REGULARITY RESULTS FOR KOLMOGOROV EQUATIONS BASED ON A BLOW-UP ARGUMENT

RISULTATI DI REGOLARITÀ PER EQUAZIONI DI KOLMOGOROV BASATI SU UN ARGOMENTO DI BLOW-UP

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ABSTRACT. We present recent results regarding the regularity theory for degenerate second order differential operators of Kolmogorov-type. In particular, we focus on Schauder estimates for classical solutions to Kolmogorov equations in non-divergence form with Dini-continuous coefficients obtained in [30] in collaboration with S. Polidoro and B. Stroffolini. Furthermore, we discuss new pointwise regularity results and a Taylor- type expansion up to second order with estimate of the rest in L^p norm, following the recent paper [14] in collaboration with E. Ipocoana. The proofs of both results are based on a blow-up technique.

SUNTO. Vengono presentati alcuni risultati recenti riguardanti la teoria della regolarità per operatori differenziali degeneri del secondo ordine di tipo Kolmogorov. In particolare, concentreremo la nostra attenzione su stime di tipo Schauder per soluzioni classiche di equazioni di Kolmogorov in forma di non divergenza con coefficienti Dini continui ottenute in [30] in collaborazione con S. Polidoro e B. Stroffolini. Inoltre, discuteremo nuovi risultati di regolarità puntuale e uno sviluppo in serie di tipo Taylor con stima del resto in norma L^p, seguendo il recente articolo [14] ottenuto in collaborazione con E. Ipocoana. Le dimostrazioni di entrambi i risultati si basano su una tecnica di tipo blow-up.

2020 MSC. Primary 35K70, 35K65, 35B65; Secondary 35B44.

KEYWORDS. Degenerate Kolmogorov equations, Regularity theory, Classical solutions, Dini continuity, Taylor formula, Pointwise regularity, BMO pointwise estimate, VMO pointwise estimate.

Bruno Pini Mathematical Analysis Seminar, Vol. 14, no. 2 (2023) pp. 139–162

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ISSN 2240-2829.

1. Introduction

In this note, we present the recent results obtained in [14, 30] about the local regularity of solutions to the second order linear differential equation

$$\mathscr{L}u := \sum_{i,j=1}^{m_0} a_{ij}(x,t)\partial^2_{x_i x_j} u + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u - \partial_t u = f, \tag{1.1}$$

where $z = (x,t) = (x_1, \ldots, x_N, t) \in \mathbb{R}^{N+1}$ and $1 \le m_0 \le N$. Furthermore, matrices $A = (a_{ij}(x,t))_{i,j=1,\ldots,m_0}$ and $B = (b_{ij})_{i,j=1,\ldots,N}$ satisfy the following structural assumptions.

(H1) For every $(x,t) \in \mathbb{R}^{N+1}$, the matrix $A(x,t)$ is symmetric and satisfies

$$\lambda |\xi|^2 \le \sum_{i,j=1}^{m_0} a_{ij}(x,t)\xi_i \xi_j \le \Lambda |\xi|^2, \quad \forall \xi \in \mathbb{R}^{m_0}$$

for some positive constants λ, Λ. The matrix B has constant entries.

We observe that, despite the degeneracy of $\mathscr{L}$ when $m_0 < N$, its first order part might induce a strong regularizing property. Indeed, it is known that, under suitable structural assumptions on the matrix B (see (1.3) below), the operator obtained from $\mathscr{L}$ by freezing the coefficients of A at some point $(x_0, t_0) \in \mathbb{R}^{N+1}$, i.e.

$$\mathscr{L}_0 u := \sum_{i,j=1}^{m_0} a_{ij}(x_0,t_0)\partial^2_{x_i x_j} u + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u - \partial_t u, \tag{1.2}$$

is hypoelliptic, meaning that every distributional solution u to $\mathscr{L}_0 u = f$ defined in some open set $\Omega \subset \mathbb{R}^{N+1}$ belongs to $C^\infty(\Omega)$ and it is a classical solution whenever $f \in C^\infty(\Omega)$. Hence, in the sequel, we rely on the following assumption.

(H2) The *constant coefficients operator* $\mathscr{L}_0$ in (1.2) is hypoelliptic and homogeneous of degree 2 with respect to the family of dilations $(\delta_r)_{r>0}$ introduced in (1.8).

