ON THE REGULARITY OF ANISOTROPIC p-LAPLACEAN OPERATORS: THE PURSUIT OF A COMPREHENSIVE THEORY OF REGULARITY

SULLA REGOLARITÀ DELL'OPERATORE p-LAPLACIANO ANISOTROPICO: ALLA RICERCA DI UNA TEORIA COMPLETA DELLA REGOLARITÀ

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Abstract. With this note, we aim at drawing a short, coherent, and compact guide of the state-of-the-art on the theory of basic regularity, such as local boundedness, local Hölder continuity, Harnack estimates and some of their consequences, in the context of solutions to anisotropic p-Laplacean operators, elliptic and parabolic.

Sunto. Con questa breve nota intendiamo proporre una panoramica, breve, coerente e concisa sullo stato dell'arte della teoria della regolarità degli operatori p-Laplaciani anisotropici sia ellittici che parabolici. Tratteremo proprietà di base quali la limitatezza locale, la continuità Hölderiana, le stime di Harnack e le loro principali conseguenze.

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1. Introduction to the problem

This note aims to be a simple guide on selected results and advances in the study of weak solutions to anisotropic equations. We present a snapshot of the current state of knowledge in this fascinating area of research, offering insights into the complexities and the unique behaviors that characterize the anisotropic parabolic equations. By addressing a few of the key findings and techniques that have emerged in this field, we hope to contribute to the ongoing dialogue on the regularity of weak solutions, adding another layer to the ever-evolving tapestry of this regularity theory that still sees most questions unanswered.

The prototype of the operators that we will address is the following

$$(\partial_t - \Delta_{\mathbf{p}})u := \partial_t u - \sum_i \partial_i(|\partial_i u|^{p_i-2}\partial_i u) = 0, \tag{1.1}$$

that, up to our knowledge, made its first appearance as the prototype of an evolution operator that is the sum of monotone operators, see for instance [52], page 186. For ease of writing, when the sum or product is carried over $\{1,\dots,N\}$, we omit the index set in the sum/product symbol. In the following, we address the question of basic regularity theory for local weak solutions to (1.1); meaning with this, their local boundedness, local Hölder continuity, and Harnack estimates. Even if much is investigated, the problem of basic regularity is still at its dawn, and the literature is fragmented and sometimes unclear. Here, we aim to draw a short guide to introduce the reader to the study of the regularity theory of anisotropic operators, intended as non-standard orthotropic operators, both elliptic and parabolic.

The terms non-standard and orthotropic have been employed mainly in the study of energy functionals as

$$I[u] := \int_\Omega f(\xi)dx, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is open and bounded, and where the convex integrand f is assumed to be a Caratheodory function satisfying

$$\frac{1}{\gamma}(|\xi|^p - 1) \leqslant f(\xi) \leqslant \gamma\left(|\xi|^q + 1\right), \tag{1.3}$$

for a constant $\gamma > 0$ and exponents $1 < p \leqslant q < \infty$ are fixed. The term orthotropic refers to the dependence of f from the vector ξ through each of its components as

$$f(\xi) = \sum_i f_i(\xi_i), \qquad f_i : \mathbb{R} \to \mathbb{R} \text{ convex } \xi = (\xi_1,\dots,\xi_N) \in \mathbb{R}^N. \tag{1.4}$$

The study of the regularity of minimizers of these kind of functionals has been initiated by Marcellini in [56], [57] and five years before from Uralt'seva, Uraldetova in [64], while already in the early seventies Kolodii in [47] took into consideration and solved the problem of the local boundedness. For instance, the prototype example of such a functional is

$$I_{\mathbf{p}}(u) = \sum_i \frac{1}{p_i}\int_\Omega |\partial_i u|^{p_i}\, dx, \qquad \Omega \subset\subset \mathbb{R}^N, \tag{1.5}$$

and its Euler-Lagrange equation is the elliptic counterpart of (1.1), i.e. the elliptic anisotropic p-Lapalcean

$$-\Delta_{\mathbf{p}} u := -\sum_i \partial_i(|\partial_i u|^{p_i-2}\partial_i u) = 0, \qquad \text{weakly in } \Omega. \tag{1.6}$$

When $p_i \equiv p$ for all $i = 1, \dots, N$, the equations (1.1)-(1.6) are not exactly the classic p-Laplacean equation, since the modulus of ellipticity depends on each directional derivatives differently. Nonetheless, the theory of basic regularity, as we intend in this note, is fairly understood and complete. Indeed, in the case of elliptic equations behaving like (1.6) with $p_i \equiv p$ for all $i = 1, \dots, N$, the local weak solutions to these equations (and moreover local minimizers of their associated functionals) belong to an energetic class called De Giorgi class of order p. The De Giorgi regularity theory had origin in the fundamental paper [25] and extended to the parabolic case in [50]. For more recent development of the theory, we refer to the seminal paper [33] and to the books [27], [41], [54] for an insight on this topic. Classically, local weak solutions of the elliptic isotropic equations enjoy local boundedness, local Hölder continuity, and, when non-negative, admit a Harnack inequality. The literature above is not exhaustive: the mentioned regularity has undergone a strong wave of research since the sixties, and still nowadays is a very active research topic. An extensive list of contributions would be nearly impossible to track and out of the scope of this note.

