# REGULARITY AND SEMIALGEBRAICITY OF SOLUTIONS OF LINEAR EQUATIONS SYSTEMS

# REGOLARITÀ E SEMIALGEBRICITÀ DI SOLUZIONI DI SISTEMI DI EQUAZIONI LINEARI

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Abstract. This work is concerned with the study of a necessary and sufficient condition for the existence of solutions with a given regularity to a system of linear equations with coefficients of given regularity. First, to properly contextualize the subject matter and to introduce crucial analytical solving tools, we go through results by C. Fefferman - J. Kollár and by C. Fefferman - G. K. Luli. Then we prove our result to determine a necessary and sufficient condition for the existence of *continuous* ($C^0$) *semialgebraic solutions* in case of a system of linear equations with *continuous semialgebraic coefficients*.

Sunto. Questo lavoro riguarda lo studio di una condizione necessaria e sufficiente per l'esistenza di soluzioni con una determinata regolaritá per un sistema di equazioni lineari a coefficienti di regolaritá data. In primo luogo, per contestualizzare adeguatamente l'argomento e introdurre strumenti analitici risolutivi cruciali, passiamo in rassegna risultati di C. Fefferman - J. Kollár e di C. Fefferman - G. K. Luli. Proseguiamo poi dimostrando un nostro risultato per determinare una condizione necessaria e sufficiente per l'esistenza di soluzioni semialgebriche continue nel caso di un sistema di equazioni lineari con coefficienti semialgebrici continui.

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## 1. Introduction

In this work we deal with the study of a necessary and sufficient condition for the existence of solutions with a given regularity to a system of linear equations

$$\sum_{j=1}^{M} A_{ij}(x)F_j(x) = f_i(x), \quad (i = 1, \ldots, N), \tag{1}$$

for unknown functions $F_1, \ldots, F_N$ with prescribed regularity. First, in order to outline the scope of investigation in which the problem is placed and to introduce crucial analytical solving techniques, we provide an overview of outcomes by C. Fefferman - J. Kollár and by C. Fefferman - G. K. Luli. Then we prove a new result to determine a necessary and sufficient condition for the existence of *continuous* ($C^0$) *semialgebraic* [1] *solutions* in case of a system of linear equations with *continuous semialgebraic coefficients* in the above system.

To set the context in which we give our contribution, we wish to recall some of the most important papers in the research area.

In "Continuous Solutions of Linear Equations" [7] C. Fefferman proved, by means of analysis techniques, a necessary and sufficient condition for the existence of a continuous solution $(\phi_1, \ldots, \phi_s)$ of the system

$$\phi = \sum_{i=1}^{s} \phi_i f_i \tag{2}$$

given the *continuous* functions $\phi$ and $f_i$. More precisely, by applying the theory of the Glaeser refinements for bundles, he showed that system (2) has a continuous solution if and only if the affine Glaeser-stable bundle associated with system (2) has no empty fiber.

Moreover J. Kollár, in the same (joint) paper [7], starting from the above result and

[1] By *semialgebraic subsets* of $\mathbb{R}^n$ we mean the smallest class of subsets of $\mathbb{R}^n$ containing all the sets $x \in \mathbb{R}^n$; $P(x) = 0$ and $x \in \mathbb{R}^n$; $P(x) > 0$, and being closed respect to the operations of complementation and finite intersection and union.

By *semialgebraic function* we mean a function which graph is a semialgebraic set.

For a more detailed description Subsection 2.2.1 and 2.2.1 in [3]

making use of algebraic geometry techniques as blowing up at singular points, proved that fixed the *polynomials* $f_1, \ldots, f_s$ and assuming system (2) has a solution, then:
1) if $\phi$ is semialgebraic then there is a solution $(\psi_1, \ldots, \psi_s)$ of $\phi = \sum_i \psi_i f_i$ such that the $\psi_i$ are also semialgebraic;
2) let $U \subset \mathbb{R}^n \setminus Z$ (where $Z := (f_1 = \ldots = f_r = 0)$) be an open set such that $\phi$ is $C^m$ on $U$ for some $1 \leq m \leq \infty$ or $m = \omega$. Then there is a solution $\psi = (\psi_1, \ldots, \psi_s)$ of $\phi = \sum_{i=1}^{s} \psi_i f_i$ such that the $\psi_i$ are also $C^m$ on $U$.

In "Solutions to a system of equations for $C^m$ functions" [9] C. Fefferman and G.K. Luli exhibited generators of the module $\mathcal{M}$ (over the ring of polynomials on $\mathbb{R}^n$) of the vectors $f := (f_1, \ldots, f_s)$ of *polynomials* $f_1, \ldots, f_s$ such that

$$\sum_{j=1}^{M} A_{ij} F_j = f_i \ (i = 1, \ldots, N),$$

(for unknown functions $F_1, \ldots, F_N \in C^m(\mathbb{R}^n)$ , $m$ fixed) admits a $C^m$ solution.

In "$C^m$ Semialgebraic Sections Over the Plane" [11] C. Fefferman and G.K. Luli showed that if $\mathcal{H}$ is a semialgebraic bundle with respect to the space of $\mathbb{R}^D$-valued functions on the plane $\mathbb{R}^2$ with continuous derivatives up to order $m$ (that space is called $C^m_{loc}(\mathbb{R}^2; \mathbb{R}^D)$) and it has a $C^m_{loc}(\mathbb{R}^2; \mathbb{R}^D)$ section, then $\mathcal{H}$ has a semialgebraic and $C^m_{loc}(\mathbb{R}^2; \mathbb{R}^D)$ section. Actually, the authors do not give an explicit method to compute that semialgebraic $C^m_{loc}(\mathbb{R}^2; \mathbb{R}^D)$ section: the $C^m_{loc}(\mathbb{R}^2; \mathbb{R}^D)$ semialgebraic section is defined as the one satisfying equations (97), (98), and (99) at p.44 of [11].

In Section 2 of this paper, first of all, to outline the general frame of the addressed issue, we present the different problems that can be posed for a system of the form (1). Then, to introduce key analytical solving techniques which are based on Fefferman-Glaeser theory and used to achieve our result (see Subsection 3.2), attention is focused on the problem solved in Fefferman-Luli [9, 10]:

**Problem:** *Suppose the $A_{ij}$ are given polynomials. For fixed $m$, the vectors $f = (f_1, \dots, f_N)$ of polynomials $f_1, \dots, f_N$ for which (1) admits a $C^m$ solution $F$ form a module $\mathcal{M}$ over the ring $\mathcal{R}$ of polynomials on $\mathbb{R}^n$ for the linearity of the system.*
*Exhibit generators for $\mathcal{M}$ .*

This problem cannot be solved using *only analysis*, because it concerns generators for a module over a polynomial ring. On the other hand, it cannot be solved using *only algebra*, because it concerns $C^m$ functions. Hence the *solving strategy* adopted by Fefferman and Luli is to make a clean *splitting into* an ***analysis and real algebraic geometry problem*** and an ***algebra problem***. After the statement of this two problems, we focus on the first one. Its solution is obtained through two theorems *Differential Characterization Theorem* and *Differential Characterization Theorem in the Compact Case* and their proofs provided in Fefferman-Luli [9] are sketched. These proofs are obtained by analysis techniques, principally:

– Gleaser refinement approach,

– theory of semialgebraic sets,

and by the characterisation of boundedness and zero-limit property of a quadratic form (to control approximation).

In Section 3 of this paper, we obtain our result: starting from the results of Fefferman-Kollár in [7], we adopt a new approach based on Fefferman's techniques of Glaeser refinement to show a more general result than the one proved by Kollár by using techniques from algebraic geometry. Considering a system of linear equations with semialgebraic (not only polynomial as in [7]) coefficients on $\mathbb{R}^n$, we get a necessary and sufficient condition for the existence of a continuous and semialgebraic solution on $\mathbb{R}^n$. This is different from what Fefferman-Luli obtained in [11] since they stated their result for solutions of regularity $C^m$ on the plane $\mathbb{R}^2$. More in depth, we prove that a continuous and semialgebraic solution on $\mathbb{R}^n$ exists if and only if there is a continuous solution i.e., if the Glaeser-stable bundle associated to the system has no empty fiber.