We remark that, if $\mathscr{L}_0$ is uniformly parabolic (i.e. $m_0 = N$ and $B \equiv \mathbb{O}$), then assumption **(H2)** is clearly satisfied, as in this case $\mathscr{L}_0$ is simply the heat operator. However, in this note we are mainly interested in the genuinely degenerate setting. Moreover, [16, Propositions 2.1 and 2.2] imply that assumption **(H2)** is equivalent to assume there exists a basis of $\mathbb{R}^N$ with respect to which B takes the form

$$B = \begin{pmatrix} \mathbb{O} & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ B_1 & \mathbb{O} & \dots & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & B_2 & \dots & \mathbb{O} & \mathbb{O} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbb{O} & \mathbb{O} & \dots & B_\kappa & \mathbb{O} \end{pmatrix}, \tag{1.3}$$

where every B_j is a $m_j \times m_{j-1}$ matrix of rank m_j, with $j = 1, 2, \dots, \kappa$,

$$m_0 \geq m_1 \geq \ldots \geq m_\kappa \geq 1 \quad \text{and} \quad \sum_{j=0}^{\kappa} m_j = N.$$

Thus, in the sequel we shall always assume that B has the canonical form (1.3). When B takes the form (1.3), the constant coefficients operator $\mathscr{L}_0$ in (1.2) belongs to the family of hypoelliptic operators considered by Hörmander in his famous work [12]. To justify this fact, let us set

$$X_i := \sum_{j=1}^{m_0} \bar{a}_{ij} \partial_{x_i}, \quad i = 1, \dots, m_0, \quad Y := \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} - \partial_t = \langle Bx, D \rangle - \partial_t,$$

where $A^{1/2} = (\bar{a}_{ij})_{i,j=1,\dots,m_0}$ is a symmetric and positive constant matrix such that $A(x_0, t_0) = A^{1/2} A^{1/2}$, while $\langle \cdot, \cdot \rangle$ and D denote the inner product and the gradient in $\mathbb{R}^N$, respectively. Then operator $\mathscr{L}$ can be written as

$$\mathscr{L} = \sum_{i=1}^{m_0} X_i^2 + Y$$

and its hypoellipticity can be read in terms of the Hörmander's condition (see [12])

$$\text{rank Lie}\,(X_1, \dots, X_{m_0}, Y)\,(x,t) = N + 1, \qquad \forall\,(x,t) \in \mathbb{R}^{N+1}.$$

Here and in the sequel, Yu will be understood as the *Lie derivative*

$$Yu(x,t) := \lim_{s \to 0} \frac{u(\exp(sB)x, t - s) - u(x,t)}{s}. \tag{1.4}$$

Moreover, as it is customary in the heat operator framework, we regard the time derivative, here generalized by the Lie derivative Y in (1.4), as a second order operator.

Degenerate operators like the one in (1.1) have applications in research areas as diverse as kinetic theory, probability theory and finance. The simplest equation of the kind (1.1) was introduced by Kolmogorov [15] in the following form

$$\mathscr{K} u := \sum_{j=1}^{m_0} \partial_{x_j}^2 u - \sum_{j=1}^{m_0} x_j \partial_{x_{m_0+j}} u - \partial_t u = \Delta_v u - \langle v, D_y \rangle u - \partial_t u = 0 \tag{1.5}$$

to describe the density u of particles having position $y \in \mathbb{R}^{m_0}$ and velocity $v \in \mathbb{R}^{m_0}$ at time t. We observe that operator $\mathscr{K}$ can be written in the form (1.1) with $\kappa = 1, m_1 = m_0$, and

$$A = \begin{pmatrix} \mathbb{I}_{m_0} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} \end{pmatrix}, \quad B = \begin{pmatrix} \mathbb{O} & \mathbb{O} \\ -\mathbb{I}_{m_0} & \mathbb{O} \end{pmatrix}$$

where $\mathbb{I}_{m_0}$ denotes the $m_0 \times m_0$ identity matrix. Equation (1.5) is usually referred to as kinetic Kolmogorov equation or frictionless Fokker-Planck equation in the kinetic literature. It is derived from Langevin dynamics, as it is the partial differential equation satisfied by the transition density of the stochastic process solving

$$\begin{cases} dP_t = \sqrt{2}\, dW_t, \\ dY_t = P_t\, dt, \end{cases}$$

where $(W_t)_{t \geq 0}$ denotes an m_0-dimensional Wiener process. Equations of the form (1.1) arise in mathematical finance as well. In particular, the following linear equation

$$S^2 \partial_{SS} V + \log(S) \partial_A V + \partial_t V = 0, \qquad (S, A, t) \in \mathbb{R}^+ \times \mathbb{R} \times]0, T[$$

appears in the Black & Scholes theory when considering the problem of the pricing of geometric average asian options, and takes the form (1.5) as we change the variable $(S, A, t) = (e^x, y, T - t)$. For the applications of operators in the form $\mathscr{L}$ to finance and to stochastic theory we refer the interested reader to the monograph [29] by Pascucci. We eventually refer to the survey articles [2, 1] for a more exhaustive description of the mathematical properties of Kolmogorov operators and of their applications, in the context of classical and weak solutions, respectively.