On the parabolic side, again when $p_i \equiv p \ \forall i = 1, \dots, N$, the theory of basic regularity for equations of the kind of (1.1) is quite well understood, again when dealing with generalizations of the classic parabolic p-Laplacian equation

$$\partial_t u - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0, \quad \text{locally weakly in} \quad \Omega, \tag{1.7}$$

where the energy is homogeneous of power p. Here, local and global boundedness for local weak solutions can be set in a fairly complete picture, too, for instance, the book [26], Chap. V. In contrast with the elliptic case, here the situation is different when considering the *degenerate* ($p > 2$) and *singular* ($1 < p < 2$) cases. Indeed, in this case, the parabolic p-Laplacian presents different forms of diffusion, and its corresponding boundedness theory needs to be studied individually in each case. Yet, in order to conclude boundedness for the singular (sub-critical) case, where $1 < p < 2N/(N+2)$, the solutions are required to have higher integrability, while in the super-critical case $p > 2N/(N+2)$ local boundedness is inherent in the definition of solutions. Hölder continuity of solutions for the evolution p-Laplacian equation (1.7), has been proven by DiBenedetto using a method called *intrinsic scaling* (see for instance [26], [65]). This method allowed the study of these equations with full quasi-linear structure. Finally, for this kind of isotropic evolutionary equations, Harnack-type inequalities have been discovered much more recently in [28], [30]. For a more complete picture, the monograph [31] and [17] for a short proof in the case of singular equations.

Remark 1.1. This correspondence between the regularity of classic p-Laplacean problems and the orthotropic yet standard ($p_i \equiv p$) equations as (1.1) is a prerogative of the basic regularity theory that we are discussing in this note. All in a rough word, it is linked to the fact that for assigned numbers $N \in \mathbb{N}, p > 1$, there exists a universal positive constant γ such that for all $\xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$ the following estimate holds true

$$\frac{1}{\gamma}\sum_i |\xi_i|^p \leqslant \|\xi\|^p \leqslant \gamma \sum_i |\xi_i|^p,$$

where,

$$\|\xi\| = \sqrt{\sum_i \xi_i^2}.$$

The situation becomes completely different when considering higher regularity such as Lipschitz or $C^{1,\alpha}$ bounds.

The local regularity theory of the parabolic p-Laplacean operator is therefore different from the one of the elliptic p-Laplacean operator. A fascinating interpretation of this phenomenon can be that this difference in local regularity theory is caused by the fact that the former operator is an anisotropic counterpart of the latter when considering time as one more variable. This has led in the past to the application of the method of *intrinsic scaling*, mainly developed by DiBenedetto for the purpose of local regularity of parabolic p-Laplacean type equations, to the resolution of the problem of local regularity for elliptic anisotropic equations (see Section 4) that admit as a prototype (1.6).

In contrast to isotropic equations, anisotropic equations describe the phenomena where diffusion or propagation rates are different in different directions; see, for instance, the seminal paper [2], the recent papers [13], [14] or the last chapters of the books [1], [9] for applications. The book [3] is a very updated reference for these kinds of anisotropic problems and their resolution through energy methods. For example, in anisotropic diffusion, the rate at which a substance spreads or diffuses varies with direction. One can observe the directional sensitivity in such equations by setting $p_i < 2$ for all $i = 1, \dots, N$, let u be a solution of (1.6) and for fixed $x \in \mathbb{R}^N, j \in \{1, ..., N\}$ if $\partial_j u$ vanishes as approaching the origin, then the directional modulus of ellipticity $|\partial_j u|^{p_j-2}$ explodes only in this direction, disregarding the others.

1.1. **Differences of full anisotropic equations with other non-standard equations.** The reader may ask herself if somehow the theory that we are about to describe fits into a more general framework whose development has undergone a more complete picture. Here, we comment on the difference between two other non-standard p-Laplacean equations that have been intensively studied. The literature we refer to is far from being complete, and the choice of the references employed is made on the taste of the authors for the sake of simplicity of exposition of the differences with the core of our research interests. On the other hand, a common feature can be found in a functional setting called Musielak-Orlicz spaces. The interested reader can consult the survey [8] or the more recent book [9] toward the theory of existence and uniqueness.

