SECOND ORDER p-EVOLUTION EQUATIONS WITH CRITICAL NONLINEARITY

EQUAZIONI DI p-EVOLUZIONE DEL SECONDO ORDINE CON NONLINEARITÀ CRITICA

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Abstract. In this paper, we study critical nonlinearities for global small data solutions to the plate equation and other second order p-evolution equations, possibly under the action of a noneffective dissipative term.

Sunto. In questo lavoro, richiamiamo alcuni recenti risultati in cui viene ottenuto l'esponente critico per la soluzione globale (in tempo) con dati sufficientemente piccoli per l'equazione della piastra e altre equazioni di p-evoluzione del secondo ordine, con nonlinearità di tipo potenza. Con l'aggiunta di un termine dissipativo noneffettivo, cioè che non cancella le oscillazioni, ma le smorza solamente, è stato recentemente mostrato come l'esponente critico rimanga lo stesso del caso non dissipativo, almeno in dimensione bassa. In questo lavoro, viene studiata una condizione integrale sul termine nonlineare che permette di distinguere precisamente la regione di esistenza globale da quella di nonesistenza globale della soluzione, raffinando i risultati sugli esponenti critici per non-linearità di tipo potenza.

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1. Introduction

It has been recently proved [29] that the critical exponent for global (in time) small data solutions to

$$\begin{cases} u_{tt} + Au = f(u), \qquad t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = 0, \\ u_t(0,x) = u_1(x), \end{cases} \tag{1}$$

where $A = (-\Delta)^\sigma$, with $\sigma > 1$ and $f(u) = |u|^p$, with $p > 1$, is

$$p_c = 1 + \frac{2}{\frac{n}{\sigma} - 1}. \tag{2}$$

In particular, in [29, Theorem 2.2] it is proved that global (in time) small data solutions to (1) exist in space dimension $n \in (\sigma, 2\sigma]$ if $p > p_c$, under the assumption

$$f(0) = 0, \qquad |f(u) - f(v)| \leq C\,|u - v|\,(|u|^{p-1} + |v|^{p-1}) \tag{3}$$

while in [17] it is proved that no global (in time) solution to (1) may exist, under a suitable data sign assumption, if $f(u) \geq C\,|u|^p$ or $f(u) \leq -C\,|u|^p$, with $1 < p \leq p_c$.

The proof of this result is based on the use of $L^1 - L^p$ estimates for the linear problem

$$\begin{cases} u_{tt} + Au = 0, \qquad t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = 0, \\ u_t(0,x) = u_1(x), \end{cases} \tag{4}$$

and on the use of Duhamel's principle and a classic contraction argument. When

$$A = \sum_{|\alpha|=2p} a_\alpha \partial_x^\alpha,$$

for some integer $p > 1$, and the roots $\lambda_\pm$ of

$$\lambda^2 + (-1)^p \sum_{|\alpha|=2p} a_\alpha \xi^\alpha = 0$$

are distinct and pure imaginary, for any $\xi \neq 0$, the equation in (4) is a p-evolution second order equation, in the sense of Petrowski. For $p = 1$, we have strictly hyperbolic equations. Since the letter p is already used in the nonlinearity (3), we replace it by σ in this paper.

When $A = (-\Delta)^\sigma$, the corresponding roots are $\pm i|\xi|^\sigma$. Moreover, $\sigma > 1$ may be non integer, unless differently specified (in this case, $(-\Delta)^\sigma g = \mathscr{F}^{-1}(|\xi|^{2\sigma}\hat{g})$). For $\sigma = 2$, the equation in (4) is often called plate equation.

The crucial aspect of $L^1 - L^p$ estimates for (4), is that they do not hold [45, 48, 53] if the quantity

$$d(p) = \frac{n}{\sigma}\left(1 - \frac{1}{p}\right) - n\left(\frac{1}{2} - \frac{1}{p}\right) \tag{5}$$

is larger than 1, due to the fact that

$$m_\sigma(\xi) = \frac{\sin|\xi|^\sigma}{|\xi|^\sigma} \notin M_1^p.$$

The notation above means that m_σ is not a multiplier from L^1 to L^p, that is, the operator $f \in \mathcal{S} \mapsto \mathscr{F}^{-1}(m_\sigma \hat{f})$ is not bounded from L^1 to L^p; equivalently, the inverse Fourier transform $\mathscr{F}^{-1}(m_\sigma)$ is not in L^p (see [37, Theorem 1.4]).