The aim of this paper is to extend a fundamental result of the classical regularity theory, namely Schauder estimates, meaning that we want to quantify the gain of regularity stemming from the equation. More precisely, we aim at establishing the regularity of the second order derivatives involved in equation (1.1) under very mild regularity assumptions on the right-hand side f. The precise statements of the estimates proved in [14, 30] will be given in Theorem 2.1 and Theorem 3.1, after introducing the needed objects coming into play. Moreover, as the proofs of both results rely on a blow-up argument, we will explain the core idea of the argument in the forthcoming sections.

1.1. **Lie Group invariance.** In this subsection, we focus on the non-Euclidean structure associated to hypoelliptic Kolmogorov operators of the form (1.2). Indeed, it is known that the natural geometry when studying operator $\mathscr{L}_0$ is determined by a suitable homogeneous Lie group structure on $\mathbb{R}^{N+1}$. More precisely, as first observed by Lanconelli and Polidoro in [16], operator $\mathscr{L}_0$ is invariant with respect to left translation in the group $\mathbb{K} = (\mathbb{R}^{N+1}, \circ)$, where the group law is defined by

$$(x,t) \circ (\xi, \tau) = (\xi + E(\tau)x, t + \tau), \qquad (x,t), (\xi,\tau) \in \mathbb{R}^{N+1}, \tag{1.6}$$

and

$$E(s) = \exp(-sB), \qquad s \in \mathbb{R}. \tag{1.7}$$

Then $\mathbb{K}$ is a non-commutative group with zero element $(0,0)$ and inverse

$$(x,t)^{-1} = (-E(-t)x, -t).$$

For a given $\zeta \in \mathbb{R}^{N+1}$ we denote by ℓ_ζ the left traslation on $\mathbb{K} = (\mathbb{R}^{N+1}, \circ)$ defined as follows

$$\ell_\zeta : \mathbb{R}^{N+1} \to \mathbb{R}^{N+1}, \quad \ell_\zeta(z) = \zeta \circ z.$$

Then operator $\mathscr{L}_0$ is left invariant with respect to the Lie product $\circ$, i.e.

$$\mathscr{L}_0 \circ \ell_\zeta = \ell_\zeta \circ \mathscr{L}_0 \qquad \textit{or, equivalently,} \qquad \mathscr{L}_0 \left(u(\zeta \circ z) \right) = \left(\mathscr{L}_0 u \right) (\zeta \circ z) \,,$$

for every u sufficiently smooth.

Another remarkable property of operator $\mathscr{L}_0$ is its dilation invariance. More precisely, the second assertion in assumption **(H2)** reads as follows

$$\mathscr{L}_0 (u \circ \delta_r) = r^2 \delta_r (\mathscr{L}_0 u), \quad \text{for every} \quad r > 0,$$

where δ_r denotes the family of dilations

$$\delta_r := \text{diag}(r\mathbb{I}_{m_0}, r^3 \mathbb{I}_{m_1}, \ldots, r^{2\kappa+1}\mathbb{I}_{m_\kappa}, r^2), \quad r > 0. \tag{1.8}$$

The *homogeneous dimension of* $\mathbb{R}^{N+1}$ *with respect to* $(\delta_r)_{r>0}$ is the integer $Q+2$, where Q is the so called *spatial homogeneous dimension of* $\mathbb{R}^{N+1}$, namely

$$Q := m + 3m_1 + \ldots + (2\kappa + 1)m_\kappa. \tag{1.9}$$

Owing to (1.8), we recall the notion of *homogeneous function* in a homogeneous group.

Definition 1.1. *We say that a function u defined on $\mathbb{R}^{N+1}$ is homogeneous of degree $\alpha \in \mathbb{R}$ if*

$$u(\delta_r(z)) = r^\alpha u(z) \qquad \text{for every } z \in \mathbb{R}^{N+1}.$$

We next introduce a homogeneous norm of degree 1 with respect to the dilations $(\delta_r)_{r>0}$ and a corresponding quasi-distance which is invariant with respect to the group operation in (1.6). We first rewrite the matrix δ_r with the equivalent notation

$$\delta_r := \text{diag}(r^{\alpha_1}, \ldots, r^{\alpha_N}, r^2), \tag{1.10}$$

where $\alpha_1, \ldots, \alpha_{m_0} = 1, \alpha_{m_0+1}, \ldots, \alpha_{m_0+m_1} = 3, \alpha_{N-m_\kappa}, \ldots, \alpha_N = 2\kappa + 1$.