The first case we consider is an operator that describes the effect of electro-rheological fluids (see, for instance, the seminal paper [60] or the book [61]),

$$-\text{div}(|\nabla u|^{p(x)-2}\nabla u) = 0, \quad \text{locally weakly in} \quad \Omega \subset\subset \mathbb{R}^N. \tag{1.8}$$

In this case, the correct functional setting is given in terms of Lebesgue-Sobolev spaces of variable exponent (see, for instance, the books [34], [42] and references therein), as

$$L^{p(\cdot)}(\Omega) = \left\{ f : \Omega \to \mathbb{R}, \text{ measurable} : \int_\Omega |f(x)|^{p(x)}\, dx < \infty \right\}.$$

In this case, assuming some continuity on the exponent $p(x)$, say, around a point x_0, the value of p must be close enough to a precise number. The modulus of ellipticity degenerates as $|\nabla u|^{p(x)-2}$ in that neighborhood. Here, we note that the equation degenerates when $|\nabla u|$ vanishes, and therefore, if there is even one i-th component $\partial_i u$ of the gradient that stays bounded away from zero, the whole equation behaves as an elliptic one. Moreover, the usual logarithmic condition that is given to the exponent $p(x)$ excludes dramatic changes in the energy density. We refer to the recent papers [44]- [43] and the literature therein for an account of recent results.

The other standard model case is the opposite of the previous discussion and takes into consideration very dramatic phase transitions. The equation, indeed, is referred to as the double-phase equation,

$$-\mathrm{div}\bigg(\varphi(x,\nabla u)\nabla u\bigg) := -\mathrm{div}\bigg((|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2})\nabla u\bigg) = 0,$$

locally weakly in $\Omega \subset\subset \mathbb{R}^N$, where $p < q$ and $a \in C^{\alpha}(\Omega)$. In this case, roughly speaking, when testing the equation with a classical truncation $(u-k)_{\pm}\zeta$ with $\zeta \in C_o^{\infty}(B_\rho)$, it is possible to bound the L^p norm of the full gradient in terms of the integral of $\varphi(x,(u-k)_{\pm})$ in $B_{2\rho}$; being this last the ball centered in the origin and of radius 2ρ. This procedure naturally falls into the consideration of the aforementioned Musielak-Orlicz spaces. Here, similarly as for the $p(x)$ case, the modulus of ellipticity φ explodes or vanishes when $|\nabla u|$ vanishes, disregarding the single components of the gradient.

On the other hand, for equations of the kind of (1.6), when a single component of the gradient vanishes, say for some $i \in \{1,\dots,N\}$, $|\partial_i u|$ vanishes, one of the terms of the sum vanishes or explodes. Therefore, the behavior mixes up in case some p_is are smaller than 2, and some are greater; moreover, the set of degeneration can be much bigger than before, and in particular, unbounded. One can, for instance, refer to the following solution of the prototype (1.6):

$$u(x) = \sum_i (\alpha_i/p_i')|x_i|^{p_i'},$$

with $p_i' = (1-1/p_i)^{-1}$ and α_i numbers such that $\sum_i |\alpha_i|^{p_i-2}\alpha_i = 0$.

1.2. **General Framework.** We may broaden our attention to quasi-linear generalizations of the prototype (1.1) as

$$\partial_t u - \operatorname{div} A(x,t,s,\xi) = B(x,t,s,\xi), \quad \text{in} \quad \Omega_T, \tag{1.9}$$

where the functions

$$\Omega_T \ni (x,t) \to \begin{cases} A(x,t,s,\xi) \in \mathbb{R}^N, \\ B(x,t,s,\xi) \in \mathbb{R}, \end{cases}$$

are measurable and satisfy the standard structural conditions

$$\begin{cases} A_i(x,t,s,\xi)\xi_i \geqslant C_o|\xi_i|^{p_i} - C, \\ |A_i(x,t,s,\xi)| \leqslant C_1|\xi|^{p_i-1} + C, \\ |B(x,t,s,\xi)| \leqslant C_2|\xi|^{p_i-1} + C, \end{cases} \tag{1.10}$$

where $C_o, C_1, C_2 > 0$ and $C \geqslant 0$ are referred to as the structural data. We observe that thanks to Remark 1.1, the stationary special case with $p_i \equiv p$ of (1.9), where we only consider spatial variables, is the quasi-linear, elliptic operator

$$-\operatorname{div} A(x,u,Du) = B(x,u,Du) \quad \text{in} \quad \Omega \subset \mathbb{R}^N, \tag{1.11}$$

where $A(x,u,Du) : \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}^N$ and $B(x,u,Du) : \Omega \times \mathbb{R}^{N+1} \to \mathbb{R}$ are assumed to be measurable functions subject to the standard structural conditions:

$$\begin{cases} A(x,u,Du) \cdot Du \geqslant C_o|Du|^p - C, \\ |A(x,u,Du)| \leqslant C_1|Du|^{p-1} + C, \\ |B(x,u,Du)| \leqslant C_2|Du|^{p-1} + C, \end{cases} \tag{1.12}$$

where $C_o, C_1, C_2 > 0$ and $C \geqslant 0$ are structural data. Such structural conditions are embodied in the elliptic p-Laplacian prototype when considering

$$A(x,u,Du) = |Du|^{p-2}Du \tag{1.13}$$

On the other hand, the anisotropic p-Laplacean operator $\Delta_{\mathbf{p}}$ of (1.6) follows the directional behaviour of the prototype

$$A_i(x,u,Du) = |\partial_i u|^{p_i-2}\partial_i u, \tag{1.14}$$

when considering (1.11) and the structure conditions (1.10) stationary in time, with $1 < p_i < \infty$, for $i \in \{1, ..., N\}$, that may be different one another.