When $d(p) < 1$, the following $L^1 - L^p$ estimates hold for the solution to (4):

$$\|u(t,\cdot)\|_{L^p} \leq C\, t^{1-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}\, \|u_1\|_{L^1}, \quad t > 0. \tag{6}$$

This result for $\sigma > 1$ is analogous to the corresponding result for $\sigma = 1$ (i.e. $A = -\Delta$, and (4) is the wave equation), thought when $\sigma = 1$ the expression for $d(p)$ in (5) is no longer valid, and is replaced by

$$d(p) = 1 - n\left(1 - \frac{1}{p}\right) - (n-1)\left(\frac{1}{2} - \frac{1}{p}\right). \tag{7}$$

This difference, which amounts to the fact that $n - 1$ in (7) replaces n in (5), is related to the different properties of the phase function $|\xi|^\sigma$ in $e^{\pm i|\xi|^\sigma}$, when $\sigma = 1$ or $\sigma \neq 1$ (for instance, the Hessian of $|\xi|^\sigma$ is singular if, and only if, $\sigma = 1$). This difference makes the case $\sigma = 1$ in (1) very peculiar with respect to all other cases $\sigma \neq 1$.

In [13], it has been shown that some $L^1 - L^p$ estimates hold for any $p \in [1, \infty]$, even if the necessary condition $d(p) \leq 1$ is violated, if a noneffective damping is added to (4) when $\sigma > 1$. The term "noneffective" was originally introduced in [56, 57] for a classical damped wave equation with time-dependent coefficients and later extended in [11] for damped σ-evolution equations. The case considered is

$$\begin{cases} u_{tt} + Au + A^{\frac{\theta}{2}} u_t = f(u), & t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = 0, \\ u_t(0,x) = u_1(x), \end{cases} \tag{8}$$

with $\theta \in [1,2]$. In the homogeneous case $\theta = 1$, the estimates are as in (6), but without the restriction $d(p) \leq 1$. In the case $\theta \in (1,2]$, the decay rate remains the same if $d(p) < 1$, but a loss of decay rate $(1+t)^{(d(p)-1)_+\left(1-\frac{1}{\theta}\right)}$ appears if $d(p) \geq 1$, possibly with an additional logarithmic loss. The same phenomenon was already observed in [51] for $p = 1$ (in odd space dimension n) and $p = \infty$, in the case $\sigma = 1$ and $\theta = 2$, and in [21] for $p = 1$ in the case $\sigma = \theta = 2$ in space dimension $n \geq 4$.

During his "Bruno Pini seminar" held on 26 January 2023, the first author announced that he and M.R. Ebert obtained an analogous result for the wave equation (i.e. $\sigma = 1$), for a class of dissipative wave equations which include (8) as a special case. In particular, they obtained the long time estimates

$$\|u(t,\cdot)\|_{L^p} \leq C\,(1+t)^{1-n\left(1-\frac{1}{p}\right)+(d(p)-1)_+\left(1-\frac{1}{\theta}\right)} \|u_1\|_{L^1},$$

with a $(\log(e+t))^{\frac{1}{2}}$ loss of decay in the case $n = p = 2$ (see [15]).

In recent years many authors have investigated the critical power p_c for nonlinearities as in (3), for damped equations

$$\begin{cases} u_{tt} + Au + A^{\frac{\theta}{2}} u_t = f(u), & t \geq 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = 0, \\ u_t(0,x) = u_1(x). \end{cases} \tag{9}$$

By critical power, we mean that for $p > p_c$ the global (in time) solution to (1) or (9) exists for sufficiently small initial data in a suitable space, whereas no global solutions exist if $1 < p < p_c$ under a suitable sign condition on the data and $f(u) \geq C|u|^p$ or $f(u) \leq -C|u|^p$. The case $p = p_c$ often belongs to the nonexistence range, though sometimes it belong to the existence range (this happens, for instance, in some models where fractional derivatives in time appears, see [7, 16, 19, 20]).

The results available in literature allow to understand how the interplay of different terms in the left-hand side of the equation influences the critical exponent p_c. It is well known that the critical exponent for the wave equation, that is problem (1) with $\sigma = 1$, is the Strauss exponent (see [32, 35, 36, 39, 40, 49, 52, 59, 61]), defined as the solution to the quadratic equation