Definition 1.2. *For every $(x,t) \in \mathbb{R}^{N+1}$ we set*

$$\|(x,t)\|_{\mathbb{K}} = |t|^{\frac{1}{2}} + |x|, \quad |x|_{\mathbb{K}} = \sum_{j=1}^{N} |x_j|^{\frac{1}{\alpha_j}} \tag{1.11}$$

where the exponents α_j, for $j = 1, \ldots, N$, were introduced in (1.10)

Owing to (1.11), we now definde the *quasi-distance* $d_{\mathbb{K}}$ by setting

$$d_{\mathbb{K}}((x,t),(\xi,\tau)) := \|(\xi,\tau)^{-1} \circ (x,t)\|_{\mathbb{K}}, \qquad (x,t),(\xi,\tau) \in \mathbb{R}^{N+1}. \tag{1.12}$$

1.2. **Outline of the paper.** This paper is structured as follows. In Section 2 we first introduce a new definition of Dini continuity naturally associated to the Lie Group structure that leaves operator (1.1) invariant. Furthermore, we present the Schauder estimates contained in [30] and the blow-up technique we rely on, together with an intrinsic Taylor formula for classical solutions to (1.1) under minimal regularity assumptions on u. Section 3 is devoted to the results contained in [14] and to the presentation of some possible further developments.

2. Dini continuity

In this section, we consider solutions to (1.1) with Dini continuous diffusion coefficients and Dini continuous right-hand side. In this setting, we derive Schauder estimates that extend the classical ones, where intrinsic Hölder continuous functions are considered. Moreover, we establish an instrinsic Taylor formula for solutions to $\mathscr{L}u = f$, which, besides being a key step in proving our Schauder estimates, is of independent interest, since it is derived under minimal regularity assumptions on u. In particular, we show that, in order to be approximated by its intrinsic Taylor polynomial of degree 2, u needs to satisfy the following requirements.

Definition 2.1. *Let Ω be an open subset of $\mathbb{R}^{N+1}$. We say that a function u belongs to $C^2_{\mathscr{L}}(\Omega)$ if u, its derivatives $\partial_{x_i}u, \partial_{x_i x_j}u$ $(i, j = 1, \dots, m_0)$ and the Lie derivative Yu defined in* (1.4) *are continuous functions in Ω. We also require, for $i = 1, \dots, m_0$, that*

$$\lim_{s\to 0} \frac{\partial_{x_i}u(\exp(sB)x, t-s) - \partial_{x_i}u(x,t)}{|s|^{1/2}} = 0, \tag{2.1}$$

uniformly for every $(x,t) \in K$, where K is a compact set $K \subset \Omega$.

Let f be a continuous function defined in Ω. We say that a function u is a classical solution to $\mathscr{L}u = f$ in Ω if u belongs to $C^2_{\mathscr{L}}(\Omega)$, and the equation $\mathscr{L}u = f$ is satisfied at every point of Ω.

In order to expose our main result concerning the regularity of the second order derivatives in equation (1.1), we first need to introduce some preliminary notation. As a first step, we introduce the sets where our local results hold true. To this end, we take advantage of the invariant structure of the constant coefficients operator $\mathscr{L}_0$ (see Subsection 1.1) in the study of the regularity of $\mathscr{L}$. Indeed, this is a standard procedure in the study of uniformly parabolic operators. In particular, owing to the quasi-distance introduced in (1.12), we define the *boxes*

$$\mathcal{Q}_r(x_0,t_0) := \{(x,t) \in \mathbb{R}^{N+1} \mid d_{\mathbb{K}}((x,t),(x_0,t_0)) < r\}. \tag{2.2}$$

We now a provide a new definition of *modulus of continuity* and *Dini continuity* which are suitable for operator $\mathscr{L}$. More precisely, we define the *modulus of continuity* of a function f defined on any set $H \subset \mathbb{R}^{N+1}$ as follows

$$\omega_f(r) := \sup_{\substack{(x,t),(\xi,\tau)\in H \\ d_{\mathbb{K}}((x,t),(\xi,\tau))<r}} |f(x,t) - f(\xi,\tau)|. \tag{2.3}$$

Definition 2.2. *A modulus of continuity ω is said Dini if it satisfies the following integral condition*

$$\int_0^1 \frac{\omega(r)}{r} dr < +\infty.$$

Accordingly, a function f is said to be Dini continuous in H if

$$\int_0^1 \frac{\omega_f(r)}{r} dr < +\infty.$$

Throughout this section, we assume that the diffusion coefficients a_{ij}'s are Dini continuous functions in the sense of Definition 2.2 and, to simplify the notation, we write

$$\omega_a(r) := \max_{i,j=1,\dots,m_0} \sup_{\substack{(x,t),(\xi,\tau)\in H \\ d_{\mathbb{K}}((x,t),(\xi,\tau))<r}} |a_{ij}(x,t) - a_{ij}(\xi,\tau)|. \tag{2.4}$$

We are now in a position to state our main result.