2. Functional Settings

In the theory of PDEs, the function space where a solution belongs is part of the problem. In order to use a variational approach, we cannot expect the solutions to be classical as, already, for the isotropic case, the maximum we can expect with rough coefficients is $u \in C^{1,\alpha}_{loc}(\Omega)$. Testing (1.1) with a compactly supported smooth function $\varphi \in C^1([0,T]; C^\infty_o(\Omega))$, integrating on Ω_T and using integration by parts, it is evident that the minimum requirement for the integrals to converge is that u has weak derivatives that have different integrabilty.

This motivates the definition of the following anisotropic spaces of locally integrable functions.

$$W^{1,\mathbf{p}}_{loc}(\Omega) = \{u \in W^{1,1}_{loc}(\Omega) \mid \partial_i u \in L^{p_i}_{loc}(\Omega)\},$$

$$L^{\mathbf{p}}_{loc}(0,T; W^{1,\mathbf{p}}_{loc}(\Omega)) = \{u \in L^1_{loc}(0,T; W^{1,1}_{loc}(\Omega)) \mid \partial_i u \in L^{p_i}_{loc}(0,T; L^{p_i}_{loc}(\Omega))\},$$

and the respective spaces of functions with zero boundary data

$$W^{1,\mathbf{p}}_o(\Omega) = \{u \in W^{1,1}_o(\Omega) \mid \partial_i u \in L^{p_i}_{loc}(\Omega)\},$$

$$L^{\mathbf{p}}_{loc}(0,T; W^{1,\mathbf{p}}_o(\Omega)) = \{u \in L^1_{loc}(0,T; W^{1,1}_o(\Omega)) \mid \partial_i u \in L^{p_i}_{loc}(0,T; L^{p_i}_{loc}(\Omega))\}.$$

We are interested in the local behavior of the solutions to (1.9), irrespective of possible prescribed boundary data, as evident from the following definition.

Definition 2.1. *A function*

$$u \in C(0,T; L^2_{loc}(\Omega)) \cap L^{\mathbf{p}}_{loc}(0,T; W^{1,\mathbf{p}}_{loc}(\Omega))$$

is called a local weak sub(super)-solution to (1.9) *in* Ω_T *if, for all times* $0 \leqslant t_1 \leqslant t_2 \leqslant T$ *and for all compact sets* $K \subset\subset \Omega$*, it satisfies the inequality*

$$\begin{aligned} \int_K u\varphi \, dx \Big|_{t_1}^{t_2} + \int_{t_1}^{t_2} \int_K \{-u\partial_\tau \varphi + \sum_i A_i(x,t,u,Du)\partial_i \varphi\} dx d\tau \\ \leqslant (\geqslant) \int_{t_1}^{t_2} \int_K B(x,t,u,Du)\varphi dx \, d\tau, \end{aligned} \tag{2.1}$$

for all non-negative test functions $\varphi \in W^{1,2}_{loc}(0,T;L^2_{loc}(\Omega)) \cap L^{\mathbf{p}}_{loc}(0,T;W^{1,\mathbf{p}}_o(\Omega))$.

This last membership of the test functions, together with the structure conditions (1.10), ensure that all the integrals in (2.1) are finite. Moreover, as test functions φ vanish along the lateral boundary of Ω_T, their integrability increases thanks to the following known embedding theorem.

Lemma 2.2. *(Anisotropic Gagliardo-Sobolev-Nirenberg, [35])*
Let $\Omega \subseteq \mathbb{R}^N$ *be a rectangular domain,* $\bar{p} < N$*, and* $\sigma \in [1, \bar{p}^*]$*. For any number* $\theta \in [0, \bar{p}/p^*]$ *define*

$$q = q(\theta, \mathbf{p}) = \theta\, \bar{p}^* + \sigma\,(1-\theta),$$

Then there exists a positive constant $c = c(N, \mathbf{p}, \theta, \sigma) > 0$ *such that*

(2.2)
$$\iint_{\Omega_T} |\varphi|^q \,dx\,dt \leqslant c\,T^{1-\theta\,\frac{\bar{p}^*}{\bar{p}}} \left(\sup_{t\in(0,T]} \int_\Omega |\varphi|^\sigma(x,t)\,dx \right)^{1-\theta} \prod_i \left(\iint_{\Omega_T} |\partial_i \varphi|^{p_i}\,dx\,dt \right)^{\frac{\theta\,\bar{p}^*}{N\,p_i}},$$

for any $\varphi \in L^1(0,T;W^{1,1}_o(\Omega))$*, being the inequality trivial when the right-hand side is unbounded, and being the numbers* $\bar{p}$*,* $\bar{p}^*$ *specified here below.*