$$(p-1)\left(\frac{n-1}{2}+\frac{1}{p}\right)=2.$$

Strauss exponent is not determined by scaling arguments. Adding a damping term, like in (9), can deeply influence the critical exponent. Already in [44], A. Matsumura proved the existence of small data global-in-time solutions to the classical damped wave equation ($\sigma = 1$ and $\theta = 0$ in (9)) in the supercritical case $p > 1+2/n$ in space dimension $n = 1, 2$. This result has been extended to any space dimension $n \geq 3$ in [55] (see also [38]), and the nonexistence counterpart has been proved in [60]. The exponent $p = 1 + 2/n$ is known as the Fujita exponent and it is the same critical power as that of the nonlinear heat equation [31]. The classical damping u_t produces a *diffusion phenomenon*, i.e., the asymptotic profile of the solution to the corresponding linear problem can be described by the solution to an heat equation with suitable initial data (see [43, 47, 58]). The critical exponent remains the same if one replaces the constant damping u_t by an effective time-dependent damping $b(t)u_t$ (see [23, 41, 42]); however, adding a mass term $m^2(t)u$ can affect the critical exponent [22, 33].

If the damping is effective ($\sigma = 1$ and $\theta \in (0, 1)$ in (9)) a *double diffusion phenomenon* appears, that is the solution to (8) may be written as the sum of two terms, whose asymptotic profiles as $t \to \infty$ are described by the solutions to the two diffusion problems [9]. As a consequence, the critical exponent is determined by a scaling argument related to its diffusive profiles [12, 24], as it was for $\theta = 0$ (though only one diffusive profile appears for $\theta = 0$), and it is of Fujita type, namely $p_c = 1 + 2/(n - \theta)$.

The techniques related to the treat the nonlinearity as a perturbation for small initial data often involves the use of appropriate decay estimates for the corresponding linear problem. It is relatively easy to use energy methods based on $L^1 - H^\kappa$ estimates to study models with an effective damping, due to the diffusive structure. Those methods however don't work very well then the critical exponent is less than 2, due to the lack of Sobolev embeddings $H^\kappa \hookrightarrow L^p$. The use of $L^1 - L^p$ estimates is therefore useful when $p_c < 2$ (see [10, 46]; see [12] for $\sigma > 1$). The use of $L^1 - L^p$ estimates is particularly easy in presence of two dissipative terms, an effective one and a non effective one, see [2, 6].

In the case of noneffective damping, when the oscillations are not canceled by the diffusive structure, one may employ Fourier analysis and stationary phase methods to obtain the $L^1 - L^p$ estimates (6), with $p \in (2, \infty]$, at least in low space dimension (namely, when $d(1,p) \leq 1$). The decay in (6) cannot be derived by the Sobolev embeddings, as for the wave equation and σ-evolution equations without damping. Therefore, for the nonlinear models related to those equations, $L^1 - L^p$ estimates (6) are also interesting when $p_c \geq 2$.

2. The critical nonlinearity

Our purpose is to obtain a sharp condition on the nonlinearity $f(u)$ in (1): we consider the critical case and we assume that there exists $\varepsilon > 0$ such that

$$f(0) = 0, \qquad |f(u) - f(v)| \leq C\,|u - v|\,(|u| + |v|)^{p_c - 1}\,\mu(|u| + |v|) \tag{10}$$

for $|u| \leq \varepsilon$ and $|v| \leq \varepsilon$, where p_c is the critical exponent in (2), and μ is an increasing function verifying the following integral condition:

$$\int_0^{2\varepsilon} \frac{\mu(\tau)}{\tau}\, d\tau < \infty. \tag{11}$$

It is clear that if μ is continuous at $\tau = 0$, then $\mu(0) = 0$ as a consequence of (11).

Remark 2.1. *If μ is $\mathcal{C}^1$ and $0 \leq \tau\mu'(\tau) \leq \mu(\tau)$ for $\tau \in [0, \varepsilon]$, then assumption (10) holds for $f(u) = |u|^{p_c}\mu(|u|)$, provided that $p_c > 1$. Indeed,*

$$f(u) - f(v) = \int_0^1 \partial_\rho f(v + \rho(u - v))\, d\rho = (u - v) \int_0^1 f'(v + \rho(u - v))\, d\rho,$$

so that

$$\begin{aligned} |f(u) - f(v)| &\leq |u - v| \int_0^1 |f'(v + \rho(u - v))|\, d\rho \\ &\leq |u - v|\,(p_c + 1) \int_0^1 |v + \rho(u - v)|^{p_c - 1}\, \mu(|v + \rho(u - v)|)\, d\rho \\ &\leq |u - v|\,(p_c + 1)\,(|u| + |v|)^{p_c - 1}\, \mu(|u| + |v|). \end{aligned}$$

This idea to use $\mu(|u|)$ to provide sharp conditions for the global existence of small data solutions, has been originally developed in [28] for the classical damped wave equation (see also [26]).