Before delving into the technical discussion we would like to point out that investigating these problems is relevant not only because that these are questions of complex resolution and therefore challenging in themselves, but also because actually these problems are deeply related to the study of the *continuous closure of ideals*, a tool developed by Brenner [1] from a problem posed by Hochster in the 1990s. It has also to be underlined that these problems have significant implications in theoretical computer science as well. Fefferman and Klartag, in fact, determined an approximation of positive functions by nonnegative semialgebraic functions [6]. In developing this Fefferman pointed out that the norm of these functions remains uniformly bounded throughout the entire approximation process, and this is a problem related to the ability of an ideal machine to store information in a self-contained manner.

## 2. $C^m$ Regularity Characterization for Solutions of a Linear System

### 2.1. **Problem Definition.**

The system of linear equations

$$\boxed{\begin{array}{l} \displaystyle (3) \qquad \sum_{j=1}^{M} A_{ij}(x)F_j(x) = f_i(x), \quad (i = 1, \dots, N), \\ \text{for unknown functions } F_1, \dots, F_N \in C^m(\mathbb{R}^n) \text{ and for fixed } m \end{array}}$$

is the object of the studies of Charles Fefferman and Garving K. Luli [9, 10]: the results of Fefferman-Luli [9] will be presented in this Section [2].

Note that since $m$ is fixed, one is not allowed to lose derivatives.

Indeed the most interesting systems of the kind (3) are the *undetermined* ones, like the following single equation

$$x^2F_1 + y^2F_2 + xyz^2F_3 = f(x, y, z) \tag{4}$$

for unknown continuous functions $F_1, F_2, F_3$ on $\mathbb{R}^3$ [4].

For a system of the form (3) the following problems can be posed:

[2] $C^m(\mathbb{R}^n)$ denotes the vector space of $m$-times continuously differentiable functions on $\mathbb{R}^n$, with no growth conditions assumed at infinity. Similarly, $C^m(\mathbb{R}^n, \mathbb{R}^D)$ denotes the space of all such $\mathbb{R}^D$-valued functions on $\mathbb{R}^n$. This notation remains in force during this section, but will be changed later on.

**Problem 1:** *Suppose the $A_{ij}$ and $f_i$ are given functions. For fixed $m$, how can one decide whether (3) admits a solution $F = (F_1, \dots, F_M) \in C^m(\mathbb{R}^n, \mathbb{R}^M)$?*

**Problem 2:** *Suppose the $A_{ij}$ are given polynomials. For fixed $m$, the vectors $f = (f_1, \dots, f_N)$ of polynomials $f_1, \dots, f_N$ for which (3) admits a $C^m$ solution $F$ form a module $\mathcal{M}$ over the ring $\mathcal{R}$ of polynomials on $\mathbb{R}^n$ for the linearity of the system.*
*Exhibit generators for $\mathcal{M}$ .*

**Problem 3:** *Suppose the $A_{ij}$ and $f_i$ are polynomials and suppose (3) admits a $C^m$ solution $F$. Can one take a $C^m$ solution $F$ to be semialgebraic?*

• For $\boldsymbol{m = 0}$, these problems were posed by Brenner [1], and Epstein-Hochster [4], and solved by Fefferman-Kollár [7] and Kollár [12].
We have solved a particular case of Problem 3 for $m = 0$, through the development of a new method which will be used in Section 3.
Indeed, for $m = 0$ the answer to Problem 3 is affirmative: $F \in C^0$ can be taken to be semialgebraic.
In Kollár-Nowak [13] an example shows that it is not always possible to take $F_1, \dots, F_N$ to be rational functions.

• For $\boldsymbol{m \geq 1}$:
− Problem 1 was solved in Fefferman-Luli [8], with no restriction on the functions $A_{ij}$ , $f_i$.
It must be pointed out that in Problem 2 and 3 one does not need to assume that the given matrix elements $A_{ij}$ are polynomials: one may take them to be (possibly discontinuous) semialgebraic functions. In fact, [9] treat Problem 2 in this more general setting and also our contribution to Problem 3 is in this setting.
− Problem 2 was solved in Fefferman-Luli [9, 10]: particularly, the studies of Fefferman-Luli [9] will be the object of Subsection 2.2.
− Problem 3 is still unsolved.

### 2.2. Solving Strategy.

This subsection provides a description of the strategy adopted in [9, 10] to solve Problem 2.

It can be noted that Problem 2 cannot be solved using *only analysis*, because it concerns generators for a module over a polynomial ring. On the other hand, it cannot be solved using *only algebra*, because it concerns $C^m$ functions. To make a clean *splitting into* an ***analysis and real algebraic geometry problem*** and an ***algebra problem***, the analogue of Problem 2 for vectors $f = (f_1, \ldots, f_N)$ of $C^\infty$ functions is posed.

**Problem 2a (*Analysis and real algebraic geometry* problem)**: Fix a *nonnegative integer* $m$ *and a matrix* $(A_{ij})$ *of semialgebraic functions on* $\mathbb{R}^n$*. Characterize all the* $f = (f_1, \ldots, f_N) \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ *for which (3) admits a* $C^m$*-solution.*

Problem 2a is a generalization of Problem 1 since the solution of Problem 2a leads to the knowledge of all the $f_j$ for which Problem 1 has a solution (and not only the given ones). In order to illustrate the result of Fefferman and Luli on Problem 2a consider the Epstein-Hochster equation (4), for unknown continuous $F_1, F_2, F_3$. It can be checked that for $f \in C^\infty$ a continuous solution exists *if and only if* $f$ satisfies

$$(5)\qquad \left[\begin{array}{ll} f(x,y,z) = \frac{\partial f}{\partial x}(x,y,z) = \frac{\partial f}{\partial y}(x,y,z) \;\; = 0 & \text{for } x = y = 0, z \in \mathbb{R} \\ \qquad\qquad\qquad \text{and} & \\ \frac{\partial^2 f}{\partial x \partial y}(x,y,z) = \frac{\partial^3 f}{\partial x \partial y \partial z}(x,y,z) = 0 & \text{at } x = y = z = 0 \end{array}\right.$$

*Proof.*

- Sufficient condition

  Taylor-expanding $f$ at $(0,0,z)$, with integral remainder, one gets

$$f(x,y,z) = \underbrace{f(0,0,z)}_{=0} + \underbrace{\frac{\partial f}{\partial x}(0,0,z)x}_{=0} + \underbrace{\frac{\partial f}{\partial y}(0,0,z)y}_{=0} + \left\langle G(x,y,z)\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle$$

where the first three terms are zero by hypothesis and $G$ is a symmetric $2 \times 2$ matrix. Therefore

$$(6)\qquad f(x,y,z) = G_{11}(x,y,z)x^2 + G_{22}(x,y,z)y^2 + 2G_{12}(x,y,z)xy$$

with

$$G_{12}(x,y,z) = C \int_0^1 (1-t)\, \frac{\partial^2 f}{\partial x \partial y}(tx,ty,tz)\, dt$$

thus

$$G_{12}(0,0,0) = C\, \frac{\partial^2 f}{\partial x \partial y}(0,0,0) \int_0^1 (1-t)\, dt$$

and

$$\frac{\partial G_{12}}{\partial z}(0,0,0) = C \int_0^1 (1-t)t\, \frac{\partial^3 f}{\partial x \partial y \partial z}(0,0,0)\, dt.$$

Then, Taylor-expand $z \longmapsto G_{12}(0,0,z)$ at $(0,0,0)$ as follows

$$G_{12}(0,0,z) = C_1 \underbrace{\frac{\partial^2 f}{\partial x \partial y}(0)}_{=0} + C_2 \underbrace{\frac{\partial^3 f}{\partial x \partial y \partial z}(0)}_{=0} z + H_{12}(0,0,z) z^2$$

where the first two terms are zero by hypothesis, hence

$$G_{12}(0,0,z) = H_{12}(0,0,z) z^2 \tag{7}$$

Besides, in order to verify the solvability of the equation (4), expand $G_{12}(x,y,z)$ in Taylor's series in $(0,0,z)$. One has

$$G_{12}(x,y,z) = G_{12}(0,0,z) + \left\langle L(x,y,z), \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle$$

where $L(x,y,z) \in \mathbb{R}^2$, thus

$$G_{12}(x,y,z) = G_{12}(0,0,z) + L_1(x,y,z)x + L_2(x,y,z)y.$$

From this, taking into account (7), one gets

$$G_{12}(x,y,z) = H_{12}(0,0,z)z^2 + L_1(x,y,z)x + L_2(x,y,z)y.$$

Replacing this result in (6) one has

$$f(x,y,z) = x^2 F_1(x,y,z) + y^2 F_2(x,y,z) + xyz^2 F_3(0,0,z) \tag{8}$$

where

$$F_1(x,y,z) = G_{11}(x,y,z) + L_1(x,y,z)y,$$

$$F_2(x,y,z) = G_{22}(x,y,z) + L_2(x,y,z)x,$$

$$F_3(0,0,z) = 2H_{12}(0,0,z).$$

The (8) proves that, assuming fulfilled hypothesis (5), a continuous solution of equation (4) exists.