2.1. **Notation.** In the rest of this paper, we assume without loss of generality (up to a change of coordinates) that the p_is are ordered, meaning with this

$$1 < p_1 \leqslant p_2 \leqslant \ldots, \leqslant p_N < \infty,$$

and $\bar{p}$ to be the harmonic mean, which is the unique number satisfying

$$\frac{1}{\bar{p}} = \frac{1}{N} \sum_i \frac{1}{p_i},$$

and $\bar{p}^*$ to be its Sobolev conjugate, i.e.

$$\bar{p}^* = \frac{\bar{p}N}{N-\bar{p}}.$$

As clear from the context, we abbreviate the index notation when it is clear that the index set ranges over $\{1,\ldots,N\}$, as

$$\prod_i f_i, \quad \sum_i f_i \qquad \text{for} \quad \prod_{i=1}^N f_i, \quad \sum_{i=1}^N f_i.$$

3. Local Boundedness

We address results of local boundedness under the assumption $\bar{p} \leqslant N$ because in the case of $\bar{p} > N$, this regularity is a consequence of Morrey embedding

$$W^{1,\mathbf{p}}(\Omega) \hookrightarrow C^{0,\alpha}(\Omega),$$

provided Ω is regular enough, as studied in [66], [5]. In this context, we focus on a few key findings. Such findings, as usual, use a chain of energy inequalities (Caccioppoli inequalities) obtained from the equation by testing with $(u-k)_{\pm}\zeta$, being $\zeta \in C_o^{\infty}(K)$ and the embedding 2.2, in a nonlinear chain *à la* De Giorgi.

In general, under no restrictions on p_i, it is not possible to conclude boundedness for weak solutions of (1.9); especially when the ratio p_N/p_1 is too large, as illustrated in the celebrated counter-example in [55]. The author examines the functional (1.2), incorporating $f(Du)$ within the form:

$$f(Du) = \frac{1}{2}\sum_{i=1}^{N-1} (\partial_i u)^2 + \frac{1}{p_N} |\partial_N u|^{p_N} .$$

Here, they set $p_i = 2$ for all i, with the exception of $p_N > 2\big(\frac{N-1}{N-3}\big) > 2$, with $N > 3$. Within this study, a minimizer of this functional is presented as:

$$u(x) = \left(c\, \frac{(x_N)^{p_N}}{\sum_{i=1}^{N-1} (x_i)^2} \right)^{\frac{1}{p_N - 2}} ,$$

where c is a positive constant. Evidently, this minimizer is unbounded as the first $(N-1)$ variables approach the origin while x_N stays positive. Another compelling counter-example is discussed in [40].

3.1. **The elliptic case.** Up to our knowledge, the first result of boundedness for solutions of equations of the type (1.11) was introduced in [47] in the year 1970. The author considers the Dirichlet problem:

$$\begin{cases} -\mathrm{div} A(x,u,Du) = B(x,u,Du) & \text{in } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where $\Omega \subset \mathbb{R}^N$ is bounded and A, B are assumed to be subject to a generalized set of structural conditions similar to (1.10). The author concludes that general weak solutions of (3.1) are bounded, giving the following estimate:

$$\sup_{\Omega} |u(x)| \leqslant C\left(||u||_{L^{sr}(\Omega)}^{\frac{1}{\theta}} + 1 \right), \tag{3.2}$$

where $\theta = (\bar{p}^* - sr)/(k - s)$ and C, r, s and k are the aforementioned structure constants, provided that $p_N < \bar{p}^*$ and u is integrable enough. Similarly, under the same assumption (namely $p_N < \bar{p}^*$), the authors in [6] show that minimizers of (1.2) with bounded boundary data are globally bounded. This result is extended to the limit case when $p_N = \bar{p}^*$ in [39]. Authors in [58], [46] only assume $p_N < (Np_1)/(N - p_1)$ to conclude the local boundedness of minimizers of (1.2) subject to a similar version of the growth conditions to (1.3).
Much more recently in [32] the authors refined the estimates of [47]. They consider local weak solutions of (1.11) within a specialized geometric setting. This consists of localizing the estimates in cubes of the shape

$$Q_\rho = \prod_j (-\rho_j, \rho_j), \quad \rho_j = \rho^{\frac{\alpha}{p_j}},$$

that we call *standard anisotropic cubes*, for a radius $\rho > 0$ such that $0 < \rho_j \leqslant \rho$ and $\alpha > 0$ chosen to accomodate the degeneracy of the equation. They conclude local boundedness for such solutions assuming that $p_N < \bar{p}^*$ with the quantitative estimate,

$$\sup_{Q_\rho} |u(x)| \leqslant 1 + C|Q_\rho|^{\frac{1}{\bar{p}^*}} ||u||_{L^{\bar{p}^*}(Q_{2\rho})}^{\beta}, \qquad \beta = \frac{\bar{p}^* - \bar{p}}{\bar{p}^* - p_N}, \tag{3.3}$$

where C is a quantitative structure constant. They also derive a similar result when $p_{\max} = \bar{p}^*$ under the integrability assumption $u \in L^{\bar{p}^*}(\Omega)$, as in this case this membership is not given for free by the anisotropic Sobolev embedding; they give a precise estimate within the same geometry. This result is an improvement of [47] in the sense that the boundedness of such local weak solutions remains invariant when the variables x_i are subject to dilations or scaling.
The author in [63], concludes global boundedness for weak solutions of a specialization of (3.1) using the method of Stampacchia [62] on a variational setting. Noteworthy, the