In particular, the integral condition in (11) allows us to obtain

$$\int_0^\infty (1+s)^{-1}\,\mu(c(1+s)^{-a})\,ds = \int_1^\infty s^{-1}\,\mu(cs^{-a})\,ds = \frac{1}{a}\int_0^c \tau^{-1}\,\mu(\tau)\,d\tau, \tag{12}$$

for any $a > 0$, where we used the change of variable $1+s \mapsto s$ first and $\tau = cs^{-a}$ later. The latter integral is finite for $c \leq 2\varepsilon$, thanks to (11). Estimate (12) will be crucial to prove the contraction argument which leads to the existence of the global small data solution to (1) and (9). It replaces a classical argument used to apply the contraction mapping principle for nonlinear problems which goes back (at least) to [50].

The integral condition (11) on μ is optimal, in the sense that if (11) is not satisfied, then any solution to (1) and (9), with $f(u) = |u|^{p_c}\mu(|u|)$, blows up in finite time, under suitable sign assumptions on the initial data. This latter result can be proved following the approach used in [28], and it will be included in a forthcoming paper, concerning nonexistence results for more general nonlinear evolution equations with Fujita type critical exponent, with nonlinearities f satisfying conditions similar to (10).

Example 2.1. *We recall that any function $\mu \in \mathcal{C}([0,\infty))$, increasing and concave, with $\mu(0) = 0$ is also called modulus of continuity. As an example,* (11) *is verified for a modulus of continuity defined for a sufficiently small τ by $(-\log\tau)^{-\gamma}$, and, more in general, by*

$$\mu(s) = \Big(-\log\tau\Big)^{-1}\Big(\log(-\log\tau)\Big)^{-1}\cdots\Big(\log^{[k]}(-\log\tau)\Big)^{-\gamma}, \quad k \in \mathbb{N},$$

where $\log^{[k]}$ denotes the composition of k $\log$, if, and only if, $\gamma > 1$.

In the following we state our main result about the global existence of small data solutions for problems (1) and (9), with f as in (10), assuming space dimension $n < 2\sigma$; for problem (9), thanks to the presence of the damping, we can prove a similar result in higher space dimension, following as in [14].

Theorem 2.1. *Assume that $1 < \sigma < n < 2\sigma$. Then, there exists $\varepsilon_1 > 0$ such that for any*

$$u_1 \in L^1 \cap L^2, \quad \textit{with} \quad \|u_1\|_{L^1\cap L^2} < \varepsilon_1,$$

where $\|\cdot\|_{L^1\cap L^2} = \|\cdot\|_{L^1} + \|\cdot\|_{L^2}$, *there exists a uniquely determined solution*

$$u \in C([0,\infty), H^\sigma) \cap C^1([0,\infty), L^2),$$

to (1) *and* (9) *with* $\theta \in [1,2]$, *with* f *as in* (10) *and* μ *satisfying* (11). *Moreover, the solution satisfies the estimate*

$$\|u(t,\cdot)\|_{L^p} \leq C(1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{p})} \|u_1\|_{L^1\cap L^2}, \quad \forall p \in [2,\infty], \tag{13}$$

and the energy estimate

$$E(t) = \|u_t(t,\cdot)\|_{L^2}^2 + \|A^{\frac{1}{2}} u(t,\cdot)\|_{L^2}^2 \leq C\,\|u_1\|_{L^1\cap L^2}^2.$$

In the case $\theta = 1$, Theorem 2.1 may be easily extended to any space dimension n. In particular, when $n \geq 2\sigma$, we assume small initial data in $L^1 \cap L^{\frac{n}{\sigma}}$ and we construct the solution in the space $u \in C([0,\infty), L^1 \cap H^\sigma \cap L^\infty) \cap C^1([0,\infty), L^2)$. The property that $u(t,\cdot) \in L^\infty$ is particularly useful to deal with μ in (10).