Theorem 2.1 (See Theorem 1.7 in [30])**.** *Let $\mathscr{L}$ be an operator in the form* (1.1) *satisfying hypotheses* **(H1)** *and* **(H2)***. Let $u \in C^2_{\mathscr{L}}(\mathcal{Q}_1(0,0))$ be a classical solution to $\mathscr{L}u = f$. Suppose that f and the coefficients a_{ij}, $i,j = 1,\dots,m_0$, are Dini continuous. Then for* *any points (x,t) and $(\xi,\tau) \in \mathcal{Q}_{\frac{1}{2}}(0,0)$ the following holds:*

$$
\begin{aligned}
|\partial^2 u(x,t) - \partial^2 u(\xi,\tau)| \le & c\Big(d \sup_{\mathcal{Q}_1(0,0)} |u| + d \sup_{\mathcal{Q}_1(0,0)} |f| + \int_0^d \frac{\omega_f(r)}{r} dr + d \int_d^1 \frac{\omega_f(r)}{r^2} dr \Big) \\
& + c\Big(\sum_{i,j=1}^{m_0} \sup_{\mathcal{Q}_1(0,0)} |\, \partial^2_{x_i x_j} u| \Big) \Big(\int_0^d \frac{\omega_a(r)}{r} dr + d \int_d^1 \frac{\omega_a(r)}{r^2} dr \Big).
\end{aligned}
$$

where $d = d_{\mathbb{K}}((x,t),(\xi,\tau))$ and ∂^2 stands either for $\partial^2_{x_i x_j}, i,j = 1,\dots,m_0$, or for Y.

Remark 2.1. *We observe that Theorem 2.1 was derived in* [30] *under the less restrictive assumption that operator $\mathscr{L}_0$ is only hypoelliptic and not dilation-invariant. However, in order to provide a coherent presentation, we here restrict ourselves to the dilation-invariant case and we refer the reader to* [30] *if interested in the more general case.*

As observed above, in order to prove the Schauder estimates presented in Theorem 2.1, we rely on an intrinsic Taylor formula that we derived for the first time in [30]. We recall that the *nth*-order intrinsic Taylor polynomial of a function u (differentiable up to order n) around the point z is defined as the unique polynomial function $P^n_z u$ of order n such that

$$
u(\zeta) - P^n_z u(\zeta) = o(d_{\mathbb{K}}(\zeta, z)^n) \quad \text{as} \quad \zeta \to z,
$$

where $d_{\mathbb{K}}$ denotes the quasi-distance defined in (1.12).

We are now in a position to state our result concerning the intrinsic second order Taylor polynomial.

Theorem 2.2 (See Theorem 1.3 in [30])**.** *Let $\mathscr{L}$ be an operator of the form* (1.1) *satisfying hypothesis* **(H1)** *and* **(H2)***. Let Ω be an open subset of $\mathbb{R}^{N+1}$ and let u be a function in $C^2_{\mathscr{L}}(\Omega)$. For every $z := (x,t) \in \Omega$ we define the second order Taylor polynomial of u around z as*

$$
\begin{aligned}
T^2_z u(\zeta) := & u(z) + \sum_{i=1}^{m_0} \partial_{x_i} u(z)(\xi_i - x_i) \\
& + \frac{1}{2} \sum_{i,j=1}^{m_0} \partial^2_{x_i x_j} u(z)(\xi_i - x_i)(\xi_j - x_j) - Yu(z)(\tau - t),
\end{aligned}
\tag{2.5}
$$

for any $\zeta = (\xi, \tau) \in \Omega$*. Indeed, we have*

$$u(\zeta) - T_z^2 u(\zeta) = o(d_{\mathbb{K}}(\zeta, z)^2) \quad as \quad \zeta \to z. \tag{2.6}$$

2.1. **Blow-up.** The approach we follow in [30] to prove Theorem 2.1 has the advantage of being quite elegant and relying on elementary properties of equation (1.1). For this reason, we here outline the proof of our main result and we refer the reader to [30] for detailed computation. We first remark that our proof of Theorem 2.1 is based on the method introduced by Safonov in [32] for the parabolic case. The core idea of Safonov's argument was adopted by Wang [33] for the study of the Poisson equation with Dini continuous right-hand side and by Imbert and Mouhot [13] for the study of kinetic Fokker–Planck equations with Hölder continuous coefficients. As we also work under the assumption of Dini continuity, we sketch the proof contained in [33] as a first step in the next paragraph, and thereafter continue with an explanation of the necessary modifications that we introduced in [30] for the study of our setting.