author only assumes the coercivity condition

$$\sum_i A_i(x,u,Du)\partial_i u \geqslant m \sum_i |\partial_i u|^{p_i},$$

for some $m \geqslant 0$ and sets $B(x,u,Du) = \sum_i \partial_i f_i$ for some integrable f_i with $u - u_0 \in W_0^{1,\mathbf{p}}(\Omega)$. Finally, in the paper [21], the authors establish local boundedness for minimizers of (1.2) in the setting of systems of anisotropic equations. The authors achieve this goal by using a distinct method, as conventional truncation methods are not applicable in the context of systems. Also, the cases of limit growth conditions have been addressed in [23], [22], [24]. A final word on generalizations: in [20], [49], [10], [48], the authors investigate the Dirichlet problem, encompassing both the functional (1.2) and the anisotropic equation (1.11), under the consideration of generalized growth conditions leading to the theory of Musielak-Orlicz spaces. Here, the power-law $f(Du) = \sum_i |\partial_i u|^{p_i}$ is substituted by a more general convex function that is called modular. See, for instance, [8] and references therein for an introductory survey on Musielak-Orlicz spaces and their applications to study equations of general growth.

3.2. **The parabolic case.** In contrast to the extensive research on the elliptic case, the study of the parabolic case has been comparatively less explored.

As far as we are aware, the first investigation of the boundedness for weak solutions of parabolic anisotropic equations as (1.9) subject to (1.10) was addressed in [59]. The authors assume the conditions $p_i \leqslant \bar{p}(1 + 2/N)$, $\forall i \in \{1, ..., N\}$ and $\bar{p} > 2N/(N+2)$ (which is, $\bar{p}$ is in some sense super-critical), and they establish that $u \in L^\infty_{\text{loc}}(\Omega_T)$. This generalizes the results applicable to the parabolic p-Laplacian case.

In [35], the authors investigate weak solutions of (1.9), with $B(x,t,s,\xi) = 0$, under the conditions (1.10). They establish the finer parabolic anisotropic Sobolev embedding, Lemma 2.2, and use it to give quantitative estimates on the supremum of the solutions (among many other results) within sets of the kind

$$Q_{\lambda,M} = \prod_i \left[-\lambda^{\frac{1}{p_i}}, \lambda^{\frac{1}{p_i}}\right] \times [T - M\lambda, T],$$

for, $M, \lambda > 0$ such that $Q_{\lambda,M} \subset \Omega_T$, provided that $\bar{p}(1 + 2/N) > \max\{2, p_N\}$. Finally, the authors in [19] study the regularity of solutions of the doubly nonlinear anisotropic equations,

$$\partial_t(|u|^{\alpha-1}u) - \text{div} A(x, t, u, Du) = 0, \quad \text{in} \quad \Omega_T, \tag{3.4}$$

with $\alpha \in (0, 1)$ and under similar structure conditions to (1.10). Note that for $\alpha = 1$, we recover the already established boundedness results from [35], [59]. The authors assume $1 < p_i < \bar{p}\big(1+\frac{\alpha+1}{N}\big)\ \forall i \in \{1, ..., N\}$, along with some other global integrability assumptions due to the double-nonlinearity, and then conclude local boundedness of the local weak solutions of (3.4). This result is a consequence of the membership of the solutions to a suitable parabolic energy class. This naturally applies for the super-critical range $\bar{p} > (N(1 + \alpha))(N + 1 + \alpha)$, while local sup-estimates are provided in the sub-critical case $\bar{p} < (N(1 + \alpha))/(N + 1 + \alpha)$ when more global integrability is available.

4. Hölder Continuity

4.1. **The elliptic case.** For the prototype equation (1.6), the Lipschitz bound of [7] provides the most general continuity result, as the authors only assume the local weak solutions to be locally bounded, disregarding the range of the degenerate p_i's considered. The authors proved that every bounded local minimizer of the functional (1.5) is locally Lipschitz. For more general assumptions on the coefficients, most available results of local Hölder continuity are achieved by adapting parabolic techniques in an elliptic setting.
In the study [53], the authors consider the following class of anisotropic equations,

$$-\sum_{i=1}^{N-1} \partial_i(A_i(x, u, Du)) - \partial^2_{NN}u = B(x, u, Du), \qquad \text{in } \Omega \subset \mathbb{R}^N. \tag{4.1}$$