Theorem 2.2. *Assume that* $\sigma \geq 1$ *and* $n \geq 2\sigma$. *Then, there exists* $\varepsilon_1 > 0$ *such that for any*

$$u_1 \in L^1 \cap L^{\frac{n}{\sigma}}, \quad \textit{with} \quad \|u_1\|_{L^1\cap L^{\frac{n}{\sigma}}} < \varepsilon_1,$$

where $\|\cdot\|_{L^1\cap L^{\frac{n}{\sigma}}} = \|\cdot\|_{L^1} + \|\cdot\|_{L^{\frac{n}{\sigma}}}$, *there exists a uniquely determined solution*

$$u \in C([0,\infty), L^1 \cap H^\sigma \cap L^\infty) \cap C^1([0,\infty), L^2),$$

to (9) *with* $\theta = 1$, *with* f *as in* (10) *and* μ *satisfying* (11). *Moreover, the solution satisfies the estimate*

$$\|u(t,\cdot)\|_{L^p} \leq C(1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{p})} \|u_1\|_{L^1\cap L^{\frac{n}{\sigma}}}, \quad \forall p \in [1,\infty], \tag{14}$$

and the energy estimate

$$E(t) = \|u_t(t,\cdot)\|_{L^2}^2 + \|A^{\frac{1}{2}} u(t,\cdot)\|_{L^2}^2 \leq C\,(1+t)^{-\frac{n}{\sigma}}\,\|u_1\|_{L^1\cap L^{\frac{n}{\sigma}}}^2.$$

In a forthcoming paper, the authors will investigate the possibility to assume condition (10) in presence of a nonlinear memory term for which the critical exponent is not obtained by scaling arguments (see [1], see also [3, 4, 8, 18, 25, 30, 34]).

3. Proof of Theorem 2.1

For a given $R>0$, we introduce the solution space

$$X=\{u\in\mathcal{C}([0,\infty),H^\sigma)\cap\mathcal{C}^1([0,\infty),L^2):\ \|u\|_X\leq R\},\quad\text{where}$$

$$\|u\|_X=\sup_{s\in[0,\infty)}\big\{(1+s)^{-1+\frac{n}{2\sigma}}\|u(s,\cdot)\|_{L^2}+(1+s)^{-1+\frac{n}{\sigma}}\|u(s,\cdot)\|_{L^\infty}+\|(u_t,A^{\frac12}u)(t,\cdot)\|_{L^2}\big\}.$$

In [29], it is proved that when $\sigma>1$ and $n<2\sigma$ in (4), the estimate

$$\|u(t,\cdot)\|_{L^p}\leq C\,(1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}\,\|u_1\|_{L^1\cap L^2}, \tag{15}$$

holds for any $p\in[2,\infty]$ (the L^2 assumption for the data is used to avoid singular power as $t\to0$, since $H^\sigma\hookrightarrow L^p$ for any $p\in[2,\infty]$, thanks to the assumption $n<2\sigma$). Under the same assumptions, in [13] it is proved that the solution to (8) also verifies (15). It is also clear that $E(t)=E(0)$ for the solution to (4), and $E(t)\leq E(0)$ for the solution to (8).

Let $K=K(t,\cdot)$ be the fundamental solution to (4) or (8). Thanks to (15) and to the energy estimate $E(t)\leq E(0)=\|u_1\|_{L^2}^2$, we find

$$\|u^{\,\mathrm{lin}}\|_X\leq C_1\|u_1\|_{L^1\cap L^2},\quad\text{where}\qquad u^{\,\mathrm{lin}}(t,x)=K(t,\cdot)*_{(x)}u_1 \tag{16}$$

is the solution to the linear problem (4) or (8), for some constant $C_1>0$ independent on u_1. A function $u\in X$ is a solution to (1) or (9) if, and only if, it satisfies

$$u(t,x)=u^{\,\mathrm{lin}}(t,x)+Fu(t,x),\quad\text{in }X, \tag{17}$$

where F is the nonlinear integral operator defined by

$$Fu(t,x)=\int_0^t K(t-s,\cdot)*_{(x)}f(u(s,\cdot))\,ds.$$

We will prove that there exists $C_2>0$ such that

$$\|Fu-Fv\|_X\leq C_2\|u-v\|_X(\|u\|_X^{p_c-1}+\|v\|_X^{p_c-1}), \tag{18}$$

for any $u,v\in X$. Due to the definition of $\|\cdot\|_X$, for any $u\in X$ and $s\geq0$ we may estimate

$$\|u(s,\cdot)\|_{L^2}\leq(1+s)^{1-\frac{n}{2\sigma}}\|u\|_X,$$

$$\|u(s,\cdot)\|_{L^\infty}\leq(1+s)^{1-\frac{n}{\sigma}}\|u\|_X,$$

so that, by interpolation, we get

$$\|u(s,\cdot)\|_{L^p} \leq (1+s)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}\|u\|_X, \qquad p\in[2,\infty]. \tag{19}$$

Thanks to (15), for any $u,v\in X$ and $p\in[2,\infty]$, we may estimate

$$\|(Fu-Fv)(t,\cdot)\|_{L^p} \leq C\int_0^t (1+t-s)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}\|f(u(s,\cdot))-f(v(s,\cdot))\|_{L^1\cap L^2}\,ds.$$