Specifically, Wang considered in [33] a solution u to the equation $\Delta u = f$ in some open set Ω. Without loss of generality, he assumed that the unit ball $B_1(0)$ is contained in Ω and he considered a sequence of Dirichlet problems defined as follows. We let $B_k = B_{\varrho^k}(0)$ be the Euclidean ball centered at the origin and of radius ϱ^k, with $\varrho = \frac{1}{2}$, and we let u_k be the solution to the Dirichlet problem

$$\Delta u_k = f(0), \quad \text{in} \quad B_k, \qquad u_k = u \quad \text{in} \quad \partial B_k.$$

As u_k is a solution to the equation with constant right-hand side, quantitative information on the derivatives of every u_k is obtained by means of elementary properties of the Laplace equation, namely the weak maximum principle, and standard a priori estimates of the derivatives, that are derived in [33] via mean value formulas. The bounds for the derivatives of u are obtained as the limit of the analogous bounds for u_k. The Taylor expansion in this step is crucial to conclude the proof. More precisely, following Safonov's argument, the idea is to show that the oscillation of the remainder of the second-order Taylor polynomial of the solution decays at rate ϱ^{2k} in a ball of radius ϱ^k.

In [30] we apply the method described above to the Kolmogorov operator $\mathscr{L}$ in (1.1), by adapting Wang's approach to the non-Euclidean structure defined in (1.6). In particular, the ball $B_{\varrho^k}(0)$ is replaced by the box $\mathcal{Q}_{\varrho^k}(0,0)$ defined in (2.2) through the dilation δ_{ϱ^k} in (1.8). Then our main objective is to estimate the quantity

$$|\partial^2 u(z) - \partial^2 u(0)| \leq |\partial^2 u_k(z) - \partial^2 u_k(0)| + |\partial^2 u_k(0) - \partial^2 u(0)| + |\partial^2 u(z) - \partial^2 u_k(z)|$$
$$=: I_1 + I_2 + I_3,$$

where z is a point sufficiently close to the origin and, as usual, ∂^2 stands either for $\partial^2_{x_i x_j}, i,j = 1,\dots,m_0$, or for Y. It is clear that terms I_2 and I_3 need to be estimated similarly, as we are evaluating the functions u and u_k at the same point. For this reason we focus on I_2 and, as a first step, we prove the following estimates, for $i,j = 1,\dots,m_0$,

$$\|u - u_k\|_\infty \leq 4\varrho^{2k}\|f - f(0)\|_\infty \leq 4\varrho^{2k}\omega_f(\varrho^k), \tag{2.7}$$

$$\|\partial^2_{x_i x_j}(u_k - u_{k+1})\|_{L^\infty(\mathcal{Q}_{k+2})} \leq C\varrho^{-2k-4} \sup_{\mathcal{Q}_{k+1}} |u_k - u_{k+1}|, \tag{2.8}$$

by means of a maximum principle (see [30, Lemma 5.1]) and a priori estimates for the derivatives of a solution to (1.1) with right-hand side equal to 0 (see [30, Propositions 3.1 and 3.2]). We remark that, in contrast to [33], we proved the a priori estimates contained in [30, Propositions 3.1 and 3.2] by taking advantage of representation formulas and properties of the fundamental solution. Hence, we obtain

$$\sum_{l=k}^{\infty} |\partial^2 u_l(0) - \partial^2 u_{l+1}(0)| \leq C \sum_{l=k}^{\infty} \omega_f(\varrho^l) \leq C \int_0^{\|z\|_{\mathbb{K}}} \frac{\omega_f(r)}{r} dr. \tag{2.9}$$

Keeping in mind (2.9), we next identify the sum of the series $\sum_{l=k}^{\infty} (\partial^2 u_l(0) - \partial^2 u_{l+1}(0))$ as

$$\sum_{l=k}^{\infty} \left(\partial^2 u_l(0) - \partial^2 u_{l+1}(0)\right) = \partial^2 u_k(0) - \partial^2 u(0). \tag{2.10}$$

To this end, for $i,j = 1,\dots,m_0$, we aim at proving

$$\lim_{k\to+\infty} \partial^2_{x_i x_j} u_k(0) = \partial^2_{x_i x_j} T_0^2 u(0), \tag{2.11}$$

where $T_0^2 u(\zeta)$ is the intrinsic second-order Taylor polynomial of u around the origin computed at $\zeta = (\xi, \tau) \in \mathcal{Q}_k$, as defined in (2.5). We now observe that, in virtue of the very definition of the Taylor polynomial in (2.5), the following holds true

$$\mathscr{L} T_0^2 u(\zeta) = \mathscr{L} u(0) = f(0) = \mathscr{L} u_k(\zeta)$$

