INTRINSIC REGULAR SURFACES IN CARNOT GROUPS

SUPERFICI INTRINSECHE REGOLARI NEI GRUPPI DI CARNOT

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Abstract. A Carnot group $\mathbb{G}$ is a simply connected, nilpotent Lie group with stratified Lie algebra. Intrinsic regular surfaces in Carnot groups play the same role as C^1 surfaces in Euclidean spaces. As in Euclidean spaces, intrinsic regular surfaces can be locally defined in different ways: e.g. as non critical level sets or as continuously intrinsic differentiable graphs. The equivalence of these natural definitions is the problem that we are studying. This is a note based on the paper [8].

Sunto. Un gruppo di Carnot è un gruppo di Lie nilpotente e semplicemente connesso che ammette una Lie algebra stratificata. Nei gruppi di Carnot, le superfici intrinseche regolari giocano lo stesso ruolo delle superfici C^1 negli spazi euclidei. Come negli spazi euclidei, esse posso essere viste in diversi modi: per esempio, come insiemi di livello non critici oppure come grafici di mappe intrinsecamente differenziabili. L'equivalenza tra queste definizioni è il problema che viene preso in esame. Questo lavoro è basato sull'articolo [8].

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1. Introduction

In the last years a systematic attempt to develop a good notion of rectifiable sets in metric space and in particular inside Carnot groups, has become the object of many studies. For a general theory of rectifiable sets in Euclidean spaces one can see [13, 12, 25] while a general theory in metric spaces can be found in [1].

Rectifiable sets are classically defined as contained in the countable union of C^1 submanifolds. In this paper we focus our attention on the natural notion of C^1 surface, inside a special class of metric spaces i.e. the Carnot groups $\mathbb{G}$ of step κ. A short description of Carnot groups is in Section 2. Here we simply recall that they are connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a step κ stratification. Through the exponential map, a Carnot group $\mathbb{G}$ can be identified with $\mathbb{R}^N$, for a certain $N > 0$, endowed with a non commutative group operation.

Euclidean spaces are commutative Carnot groups and are the only commutative ones. The simplest but, at the same time, non-trivial instances of non-Abelian Carnot groups are provided by the Heisenberg groups $\mathbb{H}^n$ (see for instance [6]).

A Carnot group $\mathbb{G}$ is endowed with a natural left-invariant metric d. Non commutative Carnot groups, endowed with their left invariant metric are not Riemannian manifolds not even locally. In fact they are particular instances of so called sub Riemannian manifolds.

Main objects of study in this paper are the notions of regular surfaces and of intrinsic graphs and their link.

Intrinsic regular surfaces in Carnot groups should play the same role as C^1 surfaces in Euclidean spaces. In Euclidean spaces, C^1 surfaces can be locally defined in different ways: e.g. as non critical level sets of C^1 functions or, equivalently, as graphs of C^1 maps between complementary linear subspaces. In Carnot groups the equivalence of these definitions is not true any more. One of the main aim of this paper is to find the additional assumptions in order that these notions are equivalent in $\mathbb{G}$. Precisely we want to generalize the results on [2] valid in Heisenberg groups to the more general setting of Carnot groups.

Here by the word *intrinsic* and *regular* we want to emphasize a privileged role played by group translations and dilations, and its differential structure as Carnot-Carathéodory manifold in a sense we will precise below.

We begin recalling that an intrinsic regular hypersurface (i.e. a topological codimension 1 surface) $S \subset \mathbb{G}$ is locally defined as a non critical level set of a C^1 intrinsic function. More precisely, there exists a continuous function $f : \mathbb{G} \to \mathbb{R}$ such that locally $S = \{p \in \mathbb{G} : f(p) = 0\}$ and the intrinsic gradient $\nabla_{\mathbb{G}} f = (X_1 f, \dots, X_m f)$ exists in the sense of distributions and it is continuous and non vanishing on S. In a similar way, a k-codimensional regular surface $S \subset \mathbb{G}$ is locally defined as a non critical level set of a C^1 intrinsic vector function $F : \mathbb{G} \to \mathbb{R}^k$.

On the other hand, the intrinsic graphs came out naturally in [16], while studying level sets of Pansu differentiable functions from $\mathbb{H}^n$ to $\mathbb{R}$. The simple idea of intrinsic graph is the following one: let $\mathbb{M}$ and $\mathbb{W}$ be complementary subgroups of $\mathbb{G}$, i.e. homogeneous subgroups such that $\mathbb{W} \cap \mathbb{M} = \{0\}$ and $\mathbb{G} = \mathbb{W} \cdot \mathbb{M}$ (here $\cdot$ indicates the group operation in $\mathbb{G}$ and 0 is the unit element), then the intrinsic left graph of $\phi : \mathbb{W} \to \mathbb{M}$ is the set

$$\mathrm{graph}\,(\phi) := \{a \cdot \phi(a) \,|\, a \in \mathbb{W}\}.$$

Hence the existence of intrinsic graphs depends on the possibility of splitting $\mathbb{G}$ as a product of complementary subgroups hence it depends on the structure of the algebra $\mathfrak{g}$.

By Implicit Function Theorem, proved in [16] for the Heisenberg group and in [17] for a general Carnot group (see also [24, Theorem 1.3]) it follows

a $\mathbb{G}$-regular surface S locally is an intrinsic graph of a suitable function ϕ.

Consequently, given an intrinsic graph $S = \mathrm{graph}\,(\phi) \subset \mathbb{G}$, the main aim of this paper is to find necessary and sufficient assumptions on ϕ in order that the opposite implication is true. More precisely, in the main result of this note, i.e., Theorem 4.1, we characterize $\mathbb{G}$-regular intrinsic graphs as graphs of uniformly intrinsic differentiable functions $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ where $\mathbb{G}$ is a step κ Carnot group, $\mathbb{W}, \mathbb{M}$ complementary subgroups, with $\mathbb{M}$ horizontal and k dimensional. This result generalizes [2, Theorem 1.2] and has proved in [8, Theorem 4.1].