Let us assume that $R\leq\varepsilon$, so that for any $u\in X$, it holds

$$|u(s,x)| \leq \|u(s,\cdot)\|_{L^\infty} \leq (1+s)^{1-\frac{n}{\sigma}}\|u\|_X \leq R\leq\varepsilon,$$

due to $n>\sigma$. We first consider $\|f(u(s,\cdot))-f(v(s,\cdot))\|_{L^1}$. Using (10), we get

$$\|f(u(s,\cdot))-f(v(s,\cdot))\|_{L^1} \leq \|(u-v)(s,\cdot)\,(|u|+|v|)^{p_c-1}(s,\cdot)\|_{L^1}\,\|\mu((|u|+|v|)(s,\cdot))\|_{L^\infty}.$$

By Hölder inequality, using (19) with $p=p_c$ (we stress that $p_c>3\geq 2$, due to $n<2\sigma$, so that we may use (19)), the first term may be estimated by

$$\begin{aligned}\|(u-v)(s,\cdot)\,(|u|+|v|)^{p_c-1}(s,\cdot)\|_{L^1} &\leq \|(u-v)(s,\cdot)\|_{L^{p_c}}\,\|(|u|+|v|)^{p_c-1}(s,\cdot)\|_{L^{p_c'}}\\ &\leq C\,(1+s)^{-1}\,\|u-v\|_X\,\big(\|u\|_X^{p_c-1}+\|v\|_X^{p_c-1}\big),\end{aligned}$$

due to the equality

$$p_c\left(1-\frac{n}{\sigma}\left(1-\frac{1}{p_c}\right)\right)=-1.$$

On the other hand, using (19) with $p=\infty$, and recalling that μ is increasing, we may estimate

$$\|\mu((|u|+|v|)(s,\cdot))\|_{L^\infty} \leq \mu((\|u\|_X+\|v\|_X)\,(1+s)^{1-\frac{n}{\sigma}}) \leq \mu(2R\,(1+s)^{1-\frac{n}{\sigma}}).$$

We now estimate

$$I_1(t)=\int_0^t (1+t-s)^{1-\frac{n}{\sigma}(1-\frac{1}{p})}\,(1+s)^{-1}\,\mu(2R\,(1+s)^{1-\frac{n}{\sigma}})\,ds.$$

We split the integral in two parts. On the one hand,

$$\begin{aligned}\int_{t/2}^t \ldots ds &\leq \mu(2R)\,(1+t/2)^{-1}\int_{t/2}^t (1+t-s)^{1-\frac{n}{\sigma}(1-\frac{1}{p})}\,ds\\ &\leq C\,\mu(2R)\,(1+t/2)^{-1}\,(1+t)^{2-\frac{n}{\sigma}(1-\frac{1}{p})} \leq C'\,(1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{p})}.\end{aligned}$$

On the other hand, using again that $R \leq \varepsilon$, thanks to (12), we get

$$\int_0^{t/2} \dots ds \leq (1+t/2)^{1-\frac{n}{\sigma}(1-\frac{1}{p})} \int_0^{t/2} (1+s)^{-1}\,\mu(2R\,(1+s)^{1-\frac{n}{\sigma}})\,ds$$
$$\leq C' t^{1-\frac{n}{\sigma}(1-\frac{1}{p})} \int_0^{\infty} (1+s)^{-1}\,\mu(2R\,(1+s)^{1-\frac{n}{\sigma}})\,ds = C''\,(1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{p})}.$$

Summarizing,

$$I_1(t) \leq C\,(1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}.$$

Now we consider $\|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^2}$. Using (10), we get

$$\|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^2} \leq \|(u-v)(s,\cdot)\,(|u|+|v|)^{p_c-1}(s,\cdot)\|_{L^2}\,\|\mu((|u|+|v|)(s,\cdot))\|_{L^\infty}.$$

By Hölder inequality, using (19) with $p = 2p_c$, the first term may be estimated by

$$\|(u-v)(s,\cdot)\,(|u|+|v|)^{p_c-1}(s,\cdot)\|_{L^2} \leq \|(u-v)(s,\cdot)\|_{L^{2p_c}}\,\|(|u|+|v|)^{p_c-1}(s,\cdot)\|_{L^{(2p_c)'}}$$
$$\leq C\,(1+s)^{-1-\frac{n}{2\sigma}}\,\|u-v\|_X\,\big(\|u\|_X^{p_c-1} + \|v\|_X^{p_c-1}\big),$$

since

$$p_c\left(1 - \frac{n}{\sigma}\left(1 - \frac{1}{2p_c}\right)\right) = -1 - \frac{n}{2\sigma}.$$