- NECESSARY CONDITION

To prove that the condition is necessary, one only needs to take derivatives of both sides of equation (4) and to evaluate both sides in the corresponding manifolds as described by system (5).

It must be pointed out that in (5) a third derivative of $f$ enters, even if here merely continuous solutions $F = (F_1, F_2, F_3) \in C^0$ are looked for.

For general systems of the kind (3), the result of Problem 2a is as follows. □

**Theorem 2.1. (Differential Characterization)** *Fix $m \geq 0$, and let $(A_{ij}(x))_{1\leq i\leq N, 1\leq j\leq M}$ be a matrix of semialgebraic functions on $\mathbb{R}^n$. Then there exist $K \in \mathbb{N}$ and linear partial differential operators $L_1, L_2, \ldots, L_K$, for which the following hold:*

- *each $L_\nu$ acts on vectors $f = (f_1, \ldots, f_N) \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$, and, for a certain $m_1 \in \mathbb{N}$ (possibly $m_1 > m$), has the form*

$$L_\nu f(x) = \sum_{i=1}^{N} \sum_{|\alpha|\leq m_1} a_{\nu i\alpha}(x)\partial^\alpha f_i(x)$$

*where the coefficients $a_{\nu i\alpha}$ are semialgebraic;*

- *let $f = (f_1, \ldots, f_N) \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$, then system (3) admits a solution $F = (F_1, \ldots, F_M) \in C^m(\mathbb{R}^n, \mathbb{R}^M)$ if and only if $L_\nu f = 0$ on $\mathbb{R}^n$ for each $\nu = 1, \ldots, K$.*

**Remark.** The existence of $K$ follows from considerations on semialgebraic sets and it could be calculated thanks to algorithms of computational algebraic geometry.

In the case of the Epstein-Hochster equation (4), with $m = 0$, the operators $L_\nu$ are

$$\mathbb{I}_{x=y=0}\partial^0,\ \mathbb{I}_{x=y=0}\partial_x,\ \mathbb{I}_{x=y=0}\partial_y,\ \mathbb{I}_{x=y=z=0}\partial^2_{xy},\ \mathbb{I}_{x=y=z=0}\partial^3_{xyz},$$

where $\mathbb{I}$ denotes the indicator function (compare with (5)).

The proof of Theorem 2.1 provided in Fefferman- Kollár [9] is constructive: in principle, one can compute the operators $L_\nu$ from the data $m$, $(A_{ij}(x))_{1\le i\le N, 1\le j\le M}$.

Theorem 2.1 can be applied to solve Problem 2. The idea is as follows.

First of all, given linear partial differential operators $L_1, \dots, L_K$ with semialgebraic coefficients (not necessarily given by Theorem 2.1), introduce the $\mathcal{R}$-module $\mathcal{M}(L_1, \dots L_K)$ of all polynomial vectors

$$f = (f_1, \dots, f_N) \text{ s.t. } L_\nu(Pf) = 0 \text{ on } \mathbb{R}^n$$

for each $\nu = 1, \dots, K$, and for each $P$ polynomial in $\mathbb{R}[x_1, \dots, x_n] = \mathcal{R}$ (the ring of polynomials on $\mathbb{R}^n$).

If the polynomial vectors $f$, annihilated by the $L_\nu$, already form an $\mathcal{R}$-module then that $\mathcal{R}$-module coincides with $\mathcal{M}(L_1, \dots, L_K)$. As a matter of fact,

- $L_\nu(f) = 0,\ \forall \nu = 1, \dots, K \quad \Rightarrow \quad f \in \mathcal{M}(L_1, \dots, L_K)$ since one can apply Theorem 2.1, after noting that if $(F_1, \dots, F_M)$ is a solution of system (3) with source term $f$ , then $(PF_1, \dots, PF_M)$ is a solution of system (3) with source term $Pf$;
- $f \in \mathcal{M}(L_1, \dots, L_K) \ \Rightarrow\ L_\nu(f) = 0,\ \forall \nu = 1, \dots, K$ since $P \equiv 1$ is a polynomial.

In particular, for the $L_1, \dots, L_K$ given by Theorem 2.1 the module $\mathcal{M}$ in Problem 2 satisfies $\mathcal{M} = \mathcal{M}(L_1, \dots, L_K)$.

From the above it follows that Problem 2 is reduced to the following problem of computational algebra.

**Problem 2b (*Algebra* problem):** Given linear partial differential operators $L_1, \cdots, L_K$ with semialgebraic coefficients, exhibit generators for the $\mathcal{R}$-module $\mathcal{M}(L_1, \cdots, L_K)$.

Therefore, by posing Problem 2a, Problem 2 is split into an analysis problem and an algebra problem. Problem 2b is solved in detail in Fefferman-Luli [10].

In the following sections we will focus on the theoretical results to the base of the resolution of Problem 2. Actually the solution of Problem 2a is provided by the complete demonstration of Theorem 2.1 given by Fefferman-Luli in [9].

2.2.1. *Outline of the Proof of the Theorem of Differential Characterization.*
Let us sketch the proof of Theorem 2.1. First of all, some notation needs to be introduced.

**Notation:**

**1.** For $x \in \mathbb{R}^n$ and $F \in C^m(\mathbb{R}^n, \mathbb{R}^D)$, write

$$J_x^{(m)} F$$

(the "$m$-jet" of $F$ at $x$) to denote the $m^{th}$ order Taylor's polynomial of $F$ at $x$. Hence

$$J_x^{(m)} F \in \mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^D)$$

where $\mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^D)$ is the vector space of all $\mathbb{R}^D$-valued polynomials of degree at most $m$ on $\mathbb{R}^n$.

**2.** If $D = 1$, write $\mathcal{P}^{(m)}(\mathbb{R}^n)$ in place of $\mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R})$.

**3.** For $F, G \in C^m(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, one has $J_x^{(m)}(FG) = J_x^{(m)} F \odot_x J_x^{(m)} G$, where

$$P \odot_x Q := J_x^{(m)}(PQ), \qquad P, Q \in \mathcal{P}^{(m)}(\mathbb{R}^n)$$

The multiplication $\odot_x$ makes $\mathcal{P}^{(m)}(\mathbb{R}^n)$ into a ring $\mathcal{R}_x^{(m)}$ (the "ring of $m$-jets at $x$"). $\mathcal{R}_x^{(m)}$ is finitely generated since

$$\{(y - x)^\alpha : \ \alpha \in (\mathbb{N} \cup \{0\})^n, \, |\alpha| \le m, \}$$

is a basis of $\mathcal{R}_x^{(m)}$.
Similarly, the multiplication

$$Q \odot_x (P_1, \dots P_M) := (Q \odot_x P_1, \dots, Q \odot_x P_M)$$

makes $\mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M)$ into an $\mathcal{R}_x^{(m)}$-module.

**Remark.** $\mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M)$, as an $\mathcal{R}_x^{(m)}$-module, is finitely generated since a basis of $\mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M)$ is

$$\{(y-x)^\alpha \otimes e_j : \ \alpha \in (\mathbb{N} \cup \{0\})^n, \, |\alpha| \leq m, \, j = 1, \ldots, M\}$$

where $e_j$ is the $j^{th}$ vector of the canonic basis of $\mathbb{R}^M$.