The rest of the paper is organized as follows. In **Section 2**, we recall the basic definitions and preliminary results that we will use later. In **Section 3**, we provide some properties of intrinsic linear maps and intrinsic differentiability ones. In **Section 4** we focus our attention on Theorem 4.1 and its corollaries.

2. Notations and preliminary results

2.1. **Carnot groups.** We begin by recalling briefly the definition of Carnot groups. For a general account see e.g. [6, 28].

A Carnot group $\mathbb{G} = (\mathbb{G}, \cdot, \delta_\lambda)$ of step κ is a connected and simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a stratification, i.e. a direct sum decomposition $\mathfrak{g} = V_1 \oplus V_2 \oplus \cdots \oplus V_\kappa$. The stratification has the further property that the entire Lie algebra $\mathfrak{g}$ is generated by its first layer V_1, the so called horizontal layer, that is

$$\begin{cases} [V_1, V_{i-1}] = V_i & \text{if } 2 \le i \le \kappa, \\ [V_1, V_\kappa] = \{0\}, \end{cases}$$

We denote by N the dimension of $\mathfrak{g}$ and by n_s the dimension of V_s.

The exponential map $\exp : \mathfrak{g} \to \mathbb{G}$ is a global diffeomorphism from $\mathfrak{g}$ to $\mathbb{G}$. Hence, if we choose a basis $\{X_1, \dots, X_N\}$ of $\mathfrak{g}$, any $p \in \mathbb{G}$ can be written in a unique way as $p = \exp(p_1 X_1 + \cdots + p_N X_N)$ and we can identify p with the N-tuple $(p_1, \dots, p_N) \in \mathbb{R}^N$ and $\mathbb{G}$ with $(\mathbb{R}^N, \cdot, \delta_\lambda)$. The identity of $\mathbb{G}$ is the origin of $\mathbb{R}^N$.

For any $\lambda > 0$, the (non isotropic) dilation $\delta_\lambda : \mathbb{G} \to \mathbb{G}$ are automorhisms of $\mathbb{G}$ and are defined as

$$\delta_\lambda(p_1, \dots, p_N) = (\lambda^{\alpha_1} p_1, \dots, \lambda^{\alpha_N} p_N),$$

where $\alpha_i \in \mathbb{N}$ is called homogeneity of the variable p_i in $\mathbb{G}$ and is given by $\alpha_i = j$ whenever $m_{j-1} < i \le m_j$ with $m_s - m_{s-1} = n_s$ and $m_0 = 0$. Hence $1 = \alpha_1 = \cdots = \alpha_{m_1} < \alpha_{m_1+1} = 2 \le \cdots \le \alpha_N = \kappa$.

The explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula. It has the form

$$p \cdot q = p + q + \mathcal{Q}(p, q) \quad \text{for all } p, q \in \mathbb{G} \equiv \mathbb{R}^N,$$

where $\mathcal{Q} = (\mathcal{Q}_1, \dots, \mathcal{Q}_N) : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ and every $\mathcal{Q}_i$ is a homogeneous polynomial of degree α_i with respect to the intrinsic dilations of $\mathbb{G}$, i.e. $\mathcal{Q}_i(\delta_\lambda p, \delta_\lambda q) = \lambda^{\alpha_i} \mathcal{Q}_i(p, q)$ for all $p, q \in \mathbb{G}$ and $\lambda > 0$.

Observe also that $\mathbb{G} = \mathbb{G}^1 \oplus \mathbb{G}^2 \oplus \cdots \oplus \mathbb{G}^\kappa$ where $\mathbb{G}^i = \exp(V_i) = \mathbb{R}^{n_i}$ is the i^{th} layer of $\mathbb{G}$ and to write $p \in \mathbb{G}$ as $(p^1, \dots, p^\kappa)$ with $p^i \in \mathbb{G}^i$. According to this

$$(1)\quad p \cdot q = (p^1 + q^1, p^2 + q^2 + \mathcal{Q}^2(p^1, q^1), \dots, p^\kappa + q^\kappa + \mathcal{Q}^\kappa((p^1, \dots, p^{\kappa-1}), (q^1, \dots, q^{\kappa-1})),$$

for every $p = (p^1, \dots, p^\kappa)$, $q = (q^1, \dots, q^\kappa) \in \mathbb{G}$. In particular $p^{-1} = (-p^1, \dots, -p^\kappa)$.

The norm of $\mathbb{R}^{n_s}$ is denoted with the symbol $|\cdot|$. For any $p \in \mathbb{G}$ the intrinsic left translation $\tau_p : \mathbb{G} \to \mathbb{G}$ are defined as

$$q \mapsto \tau_p q := p \cdot q = pq.$$

A *homogeneous norm* on $\mathbb{G}$ is a non negative function $p \mapsto \|p\|$ such that for all $p, q \in \mathbb{G}$ and for all $\lambda \geq 0$

$$\|p\| = 0 \quad \text{if and only if } p = 0,$$

$$\|\delta_\lambda p\| = \lambda \|p\|, \qquad \|p \cdot q\| \leq \|p\| + \|q\|.$$

Given any homogeneous norm $\|\cdot\|$, it is possible to introduce a distance in $\mathbb{G}$ given by

$$d(p, q) = d(p^{-1}q, 0) = \|p^{-1}q\| \qquad \text{for all } p, q \in \mathbb{G}.$$

We observe that any distance d obtained in this way is always equivalent with the Carnot-Carathéodory's distance d_{cc} of the group (see Proposition 5.1.4 and Theorem 5.2.8 [6]).

The distance d is well behaved with respect to left translations and dilations, i.e. for all $p, q, q' \in \mathbb{G}$ and $\lambda > 0$,

$$d(p \cdot q, p \cdot q') = d(q, q'), \qquad d(\delta_\lambda q, \delta_\lambda q') = \lambda d(q, q').$$

Moreover, by [6, Proposition 5.15.1], for any bounded subset $\Omega \subset \mathbb{G}$ there exist positive constants $c_1 = c_1(\Omega), c_2 = c_2(\Omega)$ such that for all $p, q \in \Omega$

$$c_1 |p - q| \leq d(p, q) \leq c_2 |p - q|^{1/\kappa},$$

and, in particular, the topology induced on $\mathbb{G}$ by d is the Euclidean topology.