This equation is a special case of (1.11), where the authors consider diffusion in only two directions; $2 < p_i = p, \forall i \in \{1, ..., N-1\}$ and $p_N = 2$ with A, B subject to (1.10) (adapted to this choice of p_i's). They assume that $2 < p \leqslant \bar{p}^*$ and $\bar{p} < N$, which guarantees that the local weak solutions are locally bounded, as seen in Section 3. For such solutions, they show a bound on their Hölder seminorm by establishing that there exists $\alpha \in (0, 1)$

such that for any $\mathcal{K} \subset \Omega$, compact set one has

$$|u(x) - u(y)| \leqslant \gamma \|u\|_{\infty,\Omega} \left(\frac{|x' - y'| + \|u\|_{\infty,\Omega}^{\frac{p-2}{p}} |x_N - y_N|^{\frac{2}{p}}}{p - \mathrm{dist}(\mathcal{K}, \partial\Omega)} \right)^{\alpha}, \tag{4.2}$$

for any $x, y \in \mathcal{K}$ and γ is positive constant. Here $x' = (x_1, \dots, x_{N-1})$ and $p - \mathrm{dist}(\mathcal{K}, \partial\Omega)$ is the intrinsic, elliptic $p-$distance from $\mathcal{K}$ to $\partial\Omega$ given by

$$p - \mathrm{dist}(\mathcal{K}, \partial\Omega) = \inf_{x \in \mathcal{K}, y \in \partial\Omega} \left(|x' - y'| + \|u\|_{\infty,\Omega}^{\frac{p-2}{p}} |x_N - y_N|^{\frac{2}{p}} \right), \tag{4.3}$$

which is a classical tool in the parabolic regularity (see [26]). Moreover, in [51], the authors extend this Hölder continuity result for the singular case where $1 < p < 2$, using a parabolic technique called *exponential shift*, reminiscent of [29]. Unfortunately, this technique only allows $p_N = 2$. In order to address the more general situation where $p_i = 2 \ \forall i \in \{1, ..., s\}, 1 < s < N$, the authors in [4], [16] investigate the following class of singular anisotropic equations,

$$\sum_{i=1}^{s} \partial_{ii}^2 u + \sum_{i=s+1}^{N} \partial_i (A_i(x, u, Du)) = 0 \qquad \text{in } \Omega \subset \mathbb{R}^N, \tag{4.4}$$

where $\Omega \subset \mathbb{R}^N$ is open and bounded. Here, A is assumed to be subject to the structure conditions (1.10) with the choice: $p_i = 2, \forall i \in \{1, .., s\}$ (called there *nondegenerate variables*) and $1 < p_i = p < 2, \forall i \in \{s+1, ..., N\}$ (*singular variables*) for $N \geqslant 2$. Using a different parabolic technique based on logarithmic estimates, the authors prove various results, including Hölder continuity for local weak solutions of (4.4) in a generalized form of (4.2).

Finally, the work [32] treats the full anisotropic case with p_is that can be all different and the operator with rough coefficients: here too the authors show Hölder continuity for local weak solutions of (1.11)-(1.10), assuming only the boundedness of such solutions, $\bar{p} < N$ and that $p_{\max} - p_{\min} \leqslant \frac{1}{q}$ for some constant $q > 1$. Unfortunately, this is a stability result, as the constant q is not quantitatively determined.

4.2. **The Parabolic case.** Unlike the elliptic case, the parabolic case has limited known results. This issue has been previously encountered with imprecise proofs or unclear geometric settings. Up to our knowledge, the only result available is in [11], and it concerns the degenerate ($p_i > 2$ for all $i = 1, \dots N$) prototype equation (1.1). There the authors show that within the condition of compactly supported evolution, i.e. $p_N < \bar{p}(1+1/N)$, it is possible to obtain various nontrivial consequences of the Harnack inequality presented in [15], such as local Hölder continuity, rigidity results and an equivalent form of Harnack inequality that is not intrinsic in time (at the price of a parabolic inclusion of the domain). Unfortunately, this result is limited to the case where the diffusion process presents a finite speed of propagation, while it would be reasonable to expect the Hölder continuity of solutions for the whole range of boundedness $p_N < \bar{p}(1 + 2/N)$, as in the isotropic case. See also [12] for another point of view through a different scaling and [45] for an investigation of continuity issues in the case of the singular anisotropic porous medium equation

$$\partial_t u = \sum_i \partial_i \left(m_i u^{m_i - 1} \partial_i u \right) \ , \qquad 0 < m_1 \leqslant \cdots \leqslant m_N < 1 \ . \tag{4.5}$$

5. Harnack inequality

Much less is known from the point of view of Harnack estimates due to the strong interaction of the competition of directional diffusion at the global scale. In the elliptic case, again, the main results are [51] with the technique of the exponential shift and [16] via logarithmic estimates. The Harnack estimates concern non-negative local weak solutions to (4.4)-1.10. We comment just on the homogeneous case, i.e., when $C = 0$ in 1.10, and we suppose for simplicity that $\Omega = \mathbb{R}^N$. If $u \geqslant 0$ is indeed such solution, and we suppose that $u(0) > 0$, then we have

$$u(0) \leqslant \gamma \inf_{Q_{\theta,\rho}} u, \qquad \text{where} \quad Q_{\theta,\rho} = B''_\theta \times B''_\rho, \tag{5.1}$$

where $B'_\theta \subset \mathbb{R}^s$ and $B''_\rho \subset \mathbb{R}^{N-s}$ are the respective balls centered in the origins and with radii $\theta, \rho > 0$ necessarily linked by the formula

$$\theta = \delta u(0)^{\frac{2-p}{p}} \rho^{\frac{p}{2}},$$

being $\delta > 0$ a constant dependent only on the structural data and $2, N, p$. This kind of geometry, being dependent on the value $u(0)$, is therefore called *intrinsic*, and for this reason, the relative estimate (5.1) is called an *intrinsic* Harnack estimate; just as in the parabolic case (see for instance [26]).