Now Fefferman-Luli [8] solution to **Problem 1** can be explained. Hence, recall Problem 1.

*Supposed $A_{ij}$ and $f_i$ are given functions. For fixed $m \geq 0$ one is asked to investigate whether system (3) has a solution $F \in C^m(\mathbb{R}^n, \mathbb{R}^M)$.*

The idea is to construct families $\mathcal{H} = (H_x)_{x \in \mathbb{R}^n}$ of affine subspaces $H_x \subset \mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M)$, such that any given $C^m$ solution $F$ of (3) necessarily satisfies

$$J_x^{(m)} F \in H_x \text{ for all } x \in \mathbb{R}^n. \tag{9}$$

To begin with, one can simply consider

$$\left[ \begin{array}{l} \hat{\mathcal{H}} = \left(\hat{H}_x\right)_{x \in \mathbb{R}^n}, \text{where} \\ \hat{H}_x = \left\{P \in \mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M) : A(x)P(x) = f(x)\right\}, \end{array} \right. \tag{10}$$

with:

$$P = \begin{bmatrix} P_1 \\ \vdots \\ P_M \end{bmatrix}, \quad A = (A_{ij}(x))_{\substack{1 \leq i \leq N, \\ 1 \leq j \leq M}} \quad f = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}.$$

Let the empty set be considered to be an affine subspace of $\mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M)$: this can already occur for $\hat{\mathcal{H}}$ given by (10) if equations (3) are inconsistent for some $x$. Note that clearly (10) cannot hold if some of the $H_x$ are empty.

The non-empty $H_x$ arising in the considered family $\mathcal{H}$ have a special form: they are translates of $\mathcal{R}_x^{(m)}$-submodules of $\mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M)$. Indeed, if

$$x \in \mathbb{R}^n, \ A(x)P_1(x) = 0 \text{ and } A(x)P_2(x) = 0,$$

where $P_1, P_2 \in \mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M)$, then

$$A(x)(P_1(x) + P_2(x)) = 0,$$
$$A(x)(\alpha P_1)(x) = A(x)(\alpha \odot_x P_1)(x) = \alpha(x)A(x)P_1(x), \quad \alpha \in \mathcal{R}_x^{(m)}.$$

since $(J_x^{(m)} P)(x) = P(x)$ for any $P \in \mathcal{P}^{(m)}(\mathbb{R}^n)$.
Thus, if

$$A(x)Q(x) = f(x),$$

where $Q \in \mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M)$, then

$$A(x)(Q(x) + P(x)) = f(x),$$

for each

$$P \in \mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M) \text{ such that } A(x)P(x) = 0$$

that means

$$Q + P \in H_x.$$

Hence,

$$(11) \qquad \left[\begin{array}{ll} \mathcal{H} = (H_x)_{x \in \mathbb{R}^n}, & \text{where } \forall x \text{ either } H_x = \emptyset \text{ or } H_x = P_x + I(x), \\ & \text{where } P_x \in \mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M) \text{ and} \\ & I(x) \in \mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M) \text{ is an } \mathcal{R}_x^{(m)}\text{-submodule.} \end{array}\right.$$

For instance, $\hat{\mathcal{H}}$ provided by (10) has this form.

Any $\mathcal{H}$ of the form (11) will be named a *bundle*, and $F \in C^m(\mathbb{R}^n, \mathbb{R}^M)$ will be called a *section* of the bundle $\mathcal{H}$ if (9) holds.

Now let us introduce further notation and definitions that will be used in this work.

**Notation and definitions:**

**1.** Any given $\mathcal{H}$ of the form (11) is named a *bundle*, and $F \in C^m(\mathbb{R}^n, \mathbb{R}^M)$ is called a *section* of the bundle $\mathcal{H}$ if (9) holds.

**2.** If $\mathcal{H} = (H_x)_{x\in\mathbb{R}^n}$ and $\mathcal{H}' = (H'_x)_{x\in\mathbb{R}^n}$ are bundles, then $\mathcal{H}'$ is named *subbundle* of $\mathcal{H}$ if $H'_x \subset H_x$ for all $x \in \mathbb{R}^n$.

To denote that $\mathcal{H}'$ is a subbundle of $\mathcal{H}$ the following notation is used

$$\mathcal{H} \supset \mathcal{H}'.$$

**3.** $H_{x_0}$ stands for the *fiber* of $\mathcal{H} = (H_x)_{x\in\mathbb{R}^n}$ at $x_0$.

It is immediate from (10) to see that a $C^m$ solution of system (3) is precisely a section of the bundle $\hat{\mathcal{H}}$: hence one can note that Problem 1 is a case of the following one.

**Problem 4:** *Given a bundle $\mathcal{H}$, decide whether $\mathcal{H}$ has a section.*

Let us have a qualitative look at how this problem can be solved: it can be solved using the notion of *Glaeser refinement.* The idea is as follows. Let $\mathcal{H} = (H_x)_{x\in\mathbb{R}^n}$ be a bundle, and let $x_0 \in \mathbb{R}^n$. Even if, by definition, any section $F$ of $\mathcal{H}$ must satisfy $J^{(m)}_{x_0}F \in H_{x_0}$, $H_{x_0}$ may contain polynomials $P_0 \in \mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M)$ that can never arise as the $m$-jet at $x_0$ of any section. In this case $\mathcal{H}$ can be replaced by a subbundle $\tilde{\mathcal{H}}$ without losing any sections. Let us see how such an $\tilde{\mathcal{H}}$ can be defined.

Once $x_0 \in \mathbb{R}^n$ and $P_0 \in H_{x_0}$ are fixed, consider a section $F$ of $\mathcal{H}$, with $J^{(m)}_{x_0}F = P_0$. Once an integer $k$ is fixed (determined by $m$, $n$, $M$), let $x_1, \dots, x_k \in \mathbb{R}^n$ lie close to $x_0$. Let us now give an idea of how $k$ can be determined. First of all, note that, to build a subbundle $\tilde{\mathcal{H}}$ of $\mathcal{H}$ such that $\mathcal{H}$ can be replaced by $\tilde{\mathcal{H}}$ without losing any sections, one need to control the growth of the $m^{th}$-order approximations of the sections of $\mathcal{H}$. Hence:

∘ if $m = 0$, only one point is needed (i.e. $k = 1$) to detect the growth of the $0^{th}$ order approximation of the sections of $\mathcal{H}$ (i.e. the behaviour of the section near $x_0$);

∘ if $m = 1$, $n$ points are needed (i.e. $k = n$) to determine the growth of the $1^{st}$ order approximation (i.e. the tangent approximation through hyperplanes) of the sections of $\mathcal{H}$, since $n$ points beyond $x_0$ are needed to determine a plane in $\mathbb{R}^n$ through $x_0$.

A more detailed explanation can be found in [5].

Now, setting

$$P_i = J_{x_i}^{(m)} F, \quad i = 1, \dots, k$$

one has

$$P_1 \in H_{x_1},\ P_2 \in H_{x_2}, \dots, P_k \in H_{x_k}$$

and

$$\sum_{0 \le i \le j \le k} \sum_{|\alpha| \le m} \left( \frac{|\partial^\alpha (P_i - P_j)(x_j)|}{|x_i - x_j|^{m-|\alpha|}} \right)^2 \xrightarrow[x_1, \dots x_k \to x_0]{} 0 \tag{12}$$

by Taylor's theorem. Note that $P_0, \dots, P_k$ enter into (12), but $P_0$ has a different role than $P_1, \dots, P_k$.

Let us prove this result in the case $m = 1$ and $|\alpha| = 0$. By Taylor-expansion one has that

$$P_j(x) = F(x_j) + F'(x_j)(x - x_j) \text{ and } P_i(x) = F(x_i) + F'(x_i)(x - x_i)$$

for each $i, j = 1, \dots, k$ with $i < j$. Hence,

$$\frac{|(P_i - P_j)(x_j)|}{|x_i - x_j|} = \left| \frac{F(x_i) - F(x_j)}{x_i - x_j} - \frac{F'(x_i)(x_j - x_i) - F'(x_j)(x_j - x_j)}{x_i - x_j} \right| . \tag{13}$$

Then note that

$$\frac{F(x_i) - F(x_j)}{x_i - x_j} \xrightarrow[x_i, x_j \to x_0]{} F'(x_0)$$

and

$$\frac{F'(x_i)(x_j - x_i) - F'(x_j)(x_j - x_j)}{x_i - x_j} = F'(x_i) \xrightarrow[x_i \to x_0]{} F'(x_0).$$

Substituting in (13) and calculating the sum for $i, j = 1, \dots, k$ with $i < j$, one gets the result (12).