We also define the distance $dist_d$ between two set $\Omega_1, \Omega_2 \subset \mathbb{G}$ by putting

$$dist_d(\Omega_1,\Omega_2) := \max\left\{ \sup_{q'\in\Omega_2} d(\Omega_1,q'),\ \sup_{Q\in\Omega_1} d(q,\Omega_2)\right\},$$

where $d(\Omega_1, q') := \inf\{d(q,q') \,:\, q\in\Omega_1\}$.

The Hausdorff dimension of $(\mathbb{G}, d)$ as a metric space is denoted *homogeneous dimension* of $\mathbb{G}$ and it can be proved to be the integer $\sum_{j=1}^{N}\alpha_j = \sum_{i=1}^{\kappa} i\ \mathrm{dim}V_i \geq N$ (see [26]).

The Haar measure of the group $\mathbb{G} = \mathbb{R}^N$ is the Lebesgue measure $d\mathcal{L}^N$. It is left (and right) invariant.

2.2. $\mathbb{C}^1_{\mathbb{G}}$ **functions and $\mathbb{G}$-regular surfaces.** (See [23, 28]). In [27] Pansu introduced an appropriate notion of differentiability for functions acting between Carnot groups. We recall this definition in the particular instance that is relevant here.

Let $\mathcal{U}$ be an open subset of a Carnot group $\mathbb{G}$. A function $f:\mathcal{U}\to\mathbb{R}^k$ is Pansu differentiable or more simply P-differentiable in $a_0\in\mathcal{U}$ if there is a homogeneous homomorphism

$$d_{\mathbf{P}}f(a_0):\mathbb{G}\to\mathbb{R}^k,$$

the Pansu differential of f in a_0, such that, for $B\in\mathcal{U}$,

$$\lim_{r\to0^+}\ \sup_{0<\|a_0^{-1}b\|<r} \frac{|f(b)-f(a_0)-d_{\mathbf{P}}f(a_0)(a_0^{-1}b)|}{\|a_0^{-1}b\|} = 0.$$

Saying that $d_{\mathbf{P}}f(a_0)$ is a homogeneous homomorphism we mean that $d_{\mathbf{P}}f(a_0):\mathbb{G}\to\mathbb{R}^k$ is a group homomorphism and also that $d_{\mathbf{P}}f(a_0)(\delta_\lambda b) = \lambda d_{\mathbf{P}}f(a_0)(b)$ for all $b\in\mathbb{G}$ and $\lambda\geq0$.

Observe that, later on in Definition 3.2, we give a different notion of differentiability for functions acting between subgroups of a Carnot group and we reserve the notation *df* or $df(a_0)$ for that differential.

We denote $C^1_{\mathbb{G}}(\mathcal{U},\mathbb{R}^k)$ the set of functions $f:\mathcal{U}\to\mathbb{R}^k$ that are P-differentiable in each $a\in\mathcal{U}$ and such that $d_{\mathbf{P}}f(a)$ depends continuously on a.

It can be proved that $f=(f_1,\dots,f_k)\in C^1_{\mathbb{G}}(\mathcal{U},\mathbb{R}^k)$ if and only if the distributional horizontal derivatives X_if_j, for $i=1\dots,m_1$, $j=1,\dots,k$, are continuous in $\mathcal{U}$. Remember that $C^1(\mathcal{U},\mathbb{R})\subset C^1_{\mathbb{G}}(\mathcal{U},\mathbb{R})$ with strict inclusion whenever $\mathbb{G}$ is not abelian (see [16, Remark 6]).

The *horizontal Jacobian* (or the *horizontal gradient* if $k = 1$) of $f : \mathcal{U} \to \mathbb{R}^k$ in $a \in \mathcal{U}$ is the matrix

$$\nabla_{\mathbb{G}} f(a) := [X_i f_j(a)]_{i=1\dots m_1, j=1\dots k}$$

when the partial derivatives $X_i f_j$ exist. Hence $f = (f_1, \dots, f_k) \in C^1_{\mathbb{G}}(\mathcal{U}, \mathbb{R}^k)$ if and only if its horizontal Jacobian exists and is continuous in $\mathcal{U}$. The *horizontal divergence* of $\phi := (\phi_1, \dots, \phi_{m_1}) : \mathcal{U} \to \mathbb{R}^{m_1}$ is defined as

$$\mathrm{div}_{\mathbb{G}} \phi := \sum_{j=1}^{m_1} X_j \phi_j$$

if $X_j \phi_j$ exist for $j = 1, \dots, m_1$.

The following proposition shows that the P-differential of a P-differentiable map f is represented by horizontal gradient $\nabla_{\mathbb{G}} f$:

Proposition 2.1. *If $f : \mathcal{U} \subset \mathbb{G} \to \mathbb{R}$ is P-differentiable at a point p and $d_{\mathbf{P}} f(p)$ is P-differential of f at p, then*

$$d_{\mathbf{P}} f(p)(q) = \nabla_{\mathbb{G}} f(p) q^1, \quad \textit{for all } q = (q^1, \dots, q^\kappa) \in \mathcal{U}.$$

Now we define co-abelian intrinsic submanifold as in [22, Definition 3.3.4]. Following the terminology of [18], we call these objects k-codimensional $\mathbb{G}$-regular surfaces.

Definition 2.1. $S \subset \mathbb{G}$ is a *k-codimensional $\mathbb{G}$-regular surface* if for every $p \in S$ there are a neighbourhood $\mathcal{U}$ of p and a function $f = (f_1, \dots, f_k) \in C^1_{\mathbb{G}}(\mathcal{U}, \mathbb{R}^k)$ such that

$$S \cap \mathcal{U} = \{q \in \mathcal{U} : f(q) = 0\},$$

and $d_{\mathbf{P}} f(q)$ is surjective, or equivalently if the $(k \times m_1)$ matrix $\nabla_{\mathbb{G}} f(q)$ has rank k, with $k < m_1$, for all $q \in \mathcal{U}$.