Now it is not necessary to employ (11) and split the integral in two parts, since

$$I_2(t) = \mu(2R)\int_0^t (1+t-s)^{1-\frac{n}{\sigma}(1-\frac{1}{p})}(1+s)^{-1-\frac{n}{2\sigma}}\,ds \leq \mu(2R)\,C\,(1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)},$$

due to

$$-1 - \frac{n}{2\sigma} < -1 < 1 - \frac{n}{\sigma}\left(1 - \frac{1}{p}\right).$$

This concludes the proof of the desired estimate

$$\|(Fu - Fv)(t,\cdot)\|_{L^p} \leq C\,(1+t)^{1-\frac{n}{\sigma}(1-\frac{1}{p})}\,\|u-v\|_X\,\left(\|u\|_X^{p_c-1} + \|v\|_X^{p_c-1}\right),$$

for any $p \in [2,\infty]$. Now we consider the energy estimate. In this case,

$$\|(\partial_t, A^{\frac{1}{2}})(Fu - Fv)(t,\cdot)\|_{L^2} \leq C\int_0^t \|f(u(s,\cdot)) - f(v(s,\cdot))\|_{L^2}\,ds.$$

Again, it is not necessary to employ (11) and split the integral in two parts, since

$$\mu(2R)\int_0^t (1+s)^{-1-\frac{n}{2\sigma}}\,ds \leq \mu(2R)\int_0^\infty (1+s)^{-1-\frac{n}{2\sigma}}\,ds = C.$$

This concludes the proof of the desired estimate

$$\|(\partial_t, A^{\frac{1}{2}})(Fu - Fv)(t,\cdot)\|_{L^2} \leq C\,\|u-v\|_X \left(\|u\|_X^{p_c-1} + \|v\|_X^{p_c-1}\right).$$

Therefore, we proved (18). The proof of the small data global existence now follows from a classic contraction argument. We define

$$R = 2C_1\,\|u_1\|_{L^1\cap L^2},$$

where C_1 is as in (16). For sufficiently small R, we get $2C_2R^{p_c-1} \leq 1/2$, where C_2 is as in (18). Namely, we assume ε_1 sufficiently small to obtain the desired smallness of R. By (16) and (18), it follows that the operator $u^{\mathrm{lin}}(t,x) + F$ maps X into itself. Due to (18), it is a contraction.

For any arbitrarily large $T > 0$, we may replace X by $X(T)$, where

$$X(T) = \{u \in \mathcal{C}([0,T], H^\sigma) \cap \mathcal{C}^1([0,T], L^2) : \ \|u\|_{X(T)} \leq R\},$$

and $\|\cdot\|_{X(T)}$ is as $\|\cdot\|_X$, but $\sup_{t\in[0,\infty)}$ is replaced by $\max_{t\in[0,T]}$. Since $X(T)$ is a Banach space, there is a unique fixed point for $u^{\mathrm{lin}}(t,x) + F$ in $X(T)$, that is, a unique solution to (17) in $X(T)$. Being T arbitrary, we find a unique solution in X. Moreover,

$$\|u\|_X \leq R = 2C_1\,\|u_1\|_{L^1\cap L^2}\,,$$

so that we get estimate (13) and the energy estimate $E(t) \leq C\,\|u_1\|_{L^1\cap L^2}$.

This concludes the proof.

4. Proof of Theorem 2.2

For a given $R > 0$, we introduce the solution space

$$X = \{u \in \mathcal{C}([0,\infty), L^1 \cap H^\sigma \cap L^\infty) \cap \mathcal{C}^1([0,\infty), L^2) : \ \|u\|_X \leq R\} \quad \text{where}$$

$$\|u\|_X = \sup_{s\in[0,\infty)} \left\{(1+s)^{-1}\|u(s,\cdot)\|_{L^1} + (1+s)^{-1+\frac{n}{\sigma}}\|u(s,\cdot)\|_{L^\infty} + (1+s)^{\frac{n}{2\sigma}}\|(u_t, A^{\frac{1}{2}}u)(t,\cdot)\|_{L^2}\right\}.$$