Now, to prove the result (12) for the case $m > 1$ and $|\alpha| \geq 0$, iteratively isolate the incremental ratios and get the difference of derivatives of maximum order calculated in $x_0$.

The above observations lead to define, for the fixed $k$ chosen above, the *Glaeser refinement of the bundle* $\mathcal{H} = (H_x)_{x\in\mathbb{R}^n}$ as

(14)

$$\left[\begin{array}{l} \qquad \mathcal{G}(\mathcal{H}) = (\tilde{\mathcal{H}}_x)_{x\in\mathbb{R}^n}, \text{ where} \\ \\ \quad \forall x_0{\in}\mathbb{R}^n,\ \tilde{H}_{x_0} \text{ consists of those } P_0{\in}H_{x_0} \text{ s.t.} \\ \min\left\{\sum_{0\leq i\leq j\leq k}\sum_{|\alpha|\leq m}\left(\frac{|\partial^\alpha(P_i-P_j)(x_0)|}{|x_i-x_j|^{m-|\alpha|}}\right)^2 : P_1\in H_{x_1},\dots,P_k\in H_{x_k}\right\} \xrightarrow[x_1,\dots x_k\to x_0]{} 0 \end{array}\right.$$

The Glaeser refinement has three basic properties:

- $\mathcal{G}(\mathcal{H})$ is a subbundle of $\mathcal{H}$.
- $\mathcal{G}(\mathcal{H})$ and $\mathcal{H}$ have the same sections, as seen above.
- $\mathcal{G}(\mathcal{H})$ can be computed from $\mathcal{H}$, thanks to the explicit nature of (14).

Note that $\mathcal{G}(\mathcal{H})$ may have empty fibers, even if $\mathcal{H}$ has none: when this happens one knows that $\mathcal{H}$ has no sections.

Now starting from a given bundle $\mathcal{H}$, an *iterated Glaeser refinement* can be performed to pass to even smaller subbundles $\mathcal{H}^{(1)}, \mathcal{H}^{(2)}, \dots$ without losing sections. Set

$$\mathcal{H}^{(0)} = \mathcal{H}$$

and set

$$\mathcal{H}^{(\ell+1)} = \mathcal{G}(\mathcal{H}^{(\ell)}), \text{ for } \ell \geq 0.$$

Hence, by a trivial induction on $\ell$, one gets

$$\mathcal{H} = \mathcal{H}^{(0)} \supset \mathcal{H}^{(1)} \supset \mathcal{H}^{(2)} \dots,$$

and yet $\mathcal{H}$ and $\mathcal{H}^{(\ell)}$ have the same sections.

Each $\mathcal{H}^{(\ell)}$ can be computed from $\mathcal{H}$.

The main result of [8] provides the solution to Problem 4.

**Solution of Problem 4:** *For a sufficiently large integer $\ell_*$ determined by $m$,$n$,$M$, the following holds.*
*Let $\mathcal{H}$ be a bundle, and let $\mathcal{H}^{(0)}, \mathcal{H}^{(1)}, \mathcal{H}^{(2)}, \dots$ be its iterated Glaeser refinements. Then $\mathcal{H}$ has a section if and only if $H^{(\ell_*)}$ has no empty fibers.*

Note that this result in particular solves Problem 1 for system (3). Thus the discussion of Problem 1 is complete.

Now, in order to apply the above to Problem 4, one has to understand how, assuming $f \in C^\infty$, the iterated Glaeser refinements arising from the bundle $\hat{\mathcal{H}}$ in (10) depend on the right-hand side $f = (f_1, \cdots, f_N)$ in (3).
This gives rise to the study of bundles of the form

$$(15)\qquad \left[\begin{array}{l} H_f = \Big(T(x) J_x^{(m_1)} f + I(x)\Big)_{x\in\mathbb{R}^n}, \text{ where} \\ \quad \bullet\, I(x) \subset \mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M) \text{ is an } \mathcal{R}_x^{(m)}\text{-submodule} \\ \quad \text{depending semialgebraically on } x. \\ \quad \bullet\, T(x) : \mathcal{P}^{(m_1)}(\mathbb{R}^n, \mathbb{R}^M) \longrightarrow \mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M) \text{ is a linear map} \\ \quad \text{depending-semialgebraically on } x. \end{array}\right.$$

It must pointed out that in this work the semialgebric dependence on a point $x$ of a module valued function of $x$ is assumed as follows.

**Definition 2.1.** *A family of submodules $(M_z)_{z\in\mathbb{R}^n}$ of the $\mathcal{R}_x^{(m)}$-module $\mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M)$, with basis $\mathcal{B} = \{(y-x)^\alpha \otimes e_j :\ \alpha \in (\mathbb{N}\cup\{0\})^n,\ |\alpha| \le m,\ j = 1, \dots, M\}$, is said to be* depending semialgebraically on $z$ *if there exists a matrix valued function $z \longmapsto A(z)$, with semialgebric entries, such that*

$$\tilde{M}_z = Im A(z), \quad z \in \mathbb{R}^n$$

*where $\tilde{M}_z$ is the set of coordinates of elements of $M_z$ over $\mathcal{B}$.*

**Remark.** The definition does not depend on $\mathcal{B}$ since the coordinate change matrix has entries belonging to $\mathcal{R}_x^{(m)}$.

Now it must be investigated how the Glaeser refinement of bundle $\mathcal{H}_f$ in (15) depends on $f \in C^\infty$. In particular, when the Glaeser refinement has no empty fibers. Under suitable assumptions on $T(x)$ in (15), the following results hold:

**Proposition 2.1.** *The fibers of $\mathcal{G}(\mathcal{H}_f)$ are all non-empty if and only if $f$ is annihilated by finitely many linear partial differential operators $L_1, \dots, L_K$ with semialgebraic coefficients.*

**Proposition 2.2.** *If the fibers of $\mathcal{G}(\mathcal{H}_f)$ are all non-empty, then $\mathcal{G}(\mathcal{H}_f)$ again has the form (15), possibly with a smaller $I(x)$, a larger $m_1$ and different $T(x)$.*

This allows to keep track of the $f$-dependence of the iterated Glaeser refinements of bundle $\hat{\mathcal{H}}$ in (10), thus proving Theorem 2.1.

Let us say a few words about the proof of Proposition 2.1 and Proposition 2.2. As in (14) a quadratic form lies at the heart of the matter, one has to understand quadratic forms acting on the jets of a function $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ at points $x_1, \dots, x_k \in \mathbb{R}^n$. More precisely, suppose one is given a positive semidefinite quadratic form

$$\left[\begin{array}{l} (P_0, P_1, \dots, P_k) \longmapsto Q_{x_0, x_1, \dots, x_k}(P_0, P_1 \dots, P_k) \\ \text{depending semialgebraically on points } x_1, \dots, x_k \in \mathbb{R}^n. \\ \text{Here, } P_0 \in \mathcal{P}^{(m)}(\mathbb{R}^n, \mathbb{R}^M), \text{while } P_1, \dots, P_k \in \mathcal{P}^{(m_1)}(\mathbb{R}^n, \mathbb{R}^N). \end{array}\right. \tag{16}$$

Fix $x_0, P_0$, and let $x_1, \dots, x_k$ vary. One has to characterize the functions $f \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ such that

$$Q_{x_0, x_1, \dots, x_k}\left(P_0, J^{(m_1)}_{x_1} f, \dots, J^{(m_1)}_{x_k} f\right) \xrightarrow[x_1, \dots x_k \to x_0]{} 0.$$

These propositions are proved by induction on the dimension of a relevant semialgebraic set. To make the induction work, one has to allow the quadratic form in (16) to depend on additional points $z_1, \dots, z_L$.