5.1. **The parabolic case.** Here again, our understanding of the properties of solutions clusters to the sole prototype degenerate parabolic equation, see [15], in the range of p_is of the finite propagation of disturbances. The Harnack estimates take the following form: let $u \geqslant 0$ be a local weak solution to (1.1) in $\mathbb{R}^N \times (0, \infty)$ with $2 < p_i < \bar{p}(1+1/N)$ for all $i = 1, \dots, N$ and let us suppose that $u(0) > 0$. Then, there exist constants $\gamma, K_1, K_2 > 0$ depending only on the structural data, such that for all $\rho > 0$, we have

$$\frac{1}{\gamma} \sup_{\mathcal{K}_\rho(\theta)} u(\,\cdot\,,\, -K_1 \theta^{2-\bar{p}} \rho^{\bar{p}}) \leqslant u(0) \leqslant \gamma \inf_{\mathcal{K}_\rho(\theta)} u(\,\cdot\,,\, K_1 \theta^{2-\bar{p}} \rho^{\bar{p}}), \tag{5.2}$$

where $\mathcal{K}_\rho(\theta)$ is the *intrinsic anisotropic cube* given by

$$\mathcal{K}_\rho(\theta) = \prod_i \left\{ |x_i| < \theta^{\frac{p_i - \bar{p}}{p_i}} \rho^{\frac{\bar{p}}{p_i}} \right\}, \quad \theta = u(0)/K_2 \,. \tag{5.3}$$

Here, we remark that, as in the case of the isotropic p-Laplacean equation, the estimate is given at different times that depend on the solution itself. On the other hand, in contrast with the isotropic case, here also the space-geometry $\mathcal{K}_\rho(\theta)$ is defined by means of the value $u(0)$, in order to accommodate the degeneracy of the anisotropy. Observe that this space geometry can become very strange: as soon as θ tends to zero, in all the directions $p_i > \bar{p}$ the cube $\mathcal{K}_\rho(\theta)$ shrinks, meanwhile in the other directions it expands. The technique performed is innovative because the aforementioned Harnack estimate is obtained via comparison with an abstract self-similar solution to the equation; even if the solution is not explicitly computed, via comparison and optimal support estimates, it is still possible to control its behavior and use it to expand the positivity of the solution. See [18] for an introduction to this approach toward the determination of Harnack estimates. Nonetheless, this technique relies strongly on the comparison principle and the natural scaling of the equation and, therefore, is unsuitable to solve the problem for more general operators as (1.9). Moreover, supposing $u \geqslant 0$ is a stationary solution to (1.1), the estimate (5.2) provides an estimate also for the solutions to the prototype elliptic equation (1.6), but unfortunately only for the parabolic range of p_is deriving from the finite speed of propagation. Some results were obtained also for anisotropic equations of porous medium type (see the recent preprint [13]).

Remark 5.1. A couple of very recent results have appeared regarding integral Harnack estimates for the singular case $1 < p_i < 2$, some integral Harnack-type estimates are found in [14], and in [13] for general operators behaving as (1.1) and the anisotropic porous medium equation (4.5). Unfortunately, these investigations answer only partially to the quest of point-wise Harnack estimates as (5.2), as they show only the following integral form for non-negative local weak solutions to equations as (1.1),

$$\sup_{0\leqslant\tau\leqslant t}\int_{\mathcal{K}_\rho(\theta(t))} u(x,\tau)\,dx \leqslant \gamma \inf_{0\leqslant\tau\leqslant t}\int_{2\mathcal{K}_\rho(\theta(t))} u(x,\tau)\,dx + \gamma\left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}, \tag{5.4}$$

for $\rho, t > 0$ and where this time $\theta(t) = (t/\rho^p)^{1/(2-\bar{p})}$ is the intrinsic scale of the geometry. See also [36], [37] for an asymptotic analysis of the behavior of solutions to these equations toward self-similarity. These L^1-L^1 Harnack-type estimates are very useful in the theory of regularity (see, for instance, [31], [17], [38]) because they allow to transport the measure information on the positivity of the solution along the time variable.

6. Conclusions and Future Perspectives

As discussed in the previous Sections, let aside the boundedness of local weak solutions, the issues of local Hölder continuity and Harnack inequality for solutions to operators (1.9) with structure conditions (1.10) are open.

We hope with this short note to attract the interest of the young research community to this domain of the regularity theory, which requires both intuition of the underlying physics of diffusion and the development of novel techniques; for these aspects, we believe it fascinating and intriguing.

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