is well-known that regular networks with straight segments are critical points of the length functional, thus steady states of the network flow. However, we expect the volume of the basin of attraction of all the many critical point of the length functional to be small in the space of networks. A first step in this direction is the quantitative estimate of the size of the basin of local minimality of regular networks with straight segments obtained by local calibrations.

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