Notice that the topological dimension of a k-codimensional $\mathbb{G}$-regular surface is $N - k$. The class of $\mathbb{G}$-regular surfaces is different from the class of Euclidean regular surfaces. Indeed, in [21], the authors give an example of $\mathbb{H}^1$-regular surfaces, in $\mathbb{H}^1$ identified with $\mathbb{R}^3$, that are (Euclidean) fractal sets. Conversely, there are continuously differentiable 2-submanifolds in $\mathbb{R}^3$ that are not $\mathbb{H}^1$-regular surfaces (see [16, Remark 6.2] and [2, Corollary 5.11]).

2.3. Complementary subgroups and graphs.

Definition 2.2. We say that $\mathbb{W}$ and $\mathbb{M}$ are *complementary subgroups in* $\mathbb{G}$ if $\mathbb{W}$ and $\mathbb{M}$ are homogeneous subgroups of $\mathbb{G}$ such that $\mathbb{W} \cap \mathbb{M} = \{0\}$ and

$$\mathbb{G} = \mathbb{W} \cdot \mathbb{M}.$$

By this we mean that for every $p \in \mathbb{G}$ there are $p_{\mathbb{W}} \in \mathbb{W}$ and $p_{\mathbb{M}} \in \mathbb{M}$ such that $p = p_{\mathbb{W}} p_{\mathbb{M}}$.

The elements $p_{\mathbb{W}} \in \mathbb{W}$ and $p_{\mathbb{M}} \in \mathbb{M}$ such that $p = p_{\mathbb{W}} \cdot p_{\mathbb{M}}$ are unique because of $\mathbb{W} \cap \mathbb{M} = \{0\}$ and are denoted components of p along $\mathbb{W}$ and $\mathbb{M}$ or projections of P on $\mathbb{W}$ and $\mathbb{M}$. The projection maps $\mathbf{P}_{\mathbb{W}} : \mathbb{G} \to \mathbb{W}$ and $\mathbf{P}_{\mathbb{M}} : \mathbb{G} \to \mathbb{M}$ defined

$$\mathbf{P}_{\mathbb{W}}(p) = p_{\mathbb{W}}, \qquad \mathbf{P}_{\mathbb{M}}(p) = p_{\mathbb{M}}, \qquad \text{for all } p \in \mathbb{G}$$

are polynomial functions (see [15, Proposition 2.2.14]) if we identify $\mathbb{G}$ with $\mathbb{R}^N$, hence are C^∞. Nevertheless in general they are not Lipschitz maps, when $\mathbb{W}$ and $\mathbb{M}$ are endowed with the restriction of the left invariant distance d of $\mathbb{G}$ (see [15, Example 2.2.15]).

Remark 2.2. The stratification of $\mathbb{G}$ induces a stratifications on the complementary subgroups $\mathbb{W}$ and $\mathbb{M}$. If $\mathbb{G} = \mathbb{G}^1 \oplus \cdots \oplus \mathbb{G}^\kappa$ then also $\mathbb{W} = \mathbb{W}^1 \oplus \cdots \oplus \mathbb{W}^\kappa$, $\mathbb{M} = \mathbb{M}^1 \oplus \cdots \oplus \mathbb{M}^\kappa$ and $\mathbb{G}^i = \mathbb{W}^i \oplus \mathbb{M}^i$. A subgroup is *horizontal* if it is contained in the first layer $\mathbb{G}^1$. If $\mathbb{M}$ is horizontal then the complementary subgroup $\mathbb{W}$ is normal.

Proposition 2.3 (see [4], Proposition 3.2)**.** *If* $\mathbb{W}$ *and* $\mathbb{M}$ *are complementary subgroups in* $\mathbb{G}$ *there is* $c_0 = c_0(\mathbb{W}, \mathbb{M}) \in (0,1)$ *such that for each* $p_{\mathbb{W}} \in \mathbb{W}$ *and* $p_{\mathbb{M}} \in \mathbb{M}$

$$c_0(\|p_{\mathbb{W}}\| + \|p_{\mathbb{M}}\|) \leq \|p_{\mathbb{W}} p_{\mathbb{M}}\| \leq \|p_{\mathbb{W}}\| + \|p_{\mathbb{M}}\|. \tag{2}$$

Definition 2.3. We say that $S \subset \mathbb{G}$ is a *left intrinsic graph* or more simply a *intrinsic graph* if there are complementary subgroups $\mathbb{W}$ and $\mathbb{M}$ in $\mathbb{G}$ and $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ such that

$$S = \operatorname{graph}(\phi) := \{a\phi(a) : \, a \in \mathcal{E}\}.$$

Observe that, by uniqueness of the components along $\mathbb{W}$ and $\mathbb{M}$, if $S = \operatorname{graph}(\phi)$ then ϕ is uniquely determined among all functions from $\mathbb{W}$ to $\mathbb{M}$.

Proposition 2.4 (see [15], Proposition 2.2.18)**.** *If S is a intrinsic graph then, for all $\lambda > 0$ and for all $q \in \mathbb{G}$, $q \cdot S$ and $\delta_\lambda S$ are intrinsic graphs. In particular, if $S = \mathrm{graph}\,(\phi)$ with $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$, then*

(1) *For all $\lambda > 0$,*

$$\delta_\lambda\,(\mathrm{graph}\,(\phi)) = \mathrm{graph}\,(\phi_\lambda),$$

where $\phi_\lambda : \delta_\lambda \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ and $\phi_\lambda(a) := \delta_\lambda \phi(\delta_{1/\lambda} a)$, for $a \in \delta_\lambda \mathcal{E}$.

(2) *For any $q \in \mathbb{G}$,*

$$q \cdot \mathrm{graph}\,(\phi) = \mathrm{graph}\,(\phi_Q),$$

where $\phi_q : \mathcal{E}_q \subset \mathbb{W} \to \mathbb{M}$ is defined as $\phi_q(a) := (\mathbf{P}_\mathbb{M}(q^{-1}a))^{-1}\phi(\mathbf{P}_\mathbb{W}(q^{-1}a))$, for all $a \in \mathcal{E}_q := \{a \,:\, \mathbf{P}_\mathbb{W}(q^{-1}a) \in \mathcal{E}\}$.