It is easy to check that, thanks to the homogeneity, the solution to (8) with $\theta = 1$ satisfies the following estimate (see, for instance, [5, 27]):

$$\|u(t,\cdot)\|_{L^p} \leq C\,t^{1-\frac{n}{\sigma}\left(\frac{1}{q}-\frac{1}{p}\right)}\,\|u_1\|_{L^q}, \quad 1 \leq q \leq p \leq \infty, \tag{20}$$

from which one easily derive the desired estimate for u^{lin} (the assumption $L^{\frac{n}{\sigma}}$ for the initial data guarantees that the L^∞ estimate is not singular as $t\to 0$):

$$\begin{aligned}\|u^{\mathrm{lin}}(t,\cdot)\|_{L^1} &\leq C\,t\,\|u_1\|_{L^1},\\ \|u^{\mathrm{lin}}(t,\cdot)\|_{L^\infty} &\leq C\,(1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}\,\|u_1\|_{L^1\cap L^{\frac{n}{\sigma}}}.\end{aligned}$$

Moreover,

$$\|(\partial_t, A^{\frac12})u(t,\cdot)\|_{L^2} \leq C\,(1+t)^{-\frac{n}{2\sigma}}\,\|u_1\|_{L^1\cap L^2}. \tag{21}$$

Following the proof of Theorem 2.1, now (16) is replaced by

$$\|u^{\mathrm{lin}}\|_X \leq C_1\|u_1\|_{L^1\cap L^{\frac{n}{\sigma}}}, \tag{22}$$

and for any $u\in X$ and $s\geq 0$ we may estimate

$$\|u(s,\cdot)\|_{L^p} \leq (1+s)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}\|u\|_X, \qquad p\in[1,\infty]. \tag{23}$$

We now employ a slightly different approach with respect to the one employed in Theorem 2.1 to estimate $\|(Fu-Fv)(t,\cdot)\|_{L^p}$.

First assume that $t\geq 2$. Inside the integral term in F, we use estimate L^1-L^p estimate ($q=1$ in (20)) for $s\in[0,t/2]$ and L^p-L^p estimate ($q=p$ in (20)) for $s\in[t/2,t]$. For the first term, we use that

$$\begin{aligned}&\int_0^{t/2}(t-s)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}\|f(u(s,\cdot))-f(v(s,\cdot))\|_{L^1}\,ds\\ &\qquad\leq (t/2)^{1-\frac{n}{\sigma}}\int_0^{t/2}\|f(u(s,\cdot))-f(v(s,\cdot))\|_{L^1}\,ds,\end{aligned}$$

then we estimate the latter integral by a constant as in the proof of Theorem 2.1, assuming that $R\leq\varepsilon$. We stress that it is not necessary to verify that $p_c\geq 2$, since (23) holds for any $p\geq 1$. For the second term, we use (20) to get

$$\|f(u(s,\cdot))-f(v(s,\cdot))\|_{L^p}\leq C\,\mu(2R)\,(1+s)^{-1-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}\,\|u-v\|_X\left(\|u\|_X^{p_c-1}+\|v\|_X^{p_c-1}\right),$$

so that

$$\begin{aligned}\int_{t/2}^{t}&(t-s)\,\|f(u(s,\cdot))-f(v(s,\cdot))\|_{L^p}\,ds\\ &\leq C\,(1+t)^{-1-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}\int_{t/2}^{t}(t-s)\,ds\,\|u-v\|_X\left(\|u\|_X^{p_c-1}+\|v\|_X^{p_c-1}\right)\\ &\leq C\,(1+t)^{1-\frac{n}{\sigma}\left(1-\frac{1}{p}\right)}\|u-v\|_X\left(\|u\|_X^{p_c-1}+\|v\|_X^{p_c-1}\right)\end{aligned}$$

For $t\in[0,2]$, we use L^p-L^p estimates in the whole integral to get the obvious estimate

$$\int_0^2(t-s)\,\|f(u(s,\cdot))-f(v(s,\cdot))\|_{L^p}\,ds\leq C\,\|u-v\|_X\left(\|u\|_X^{p_c-1}+\|v\|_X^{p_c-1}\right).$$

We proceed in a similar way for the energy, using

$$\begin{aligned}\|(\partial_t,A^{\frac{1}{2}})(f(u(t,\cdot))-f(v(t,\cdot)))\|_{L^2}\leq&\int_0^{t/2}(t-s)^{-\frac{n}{2\sigma}}\|(f(u(s,\cdot))-f(v(s,\cdot)))\|_{L^1}ds\\ &+\int_{t/2}^{t}\|(f(u(s,\cdot))-f(v(s,\cdot)))\|_{L^2}ds.\end{aligned}$$

This concludes the proof of (18). The end of the proof is as in the proof of Theorem 2.1.

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