and therefore $T_0^2 u - u_k$ is a solution to (1.1) with 0 right hand-side and we can apply again the a priori estimates contained in [30, Propositions 3.1 and 3.2]. Furthermore, from (2.6), it follows

$$\sup_{\zeta \in \mathcal{Q}_k} |u - T_0^2 u| = o(\varrho^{2k}). \tag{2.12}$$

Estimates (2.12) and (2.8) finally yield

$$|\partial^2_{x_i x_j}(u_k - T_0^2 u)(0)| \leq C \varrho^{-2k} \sup_{\mathcal{Q}_k} |u_k - T_0^2 u| + O(\varrho^k) \leq C \varrho^{-2k} o(\varrho^{2k}) + O(\varrho^k) = o(1),$$

where, as usual, the indexes i and j range from 1 to m_0. Thus, for any $i, j = 1, \ldots, m_0$ we have showed that (2.11) holds true. Repeating the same argument for the vector field Y, and using again Theorem 2.2, we obtain

$$I_2 \leq \sum_{l=k}^{\infty} |\partial^2 u_l(0) - \partial^2 u_{l+1}(0)| \leq C \int_0^{\|z\|_{\mathbb{K}}} \frac{\omega_f(r)}{r} dr,$$

for $k \geq 1$ such that $\varrho^{k+4} \leq \|z\|_{\mathbb{K}} \leq \varrho^{k+3}$. We now briefly explain how to take care of the term I_1. As in I_1 we are evaluting u and u_k in two different points, we take advantage of a mean value formula, that we derived ad hoc in [30, Proposition 3.5], to obtain

$$\begin{aligned} |\partial^2_{x_i x_j}(u_{k+1} - u_k)(z) - \partial^2_{x_i x_j}(u_{k+1} - u_k)(0)| &\leq \frac{C}{\varrho^k} \|z\|_{\mathbb{K}} \|\partial^2_{x_i x_j}(u_{k+1} - u_k)\|_{L^\infty(\mathcal{Q}_{k+1})} \\ &\leq C \|z\|_{\mathbb{K}} \varrho^{-k} \omega_f(\varrho^k), \end{aligned} \tag{2.13}$$

for $i, j = 1, \ldots, m_0$. We observe that, in the passage from the first to the second line in (2.13), we used once again (2.8). We repeat the same argument for the vector field Y and we infer

$$|Y(u_{k+1} - u_k)(z) - Y(u_{k+1} - u_k)(0)| \leq \frac{C}{\varrho^k} \|z\|_{\mathbb{K}} \|Y(u_{k+1} - u_k)\|_{L^\infty(\mathcal{Q}_{k+1})} \leq C \|z\|_{\mathbb{K}} \varrho^{-k} \omega_f(\varrho^k).$$

Hence, since $u_k(z) - u_k(0) = u_0(z) - u_0(0) + \sum_{j=0}^{k-1} ((u_{j+1} - u_j)(0) - (u_{j+1} - u_j)(z))$, we have

$$\begin{aligned} I_1 &\le |\partial^2 u_0(z) - \partial^2 u_0(0)| + \sum_{j=0}^{k-1} |\partial^2 (u_{j+1} - u_j)(z) - \partial^2 (u_{j+1} - u_j)(0)| \\ &\le C\|z\|_{\mathbb{K}} \big(\|u_0\|_{L^\infty(\mathcal{Q}_0)} + C \sum_{j=0}^{k-1} \varrho^{-j} \omega_f(\varrho^j) \big) \\ &\le C\|z\|_{\mathbb{K}} \big(\|u\|_{L^\infty(\mathcal{Q}_1(0))} + \|f\|_{L^\infty(\mathcal{Q}_1(0))} + C \int_{\|z\|_{\mathbb{K}}}^1 \frac{\omega_f(r)}{r^2} \big). \end{aligned}$$

These are the main steps of the blow-up technique, and for more specific details we refer the reader to the proof of [30, Theorem 1.6].

2.2. **Comparison with existing results.** We compare our main findings with the current literature on this subject. As in the parabolic case, the classical theory regarding Schauder estimates for degenerate Kolmogorov operators is developed for spaces of Hölder continuous functions. Since we rely on the non-Euclidean structure defined in (1.6), we need to consider functions which are Hölder continuous with respect to the quasi-distance in (1.12), i.e. functions which are Hölder continuous *intrinsically*. More precisely, we say that a function f defined on $H \subset \mathbb{R}^{N+1}$ is Hölder continuous with respect to the distance (1.12) if

$$|f(x,t) - f(\xi,\tau)| \le M\, d_{\mathbb{K}}((x,t),(\xi,\tau))^\alpha, \qquad \text{for every} \quad (x,t),(\xi,\tau) \in H, \tag{2.14}$$

for some constants $M > 0$ and $\alpha \in (0,1]$. In this case we write $f \in C_L^{0,\alpha}(H)$ and we let

$$\|f\|_{C_L^{0,\alpha}(H)} = \sup_H |f| + \inf \big\{ M \ge 0 \mid (2.14) \text{ holds} \big\}.$$

When $\alpha < 1$ we write $C_L^\alpha(H)$ instead of $C_L^{0,\alpha}(H)$. Finally, in the same spirit of (2.4), we set

$$\|a\|_{C_L^\alpha(H)} = \max_{i,j=1,\dots,m_0} \|a_{ij}\|_{C_L^\alpha(H)}.$$

Then, as a direct consequence of Theorem 2.1, we have the following corollary.