The following notion of intrinsic Lipschitz function appeared for the first time in [16] and was studied, more diffusely, in [9, 10, 14, 15, 18, 29]. Intrinsic Lipschitz functions play the same role as Lipschitz functions in Euclidean context but they are different (see [28, Example 4.58]). Recently, in [11] the authors generalized this concept in the metric setting.

Definition 2.4. Let $\mathbb{W}, \mathbb{M}$ be complementary subgroups in $\mathbb{G}$, $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$. We say that ϕ is *intrinsic C_L-Lipschitz in $\mathcal{E}$*, or simply intrinsic Lipschitz, if there is $C_L \geq 0$ such that

$$\|\mathbf{P}_\mathbb{M}(q^{-1}q')\| \leq C_L \|\mathbf{P}_\mathbb{W}(q^{-1}q')\|, \qquad \text{for all } q, q' \in \mathrm{graph}\,(\phi).$$

$\phi : \mathcal{E} \to \mathbb{M}$ is locally intrinsic Lipschitz in $\mathcal{E}$ if ϕ is intrinsic Lipschitz in $\mathcal{E}'$ for every $\mathcal{E}' \subset\subset \mathcal{E}$.

Remark 2.5. In this paper, we are interested mainly in the special case when $\mathbb{M}$ is a horizontal subgroup and consequently $\mathbb{W}$ is a normal subgroup. Under these assumptions, for all $p = a\phi(a), q = b\phi(b) \in \mathrm{graph}\,(\phi)$ we have

$$\mathbf{P}_\mathbb{M}(p^{-1}q) = \phi(a)^{-1}\phi(b), \quad \mathbf{P}_\mathbb{W}(p^{-1}q) = \phi(a)^{-1}a^{-1}b\phi(a).$$

Hence, if $\mathbb{M}$ is a horizontal subgroup, $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ is intrinsic Lipschitz if

$$\|\phi(a)^{-1}\phi(b)\| \leq C_L \|\phi(a)^{-1}a^{-1}b\phi(a)\| \qquad \text{for all } a, b \in \mathcal{E}.$$

Moreover, if ϕ is intrinsic Lipschitz then $\|\phi(a)^{-1}a^{-1}b\phi(a)\|$ is comparable with $\|p^{-1}q\|$. Indeed from (2)

$$\begin{aligned} c_0\|\phi(a)^{-1}a^{-1}b\phi(a)\| &\leq \|p^{-1}q\| \\ &\leq \|\phi(a)^{-1}a^{-1}b\phi(a)\| + \|\phi(a)^{-1}\phi(b)\| \\ &\leq (1+C_L)\|\phi(a)^{-1}a^{-1}b\phi(a)\|. \end{aligned}$$

The quantity $\|\phi(a)^{-1}a^{-1}b\phi(a)\|$, or better a symmetrized version of it, can play the role of a ϕ dependent, quasi distance on $\mathcal{E}$. See e.g. [2].

In Euclidean spaces, i.e. when $\mathbb{G}$ is $\mathbb{R}^N$ and the group operation is the usual Euclidean sum of vectors, intrinsic Lipschitz functions are the same as Lipschitz functions. On the contrary, when $\mathbb{G}$ is a general non commutative Carnot group and $\mathbb{W}$ and $\mathbb{M}$ are complementary subgroups, the class of intrinsic Lipschitz functions from $\mathbb{W}$ to $\mathbb{M}$ is different from the class Lipschitz functions (see [14, Example 2.3.9]). More precisely, if $\phi : \mathbb{W} \to \mathbb{M}$ is intrinsic Lipschitz then in general does not exists a constant C such that

$$\|\phi(a)^{-1}\phi(b)\| \leq C\|a^{-1}b\| \quad \text{for } a, b \in \mathbb{W},$$

not even locally. Nevertheless the following weaker result holds true:

Proposition 2.6 (see [15], Proposition 3.1.8)**.** *Let $\mathbb{W}, \mathbb{M}$ be complementary subgroups in a step κ Carnot group $\mathbb{G}$. Let $\phi : \mathcal{E} \subset \mathbb{W} \to \mathbb{M}$ be an intrinsic C_L-Lipschitz function. Then, for all $r > 0$,*

(1) *there is $C_1 = C_1(\phi, r) > 0$ such that*

$$\|\phi(a)\| \leq C_1 \quad \textit{for all } a \in \mathcal{E} \textit{ with } \|a\| \leq r$$

(2) *there is $C_2 = C_2(C_L, r) > 0$ such that ϕ is locally $1/\kappa$-Hölder continuous i.e.*

$$\|\phi(a)^{-1}\phi(b)\| \leq C_2\|a^{-1}b\|^{1/\kappa} \quad \textit{for all } a, b \textit{ with } \|a\|, \|b\| \leq r.$$

3. Intrinsic differentiability

Here we recall a different notion of differentiability, the so called *intrinsic differentiability* that is, by its very definition, invariant under translations. A function is intrinsic differentiable if it is locally well approximated by *intrinsic linear* functions that are functions whose graph is a homogeneous subgroup in $\mathbb{G}$.

Definition 3.1. Let $\mathbb{W}$ and $\mathbb{M}$ be complementary subgroups in $\mathbb{G}$. Then $\ell : \mathbb{W} \to \mathbb{M}$ is *intrinsic linear* if ℓ is defined on all of $\mathbb{W}$ and if graph$\,(\ell)$ is a homogeneous subgroup of $\mathbb{G}$.

Intrinsic linear functions can be algebraically caracterized as follows.