Let us establish the following variant of Theorem 2.1 in the compact case.

**Theorem 2.2. (Differential Characterization in the Compact Case)** *Let $E \subset \mathbb{R}^n$ be compact, semialgebraic. Let $A(x)$ be an $N \times M$ matrix of semialgebraic functions* *defined on $E$. Let $m \geq 0$ be given. Then there exist linear partial differential operators $L_\nu (1 \leq \nu \leq \nu_{\max})$, for which the following facts hold:*

- *each $L_\nu$ has semialgebraic coefficients and carries functions in $C^\infty(\mathbb{R}^n, \mathbb{R}^N)$ to scalar-valued functions on $\mathbb{R}^n$,*
- *let $f = (f_1, \dots, f_N) \in C^\infty(\mathbb{R}^n, \mathbb{R}^N)$, then there exist*

$$F_1, \dots, F_M \in C^m(\mathbb{R}^n)$$

*such that*

$$\sum_{j=1}^{M} A_{ij}(x) F_j(x) = f_i(x), \quad \text{for all } x \in E, i = 1, \dots, N,$$

*if and only if $L_\nu f = 0$ on $E$ for all $\nu = 1, \dots, \nu_{\max}$.*

In Fefferman-Luli [9], it is treated how to pass from the compact case to the non-compact case, and thus establish Theorem 2.1.

## 3. Semialgebraic and Continuous Solution of Linear Equation with Semialgebraic Coefficients

This investigation deal with the open problem of obtaining by analytical techniques necessary and sufficient conditions for the existence of a $C^m$ and semialgebraic solution of a system of linear equation with semialgebraic coefficients. In case $m = 0$ of a system with polynomial coefficients, the problem was solved by Fefferman-Kollár [7] and Kollár [12] using algebraic techniques for systems with polynomial coefficients. In this study, by a new approach based on Fefferman's analytic techniques of Glaeser's refinements, we solve the problem for the case of a $C^0$ and semialgebraic solution on a general $n$-dimension space $\mathbb{R}^n$, extending Fefferman- Kollár's result to the case of a system with semialgebraic (not only polynomial as in [7]) coefficients.

More in detail by a new approach, based on Fefferman's techniques, for $m = 0$ we generalize and solve the problem of determining necessary and sufficient conditions for the existence of a continuous and semialgebraic solution of

$$\sum_{j=1}^{M} A_{ij} F_j = f_i \ (i = 1, \dots, N), \tag{17}$$

where $A_{ij}$ and $f_j$ are given *polynomials* and (17) admits a continuous solution. More precisely, we prove that if a semialgebraic bundle associated to a system with coefficients and right-hand side that are semialgebraic (but not necessarily continuous) on $\mathbb{R}^n$ has a continuous section then it has also a continuous and semialgebraic section. We show it without employing the algebraical blow-up theory but only by using the analytical Fefferman-Glaeser theory with the aim of determining an explicit method to construct a continuous and semialgebraic section.

Let us go through a deeper description of the problem we deal with. We consider a *semialgebraic compact metric space* $Q \subseteq \mathbb{R}^n$ and a *system of linear equations*

$$A\left(x\right)\phi\left(x\right)=\gamma\left(x\right),\quad x=\left(x_1,\ldots,x_n\right)\in Q \tag{18}$$

*where*

$$Q \ni x \longmapsto A(x) = (a_{ij}(x)) \in M_{r,s}(\mathbb{R})$$

*is* semialgebraic, *with* $M_{r,s}(\mathbb{R})$ *denoting the set of real* $r \times s$ *matrices and*

$$Q \ni x \longmapsto \gamma(x) \in \mathbb{R}^r, \gamma(x) = \begin{bmatrix} \gamma_1(x) \\ \vdots \\ \gamma_r(x) \end{bmatrix} \in \mathbb{R}^r$$

*being themselves* semialgebraic functions *on* $Q \subseteq \mathbb{R}^n$.

Our aim is to find a *necessary and sufficient condition* for the existence of a solution $Q \ni x \longmapsto \phi\left(x\right) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_s(x) \end{bmatrix} \in \mathbb{R}^s$ of system (18), with the $\phi_i : Q \to \mathbb{R}$ *continuous* and *semialgebraic.*

We notice that the semialgebraicity of $Q$ is a necessary condition for the existence of a semialgebraic solution of system (18) by the definition of semialgebraic function (i.e. a function with semialgebraic graph) and by the Tarski-Seidenberg theorem [3].

The plan of this section is the following. In Subsection 3.1 we fix some notations and give some definitions that will be used in Subsection 3.2.
In Subsection 3.2 we prove that if a semialgebraic bundle associated to a system of semialgebraic (but not necessarily continuous) function on a semialgebraic compact set $Q$ has a continuous section then it has also a semialgebraic and continuous one. The main idea is to prove the result by an induction argument on the dimension $d$ of $Q$. (We recall that the dimension of a semialgebraic set $E \subset \mathbb{R}^n$ is the maximum of the dimensions of all the embedded, not necessarily compact, submanifolds of $\mathbb{R}^n$ that are contained in $E$.) In fact, for the case $d = 1$ we use the fact that a semialgebraic function on a subset of $\mathbb{R}$ has finitely many isolated discontinuity points (the set on which a semialgebraic function is not continuous is a semialgebraic subset of its domain of strictly lower dimension and a semialgebraic set of dimension 0 is finite i.e. it is made by finitely many isolated points). Hence, we construct a local semialgebraic and continuous section of the bundle on a neighbourhood of each point of $Q$ and we glue the semialgebraic and continuous sections by a semialgebraic and continuous partition of the unity. Next, in the case $d \geq 2$, by induction hypothesis there is a continuous and semialgebraic section on an appropriate compact subset of $Q$ of dimension $\leq d-1$ (which will be defined in the proof of Theorem 3.1) and we extend it thanks to a semialgebraic version of Tietze-Uryshon Theorem. Finally, we need to compute the projection of the extension on the fibers of $\mathcal{H}^{\mathrm{Gl}}$ (i.e. the Glaeser-stable bundle associated to the system (18)) to obtain a continuous and semialgebraic section of $\mathcal{H}^{\mathrm{Gl}}$ i.e. a semialgebraic and continuous solution of (18).

[3]**Tarski-Seidenberg Theorem** *Let $A$ a semialgebraic subset of $\mathbb{R}^{n+1}$ and $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$, the projection on the first $n$ coordinates. Then $\pi(A)$ is a semialgebraic subset of $\mathbb{R}^n$.*

**Corollary** *If $A$ is a semialgebraic subset of $\mathbb{R}^{n+k}$, its image by the projection on the space of the first $n$ coordinates is a semialgebraic subset of $\mathbb{R}^n$.*

The result of Subsection 3.2 is obtained without the use of algebraic geometrical tools, but only by the analysis techniques such as the Glaeser refinement and the theory of bundle sections developed by Fefferman. This result gives an explicit method for the construction of a semialgebraic continuous solution of system (18) by finitely many induction steps.

### 3.1. **The setting.**

Let us start by setting some notations and definitions that will be used to pursue our goal. We shall endow every $\mathbb{R}^s$ used here with euclidean norm.

> **Notation:** Let $V \subseteq \mathbb{R}^s$ be an affine space in $\mathbb{R}^s$ and $w \in \mathbb{R}^s$. We denote the *projection of $w$ on $V$* (i.e. the point $v \in V$ that makes the euclidean norm of $v - w$ as small as possible) by $\Pi_V w$.

Let us consider a *singular affine bundle* (or *bundle* for short) (see [7]), meaning a family $\mathcal{H} = (H_x)_{x \in Q}$ of affine subspaces $H_x \subseteq \mathbb{R}^s$, parametrized by the points $x \in Q$. The affine subspaces

$$H_x = \{\lambda \in \mathbb{R}^s : A(x)\,\lambda = \gamma(x)\}, \quad x \in Q$$

are the *fibers* of the bundle $\mathcal{H}$. (Here, we allow the empty set $\emptyset$ and the whole space $\mathbb{R}^s$ as affine subspaces of $\mathbb{R}^s$.)