Proposition 3.1 (see [14], Propositions 3.1.3 and 3.1.6)**.** *Let* $\mathbb{W}$ *and* $\mathbb{M}$ *be complementary subgroups in* $\mathbb{G}$*. Then* $\ell : \mathbb{W} \to \mathbb{M}$ *is intrinsic linear if and only if*

$$\ell(\delta_\lambda a) = \delta_\lambda(\ell(a)), \qquad \text{for all } a \in \mathbb{W} \text{ and } \lambda \geq 0$$
$$\ell(ab) = (\mathbf{P}_{\mathbb{H}}(\ell(a)^{-1}b))^{-1}\ell(\mathbf{P}_{\mathbb{W}}(\ell(a)^{-1}b)), \qquad \text{for all } a, b \in \mathbb{W}.$$

Moreover any intrinsic linear function ℓ *is a polynomial function and it is intrinsic Lipschitz with Lipschitz constant* $C_L := \sup\{\|\ell(a)\| \,:\, \|a\| = 1\}$*. Note that* $C_L < +\infty$ *because* ℓ *is continuous. Moreover*

$$\|\ell(a)\| \leq C_L\|a\|, \qquad \text{for all } a \in \mathbb{W}.$$

In particular, if $\mathbb{W}$ *is normal in* $\mathbb{G}$ *then* $\ell : \mathbb{W} \to \mathbb{M}$ *is intrinsic linear if and only if*

$$\begin{aligned} &\ell(\delta_\lambda a) = \delta_\lambda(\ell(a)), \qquad \text{for all } a \in \mathbb{W} \text{ and } \lambda \geq 0 \\ &\ell(ab) = \ell(a)\ell\big(\ell(a)^{-1}b\ell(a)\big), \qquad \text{for all } a, b \in \mathbb{W}. \end{aligned} \tag{3}$$

We use intrinsic linear functions to define intrinsic differentiability as in the usual definition of differentiability.

Definition 3.2. Let $\mathbb{W}$ and $\mathbb{M}$ be complementary subgroups in $\mathbb{G}$ and let $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{M}$ with $\mathcal{O}$ open in $\mathbb{W}$. For $a \in \mathcal{O}$, let $p := a \cdot \phi(a)$ and $\phi_{p^{-1}} : \mathcal{O}_{p^{-1}} \subset \mathbb{W} \to \mathbb{M}$ be the shifted function defined in Proposition 2.4. We say that ϕ is *intrinsic differentiable in* a if the shifted function $\phi_{p^{-1}}$ is intrinsic differentiable in 0, i.e. if there is a intrinsic linear $d\phi_a : \mathbb{W} \to \mathbb{M}$ such that

$$\lim_{r \to 0^+} \sup_{0<\|b\|<r} \frac{\|d\phi_a(b)^{-1}\phi_{p^{-1}}(b)\|}{\|b\|} = 0. \tag{4}$$

The function $d\phi_a$ is the *intrinsic differential of* ϕ *at* a.

Remark 3.2. Definition 3.2 is a natural one because of the following observations.

(i) If ϕ is intrinsic differentiable in $a \in \mathcal{O}$, there is a unique intrinsic linear function $d\phi_a$ satisfying (4). Moreover ϕ is continuous at a. (See Theorem 3.2.8 and Proposition 3.2.3 in [14]).

(ii) The notion of intrinsic differentiability is invariant under group translations. Precisely, let $p := a\phi(a), q := b\phi(b)$, then ϕ is intrinsic differentiable in a if and only if $\phi_{qp^{-1}} := (\phi_{p^{-1}})_q$ is intrinsic differentiable in b.

(iii) The analytic definition of intrinsic differentiability of Definition 3.2 has an equivalent geometric formulation. Indeed intrinsic differentiability in one point is equivalent to the existence of a tangent subgroup to the graph (see [14, Theorem 3.2.8]). Let $\phi : \mathbb{W} \to \mathbb{M}$ be such that $\phi(0) = 0$. We say that an homogeneous subgroup $\mathbb{T}$ of $\mathbb{G}$ is a tangent subgroup to graph (ϕ) in 0 if

(1) $\mathbb{T}$ is a complementary subgroup of $\mathbb{M}$
(2) in any compact subset of $\mathbb{G}$

$$\lim_{\lambda \to \infty} \delta_\lambda (\text{graph}\,(\phi)) = \mathbb{T}$$

in the sense of Hausdorff convergence.

Moreover in [14, Theorem 3.2.8], the authors show that ϕ is intrinsic differentiable in 0 if and only if graph (ϕ) has a tangent subgroup $\mathbb{T}$ in 0 and in this case $\mathbb{T} = \text{graph}\,(d\phi_0)$.

In addition to pointwise intrinsic differentiability, we are interested in an appropriate notion of continuously intrinsic differentiable functions. For functions acting between complementary subgroups, one possible way is to introduce a stronger, i.e. uniform, notion of intrinsic differentiability in the general setting of Definition 3.2.

Definition 3.3. Let $\mathbb{W}$ and $\mathbb{M}$ be complementary subgroups in $\mathbb{G}$ and $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{M}$ with $\mathcal{O}$ open in $\mathbb{W}$. For any $a \in \mathcal{O}$, let $p := a \cdot \phi(a)$ and $\phi_{p^{-1}} : \mathcal{O}_{p^{-1}} \subset \mathbb{W} \to \mathbb{M}$ be the shifted function defined in Proposition 2.4. We say that ϕ is *uniformly intrinsic differentiable in* $a_0 \in \mathcal{O}$ or ϕ is *u.i.d. in* a_0 if there exist a intrinsic linear function

$d\phi_{a_0} : \mathbb{W} \to \mathbb{M}$ such that

$$\lim_{r\to 0^+} \sup_{\|a_0^{-1}a\|<r} \sup_{0<\|b\|<r} \frac{\|d\phi_{a_0}(b)^{-1}\phi_{p^{-1}}(b)\|}{\|b\|} = 0. \tag{5}$$

Analogously, ϕ is u.i.d. in $\mathcal{O}$ if it is u.i.d. in every point of $\mathcal{O}$.

Remark 3.3. We recall that in [4, Definition 3.16] the authors give another notion of uniformly intrinsic differentiable map. It is possible compare these two notions; indeed, for example, Proposition 3.7 (3) is in the definition of u.i.d. for [4]. Moreover, it is clear, taking $a = a_0$ in (5), that if ϕ is uniformly intrinsic differentiable in a_0 then it is intrinsic differentiable in a_0 and $d\phi_{a_0}$ is the intrinsic differential of ϕ at a_0 (that is the first point of the definition for [4]).

Remark 3.4. In Heisenberg groups, it is known after the results in [2, 5] that the intrinsic differentiability of ϕ is equivalent to the existence and continuity of suitable 'derivatives' $D^\phi\phi$ of ϕ. The non linear first order differential operators D^ϕ_j were introduced by Serra Cassano et al. in the context of Heisenberg groups $\mathbb{H}^n$ (see [28] and the references therein) and, in the first Heisenberg group $\mathbb{H}^1$, $D^\phi\phi$ reduces to the classical Burgers' equation. This issue has been fully addressed in [3] in the general context of Carnot groups (see also [8]).

From now on we restrict our setting studying the notions of intrinsic differentiability and of uniform intrinsic differentiability for functions $\phi : \mathbb{W} \to \mathbb{H}$ when $\mathbb{H}$ is a horizontal subgroup. When $\mathbb{H}$ is horizontal, $\mathbb{W}$ is always a normal subgroup since, as observed in Remark 2.2, it contains the whole strata $\mathbb{G}^2, \dots, \mathbb{G}^\kappa$. In this case, the more explicit form of the shifted function $\phi_{p^{-1}}$ allows a more explicit form of equations (4) and (5).

First we observe that, when the target space is horizontal, intrinsic linear functions are Euclidean linear functions from the first layer of $\mathbb{W}$ to $\mathbb{H}$. The analogous of the following proposition is [4, Proposition 3.23] in the Heisenberg groups.

Proposition 3.5 (see [8], Proposition 3.4)**.** *Let $\mathbb{W}$ and $\mathbb{H}$ be complementary subgroups in $\mathbb{G}$ with $\mathbb{H}$ horizontal. Then an intrinsic linear function $\ell : \mathbb{W} \to \mathbb{H}$ depends only on the variables in the first layer $\mathbb{W}^1 := \mathbb{W} \cap \mathbb{G}^1$ of $\mathbb{W}$. That is*

$$\ell(a) = \ell(a^1, 0, \dots, 0), \qquad \text{for all } a = (a^1, \dots, a^\kappa) \in \mathbb{W}. \tag{6}$$

Moreover there is $C_L \geq 0$ such that, for all $a \in \mathbb{W}$,

$$\|\ell(a)\| \leq C_L \|(a^1, 0, \dots, 0)\| \tag{7}$$

and $\ell_{|\mathbb{W}^1} : \mathbb{W}^1 \to \mathbb{H}$ is Euclidean linear.

Finally if $k < m_1$ is the dimension of $\mathbb{H}$ and if, without loss of generality, we assume that

$$\mathbb{H} = \{p = (p_1, \dots, p_N) : p_{k+1} = \dots = p_N = 0\}, \quad \mathbb{W} = \{p = (p_1, \dots, p_N) : p_1 = \dots = p_k = 0\}$$

then there is a $k \times (m_1 - k)$ matrix $\mathcal{L}$ such that

$$\ell(a) = \big(\mathcal{L}(a_{k+1}, \dots, a_{m_1})^T, 0, \dots, 0\big) \tag{8}$$

for all $a = (a_1, \dots, a_N) \in \mathbb{W}$.

Keeping in mind this special form of intrinsic linear functions we obtain the following special form of intrinsic differentiability. The reader can see Proposition 3.25 (ii) and Proposition 3.26 (ii) in [4] for the Heisenberg groups.

Proposition 3.6 ([8], Proposition 3.5)**.** *Let $\mathbb{H}$ and $\mathbb{W}$ be complementary subgroups of $\mathbb{G}$, $\mathcal{O}$ open in $\mathbb{W}$ and $\mathbb{H}$ horizontal. Then $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$ is intrinsic differentiable in $a_0 \in \mathcal{O}$ if and only if there is a intrinsic linear $d\phi_{a_0} : \mathbb{W} \to \mathbb{H}$ such that*

$$\lim_{r \to 0^+} \sup_{0 < \|a_0^{-1} b\| < r} \frac{\|\phi(b) - \phi(a_0) - d\phi_{a_0}(a_0^{-1} b)\|}{\|\phi(a_0)^{-1} a_0^{-1} b \phi(a_0)\|} = 0.$$

Analogously, ϕ is uniformly intrinsic differentiable in $a_0 \in \mathcal{O}$ *if there is a intrinsic linear $d\phi_{a_0} : \mathbb{W} \to \mathbb{H}$ such that*

$$\lim_{r \to 0^+} \sup_{\|a_0^{-1} a\| < r} \sup_{0 < \|a^{-1} b\| < r} \frac{\|\phi(b) - \phi(a) - d\phi_{a_0}(a^{-1} b)\|}{\|\phi(a)^{-1} a^{-1} b \phi(a)\|} = 0$$

where r is small enough so that $\mathcal{U}(a_0, 2r) \subset \mathcal{O}$.

We conclude this section by giving a regularity result. The point (1) of the following proposition states precisely a natural relation between uniform intrinsic differentiability and intrinsic Lipschitz continuity (see [4, Proposition 3.30] for the Heisenberg groups). The point (2) is a generalization of what was previously known for u.i.d. functions in Heisenberg groups (see [2, Theorem 1.3]).

Proposition 3.7 (see [8], Proposition 3.7)**.** *Let $\mathbb{H}$, $\mathbb{W}$ be complementary subgroups of $\mathbb{G}$ with $\mathbb{H}$ horizontal. Let $\mathcal{O}$ be open in $\mathbb{W}$ and $\phi : \mathcal{O} \to \mathbb{H}$ be u.i.d. in $\mathcal{O}$. Then*

(1) *ϕ is intrinsic Lipschitz continuous in every relatively compact subset of $\mathcal{O}$.*

(2) *$\phi \in h^{1/\kappa}_{loc}(\mathcal{O})$, that is $\phi \in C(\mathcal{O}, \mathbb{R})$ and for all $\mathcal{F} \subset\subset \mathcal{O}$ and $a, b \in \mathcal{F}$*

$$\lim_{r \to 0^+} \sup_{0<\|a^{-1}b\|<r} \frac{\|\phi(b) - \phi(a)\|}{\|a^{-1}b\|^{1/\kappa}} = 0. \tag{9}$$

(3) *the function $a \mapsto d\phi_a$ is continuous in $\mathcal{O}$.*

4. $\mathbb{G}$-regular surfaces

The main result of this note is Theorem 4.1 where we prove that, if $\mathbb{H}$ is a horizontal subgroup, the intrinsic graph of $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$ is a $\mathbb{G}$-regular k-codimensional surface if and only if ϕ is uniformly intrinsic differentiable in $\mathcal{O}$. The proof of this theorem requires both Whitney's Extension Theorem and Implicit Function Theorem ([24, Theorem 1.3]) in Carnot groups. The proof of Whitney's Extension Theorem can be found in [19] for Carnot groups of step two only, but it is identical for general Carnot groups (see [7, Theorem 2.3.8]).