Now we call $\mathcal{H}^{(k)}$ the $k$-th *Glaeser refinement* of $\mathcal{H}$ i.e. $\mathcal{H}^{(0)} := \mathcal{H}$ and for all $k \geq 1$ the fibers of $\mathcal{H}^{(k)}$ are

$$H_x^{(k)} := \{\lambda \in H_x^{(k-1)};\ \mathrm{dist}(\lambda, H_y^{(k-1)}) \longrightarrow 0 \text{ as } y \longrightarrow x\ \ (y \in Q)\},$$

for all $x \in Q$ (see Chapter 2 of [7]). We notice that $\mathcal{H}^{(k)}$ is a subbundle of $\mathcal{H}^{(k-1)}$ for all $k \geq 1$. By Lemma 5 of [7] the procedure of refinement leads to a Glaeser-stable refinement of $\mathcal{H}$ i.e. there is a $r \in \mathbb{N}$ such that $\mathcal{H}^{(2r+1)} = \mathcal{H}^{(2r+2)} = \cdots$. We denote $\mathcal{H}^{(2r+1)}$ by $\mathcal{H}^{\mathrm{Gl}}$ and we will call it the *Glaeser-stable refinement* of $\mathcal{H}$ (its fibers will respectively be denoted by $H_x^{\mathrm{Gl}}$ for all $x \in Q$). Notice that the projection on the fibers of $\mathcal{H}^{\mathrm{Gl}}$ is not linear as the fibers are affine spaces and not vector spaces.

Given a continuous solution $f$ of system (19) we define

$$Q \ni y \longmapsto \omega(y) := \Pi_{\left(H_y^{\mathrm{Gl}}\right)^{\perp}} f(y),$$

and we notice that $\omega$ does not depend on the choice of the continuous solution $f$. More precisely, for all $y \in Q$ the value $\omega(y)$ can be computed by projecting 0 on $H_y^{\mathrm{Gl}}$. Moreover, we define

$$Q \ni y \longmapsto \tilde{\Pi}_1(y)v := \Pi_{\left(H_y^{\mathrm{Gl}}\right)^{\perp}} v,$$

for all $v \in \mathbb{R}^s$. We say that $\tilde{\Pi}_1$ is continuous if $y \longmapsto \tilde{\Pi}_1(y)e_j$ is continuous for all $j = 1, ..., s$ with $(e_1, ..., e_s)$ the canonical basis of $\mathbb{R}^s$.

### 3.2. Existence of a continuous semialgebraic solution.

In this subsection we prove that system (18) on a semialgebraic compact space $Q \subseteq \mathbb{R}^n$ has a semialgebraic and continuous solution if and only if it has a continuous one. We do it by induction on the dimension of $Q$ on which the problem is defined.

**Theorem 3.1.** *Consider a semialgebraic compact metric space $Q \subseteq \mathbb{R}^n$ and a system of linear equations*

$$A(x)\,\phi(x) = \gamma(x), \quad x \in Q, \tag{19}$$

*where the entries of*

$$A(x) = (a_{ij}(x_1, \dots, x_n)) \in M_{r,s}(\mathbb{R}) \quad \text{and} \quad \gamma(x) = (\gamma_i(x)) \in \mathbb{R}^r$$

*are themselves semialgebraic functions on $\mathbb{R}^n$.*
*Then system* (19) *has a continuous semialgebraic solution $\phi : Q \to \mathbb{R}^s$ if and only if $\mathcal{H}^{\mathrm{Gl}}$ has no empty fiber.*

*Proof.* We start by proving the forward implication which is trivial. In fact, if system (19) has a continuous solution then $\mathcal{H}^{\mathrm{Gl}}$ has no empty fiber (see [7]).

Now, we prove the reverse implication. To do it we proceed by induction on the dimension $d \in \{1, ..., n\}$ of $Q$. (We notice that if $d = 0$ then $Q$ is a finite set and, hence, any selection of $\mathcal{H}^{\mathrm{Gl}}$ is a semialgebraic and continuous section of $\mathcal{H}^{\mathrm{Gl}}$. A selection of $\mathcal{H}^{\mathrm{Gl}}$

exists since $\mathcal{H}^{\mathrm{Gl}}$ has no empty fiber.) Actually, before starting the proof by induction, we need to show that $\omega$ is semialgebraic. To do this we need to verify that the set

$$\mathcal{H}_Q^{\mathrm{Gl}} := \{(x,v) \in \mathbb{R}^n_x \times \mathbb{R}^s_v;\ x \in Q,\ v \in H_x^{\mathrm{Gl}}\} \text{ is semialgebraic.}$$

Hence, we prove by induction on $k \geq 0$ that

$$\mathcal{H}_Q^{(k)} := \{(x,v) \in \mathbb{R}^n_x \times \mathbb{R}^s_v;\ x \in Q,\ v \in H_x^{(k)}\}$$

is semialgebraic for all $k \geq 0$. In fact,

$$\mathcal{H}_Q^{(0)} = \{(x,v) \in Q \times \mathbb{R}^s;\ A(x)v = \gamma(x)\}$$

is semialgebraic and after supposing that $\mathcal{H}_Q^{(k-1)}$ is semialgebraic ($k \geq 1$), $\mathcal{H}_Q^{(k)}$ can be rewritten as

$$\begin{aligned}\{(x,v) \in \mathcal{H}_Q^{(k-1)};\ &\forall\varepsilon > 0,\ \exists\delta > 0,\\ &\forall(y,v') \in (B(x,\delta) \times \mathbb{R}^s) \cap \mathcal{H}_Q^{(k-1)} :\ \left\|v - v'\right\| < \varepsilon\},\end{aligned}$$

that is semialgebraic by elimination of quantifiers. Now, $\omega$ has graph given by

$$\{(x,v) \in \mathcal{H}_Q^{\mathrm{Gl}};\ \nexists(x',v') \in \mathcal{H}_Q^{\mathrm{Gl}},\ x' = x,\ \|v\| > \left\|v'\right\|\}, \tag{20}$$

with $\|\cdot\|$ the euclidean norm on $\mathbb{R}^s$ and, hence, it is semialgebraic by elimination of quantifiers.

In a similar way we have that if $(e_1, ..., e_s)$ is the canonical basis of $\mathbb{R}^s$ then $y \longmapsto \tilde{\Pi}_1(y)e_j$ is semialgebraic for all $j$ since its graph can be written as

$$\{(y,v) \in Q \times \mathbb{R}^s;\ \exists(x',v') \in \left(\mathcal{H}^{\mathrm{Gl}} - \omega\right)_Q,\ x' = y,\ \left\langle v',v\right\rangle = 0,\ v + v' = e_j\},$$

which is semialgebraic by elimination of quantifiers because

$$\left(\mathcal{H}^{\mathrm{Gl}} - \omega\right)_Q := \{(x,v) \in Q \times \mathbb{R}^s;\ \exists(x',v') \in \mathcal{H}_Q^{\mathrm{Gl}} - \omega,\ x' = x,\ v = v' - \omega(x)\}$$

is semialgebraic again by elimination of quantifiers.

Now, we are ready to begin the proof by induction.