Theorem 4.1 (see [8], Theorem 4.1)**.** *Let $\mathbb{W}$ and $\mathbb{H}$ be complementary subgroups of a Carnot group $\mathbb{G}$ with $\mathbb{H}$ horizontal and k dimensional. Let $\mathcal{O}$ be open in $\mathbb{W}$, $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$ and $S :=$ graph (ϕ). Then for every $a_0 \in \mathcal{O}$ the following are equivalent:*

(1) *there are a neighbourhood $\mathcal{U}$ of $a_0 \cdot \phi(a_0)$ and $f \in C^1_{\mathbb{G}}(\mathcal{U}; \mathbb{R}^k)$ such that*

$$S \cap \mathcal{U} = \{p \in \mathcal{U} : f(p) = 0\}$$

$$d_{\mathbf{P}} f(q)_{|\mathbb{H}} : \mathbb{H} \to \mathbb{R}^k \quad \textit{is bijective for all } q \in \mathcal{U}$$

and $q \mapsto \left(d_{\mathbf{P}} f(q)_{|\mathbb{H}}\right)^{-1}$ is continuous.

(2) *ϕ is u.i.d. in a neighbourhood $\mathcal{O}' \subset \mathcal{O}$ of a_0.*

Moreover, if (1) *or equivalently* (2)*, hold then, for all $a \in \mathcal{O}$ the intrinsic differential $d\phi_a$ is*

$$d\phi_a = -\left(d_{\mathbf{P}} f(a\phi(a))_{|\mathbb{H}}\right)^{-1} \circ d_{\mathbf{P}} f(a\phi(a))_{|\mathbb{W}}.$$

Remark 4.2. If, without loss of generality, we choose a base $X_1, \dots, X_N$ of $\mathfrak{g}$ such that $X_1, \dots, X_k$ are horizontal vector fields, $\mathbb{H} = \exp(\text{span}\{X_1, \dots, X_k\})$ and $\mathbb{W} = \exp(\text{span}\{X_{k+1}, \dots, X_N\})$ then

$$\mathbb{H} = \{p \in \mathbb{G} : p_{k+1} = \dots = p_N = 0\} \qquad \mathbb{W} = \{p \in \mathbb{G} : p_1 = \dots = p_k = 0\},$$

and, if $f = (f_1, \dots, f_k)$, then $\nabla_{\mathbb{G}} f = (\, \mathcal{M}_1 \mid \mathcal{M}_2 \,)$ where

$$\mathcal{M}_1 := \begin{pmatrix} X_1 f_1 \dots X_k f_1 \\ \vdots \quad \ddots \quad \vdots \\ X_1 f_k \dots X_k f_k \end{pmatrix}, \qquad \mathcal{M}_2 := \begin{pmatrix} X_{k+1} f_1 \dots X_{m_1} f_1 \\ \vdots \quad \ddots \quad \vdots \\ X_{k+1} f_k \dots X_{m_1} f_k \end{pmatrix}.$$

Moreover, for all $q \in \mathcal{U}$, for all $a \in \mathcal{O}$ and for all $p \in \mathbb{G}$

$$(d_{\mathbf{P}} f(q))\,(p) = (\nabla_{\mathbb{G}} f(\,q))p^1$$

and the intrinsic differential is

$$\begin{aligned} d\phi_a(b) &= \left(\left(\nabla^{\phi}\phi(a)\right)(b_{k+1}, \dots, b_{m_1})^T, 0, \dots, 0\right) \\ &= \left(\left(-\mathcal{M}_1(a\phi(a))^{-1}\mathcal{M}_2(a\phi(a))\right)(b_{k+1}, \dots, b_{m_1})^T, 0, \dots, 0\right), \end{aligned} \tag{10}$$

for all $b = (b_1, \dots, b_N) \in \mathbb{W}$.

An immediate corollary of the Theorem 4.1 is the following.

Corollary 4.3. *Under the same assumptions of Theorem* 4.1*, if* $S := \text{graph}\,(\phi)$ *satisfies the condition* (1) *of Theorem* 4.1*, then*

(1) *the function* $b \mapsto d\phi(b)$ *is continuous in* $\mathcal{O}$*.*
(2) $\phi \in h^{1/\kappa}_{loc}(\mathcal{O})$*, that is* $\phi \in C(\mathcal{O}, \mathbb{R})$ *and for all* $\mathcal{F} \subset\subset \mathcal{O}$ *and* $a, b \in \mathcal{F}$

$$\lim_{r \to 0^+} \sup_{0 < \|a^{-1}b\| < r} \frac{\|\phi(b) - \phi(a)\|}{\|a^{-1}b\|^{1/\kappa}} = 0.$$

We conclude this note observing that u.i.d. functions do exist. In particular, when $\mathbb{H}$ is a horizontal subgroup, $\mathbb{H}$ valued Euclidean C^1 functions are u.i.d.

Theorem 4.4 (see [8], Theorem 4.9)**.** *If* $\mathbb{W}$ *and* $\mathbb{H}$ *are complementary subgroups of a Carnot group* $\mathbb{G}$ *with* $\mathbb{H}$ *horizontal and* k *dimensional. If* $\mathcal{O}$ *is open in* $\mathbb{W}$ *and* $\phi : \mathcal{O} \subset \mathbb{W} \to \mathbb{H}$ *is such that* $\phi \in C^1(\mathcal{O}, \mathbb{H})$ *then* ϕ *is u.i.d. in* $\mathcal{O}$*.*

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