We start with the case $d = 1$. For every given $x \in Q$ there exists $v_x \in H_x^{\mathrm{Gl}} \neq \emptyset$ and a ball $B(x, r_{v_x}) \subseteq \mathbb{R}^n$ such that

$$\begin{aligned} Q \cap B(x, r_{v_x}) \ni y \longmapsto \tilde{\gamma}_{v_x}(y) :=& \mathit{\Pi}_{H_y^{\mathrm{Gl}}} v_x \\ =& \omega(y) + v_x - \tilde{\mathit{\Pi}}_1(y) v_x \end{aligned}$$

is semialgebraic since $\omega$ and $\tilde{\mathit{\Pi}}_1$ are semialgebraic. We show that $\tilde{\gamma}_{v_x}$ is continuous for $r_{v_x}$ small enough. In fact, if we suppose by contradiction that there is no $r_{v_x}$ such that $\tilde{\gamma}_{v_x}$ is continuous then for all $n \in \mathbb{N}$ there is $y_n \in B(x, \frac{r_{v_x}}{n})$ such that $\tilde{\gamma}_{v_x}$ is discontinuous at $y_n$. Hence, there are two possibilities:

1. $\forall n \in \mathbb{N}$, $y_n \neq x$. A semialgebraic function is real analytic on the complementary of a semiagebraic set of dimension strictly less than the one of its domain. (In fact, the domain of a semialgebraic function is semialgebraic by the Tarski-Seidenberg theorem.) Thus, since $\gamma_{v_x}$ is semialgebraic on $Q$ the discontinuity points of $\gamma_{v_x}$ are finitely many. Hence, we come to a contradiction;
2. $\exists \overline{n} \in \mathbb{N}$ such that $y_{\overline{n}} = x$. Now, since $v_x \in H_x^{\mathrm{Gl}}$ on the one hand

$$\operatorname{dist}(v_x; H_y^{\mathrm{Gl}}) \underset{y \to x}{\longrightarrow} 0,$$

and, on the other,

$$\left\| \Pi_{H_y^{\mathrm{Gl}}} v_x - v_x \right\| = \operatorname{dist}(v_x, H_y^{\mathrm{Gl}}).$$

Therefore

$$\Pi_{H_y^{\mathrm{Gl}}} v_x \underset{y \to x}{\longrightarrow} v_x \underset{\substack{\uparrow \\ v_x \in H_x^{\mathrm{Gl}}}}{=} \Pi_{H_x^{\mathrm{Gl}}} v_x.$$

This is impossible since $\gamma_{v_x}$ would be continuous at $x$, contrary to the assumption. Thus $\tilde{\gamma}_{v_x}$ is continuous upon possibly reducing the ball radius $r_{v_x}$.

Now we glue these local solutions thanks to a semialgebraic continuous partition of the unity. In fact, we notice that the set of balls $\{B(x, \overline{r}_{v_x})\}_{x \in Q}$, where $v_x$ is chosen in $H_x^{\mathrm{Gl}}$, is an open cover of the compact space $Q$. Then there is $N$ such that $\{B(x_i, \overline{r}_{v_{x_i}})\}_{i=1,\dots,N}$ is an open cover of $Q$. Consider

$$\mu_{(x,r)}(y) := \begin{cases} \sqrt{r^2 - \|y - x\|^2} & \text{for } y \in B(x, r), \\ 0 & \text{for } y \notin B(x, r). \end{cases} \tag{21}$$

Notice that $\mu_{(x,r)}(y)$ is semialgebraic and continuous on $Q$, $\forall x \in Q$, $\forall r \in \mathbb{R}^+$ and that $\sum_{i=1}^{N} \mu_{(x_i,\overline{r}_{v_{x_i}})}(y) > 0$ for each $y \in Q$ as $\mu_{(x,r)}(y) \geq 0$ for every $y \in Q$ and $\mu_{(x,r)}(y) > 0$ for all $y \in B(x,r)$. Moreover, for all $y \in Q$ there is $B(x_i, \overline{r}_{v_{x_i}})$ as above such that $y \in B(x_i, \overline{r}_{v_{x_i}})$ since $\{B(x_i, \overline{r}_{v_{x_i}})\}_{i=1,\ldots,N}$ is an open covering of $Q$. Hence the function

$$Q \ni y \longmapsto \phi(y) := \frac{1}{\sum_{i=1}^{N} \mu_{(x_i,\overline{r}_{v_{x_i}})}(y)} \sum_{j=1}^{N} \mu_{(x_j,\overline{r}_{v_{x_j}})}(y) \Pi_{H_y^{\mathrm{Gl}}} v_{x_j}$$

is a semialgebraic and continuous solution of the system on $Q$. (We also notice that $\phi(y) \in H_y^{\mathrm{Gl}}$ for all $y \in Q$.)

Next, we suppose that $Q$ is a semialgebraic subset of dimension $\tilde{d} \leq n$ and that we can write a semialgebraic and continuous section of $\mathcal{H}^{\mathrm{Gl}}$ on any compact semialgebraic subset of $Q$ of dimension $d \leq \tilde{d} - 1$ of $Q$. We want to construct a semialgebraic and continunous section of $\mathcal{H}^{\mathrm{Gl}}$ on $Q$. We will call $U$ the subset of $Q$ where $\omega$ or $\tilde{\Pi}_1$ is not continuous. Since we proved that $\omega$ and $y \longmapsto \tilde{\Pi}_1(y)e_j$ are semialgebraic (for all $j$), $U$ is a semialgebraic set of dimension $\leq \tilde{d} - 1$. (A zero-dimensional semialgebraic subset of $\mathbb{R}^n$ is finite. A one-dimensional semialgebraic subset of $\mathbb{R}^n$ is a union of finitely many real-analytic arcs and finitely many points. See Chapter 2 of [2].)

Thus, by inductive hypothesis there is a semialgebraic and continuous section $S$ of $\mathcal{H}^{\mathrm{Gl}}$ on $\overline{U}$. In fact, $\overline{U}$ is a compact semialgebraic subset of $Q$ of dimension $\tilde{d} - 1$. Now, $S$ can be extended to a semialgebraic and continuous function on $Q$ by Proposition 2.6.9 at p. 45 of [2] which is a semialgebraic version of Tietze-Uryshon Theorem and we will call $S$ that extension again. Actually, $S$ is defined on $Q$, but it is a section of $\mathcal{H}^{\mathrm{Gl}}$ only on $\overline{U}$. Hence, we compute the projection of $S$ on the fibers of $\mathcal{H}^{\mathrm{Gl}}$ i.e.

$$\begin{aligned} Q \ni y \longmapsto \sigma(y) :=& \Pi_{H_y^{\mathrm{Gl}}} S(y) \\ =& \omega(y) + S(y) - \tilde{\Pi}_1(y)S(y). \end{aligned}$$

We notice that $\sigma$ is semialgebraic and also that $\sigma$ is continuous on $Q \setminus \overline{U}$ since $\omega$ and $y \longmapsto \tilde{\Pi}_1(y)S(y)$ is continuous on $Q \setminus \overline{U}$. Moreover, $\sigma$ is continuous on $\overline{U}$ since $S(x) \in H_x^{\mathrm{Gl}}$

for all $x \in \overline{U}$ and, hence, we can proceed as done to prove that $x$ is not a discontinuity point for $\tilde{\gamma}_{v_x}$ in the case $d = 1$. In fact, for all $x \in \overline{U}$ and all $y \in Q$

$$\begin{aligned}\left\| \Pi_{H_y^{\mathrm{Gl}}} S(y) - \underbrace{\Pi_{H_x^{\mathrm{Gl}}} S(x)}_{=S(x) \in H_x^{\mathrm{Gl}}} \right\| &\leq \left\| \Pi_{H_y^{\mathrm{Gl}}} S(y) - S(y) \right\| + \|S(y) - S(x)\| \\ &\leq \left\| \Pi_{H_y^{\mathrm{Gl}}} S(x) - S(y) \right\| + \|S(y) - S(x)\| \\ &\leq \left\| \Pi_{H_y^{\mathrm{Gl}}} S(x) - S(x) \right\| + 2\|S(y) - S(x)\|,\end{aligned}$$

where the second inequality follows from the minimal distance property of the projection (we notice that $\Pi_{H_y^{\mathrm{Gl}}} S(x) \in H_y^{\mathrm{Gl}}$). Now, $\|S(y) - S(x)\| \underset{y \to x}{\longrightarrow} 0$ by the continuity of $S$ and $\left\| \Pi_{H_y^{\mathrm{Gl}}} S(x) - S(x) \right\| = \mathrm{dist}(S(x), H_y^{\mathrm{Gl}}) \underset{y \to x}{\longrightarrow} 0$ since $S(x) \in H_x^{\mathrm{Gl}}$.

The proof is complete. □

**Remark.** Since the absence of empty fiber of $\mathcal{H}^{\mathrm{Gl}}$ is equivalent to the existence of a continuous section of $\mathcal{H}^{\mathrm{Gl}}$ (see [7]) and, hence, of $\mathcal{H}$, we have just proved that system (19) on a semialgebraic compact space $Q \subseteq \mathbb{R}^n$ has a semialgebraic and continuous solution if and only if it has a continuous one.

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