Corollary 2.1. *Let $u \in C^2_{\mathscr{L}}(\mathcal{Q}_1(0,0))$ be a classical solution to $\mathscr{L}u = f$. Suppose that f and the coefficients a_{ij}'s, $i,j = 1,\dots,m_0$, belong to $C^{0,\alpha}_L(\mathcal{Q}_1(0,0))$. Then for any points (x,t) and $(\xi,\tau) \in \mathcal{Q}_{\frac{1}{2}}(0,0)$ the following holds:*

$$\begin{aligned}
|\partial^2 u(x,t) - \partial^2 u(\xi,\tau)| \leq & c\, d^\alpha \Bigg(\sup_{\mathcal{Q}_1(0,0)} |u| + \frac{\|f\|_{C^\alpha_L(\mathcal{Q}_1(0,0))}}{\alpha(1-\alpha)} \\
& + \sum_{i,j=1}^{m_0} \sup_{\mathcal{Q}_1(0,0)} |\, \partial^2_{x_i x_j} u| \frac{\|a\|_{C^\alpha_L(\mathcal{Q}_1(0,0))}}{\alpha(1-\alpha)} \Bigg), \qquad \text{if} \quad \alpha < 1, \\
|\partial^2 u(x,t) - \partial^2 u(\xi,\tau)| \leq & c\, d \Bigg(\sup_{\mathcal{Q}_1(0,0)} |u| + \|f\|_{C^{0,1}_L(\mathcal{Q}_1(0,0))} |\log\ d| \\
& + \Bigg(\sum_{i,j=1}^{m_0} \sup_{\mathcal{Q}_1(0,0)} |\, \partial^2_{x_i x_j} u| \Bigg) \|a\|_{C^{0,1}_L(\mathcal{Q}_1(0,0))} |\log\ d| \Bigg), \quad \text{if} \quad \alpha = 1.
\end{aligned}$$

We remark that, for $\alpha < 1$, Corollary 2.1 restores the Schauder estimates previously proved by Manfredini in [22, Theorem 1.4] for the dilation-invariant case, and then by Di Francesco and Polidoro in [9, Theorem 1.3] for the not dilation-invariant case. We also recall that Schauder estimates in the framework of semigroups have been proved by Lunardi [21], Lorenzi [19], Priola [31]. Theorem 2.1 improves the ones contained in the aforementioned papers in two directions. First of all, we weaken the regularity assumption on f and on the coefficients a_{ij}'s. Second, we are able to establish Schauder estimates for $\alpha = 1$, extending the results of the aforementioned articles, where $\alpha < 1$.

More recently, partial Schauder estimates for the second order derivatives of u, together with local Hölder continuity in the joint variables, were proved by Biagi and Bramanti in [4]. We also quote the recent paper [20] by Lucertini, Pagliarani and Pascucci, where the authors established Schauder estimates for Kolmogorov equations with coefficients that are Hölder continuous in space, and only measurable in time. As far as Dini continuity assumptions are concerned, we recall that partial continuity estimates on $\partial^2_{x_i x_j} u, i,j = 1,\dots,m_0$, and Yu were proved in [5] by Biagi, Bramanti and Stroffolini under the assumption that the coefficients a_{ij}'s and f are bounded and Dini continuous in the spatial variables, but only measurable and bounded in time. Partial Schauder estimates for degenerate Kolmogorov-Fokker-Planck operators with coefficients lying in suitable anisotropic Hölder spaces were studied also in the recent paper [7] by Chaudru de Raynal, Honorè and Menozzi.

With regards to kinetic Schauder estimates, we recall that in the case of one commutator and Hölder continuous coefficients and right-hand side, Imbert and Mouhot proved in [13] Schauder estimates for linear kinetic Fokker– Planck equations, as well as well for a toy nonlinear kinetic model. Schauder estimates for kinetic equations (and in particular for linear kinetic Fokker-Planck equations in trace-form) were also obtained by Henderson and Snelson in [11], where they are crucial in deriving a C^∞-smoothing estimate for the inhomogeneous Landau equation. Finally, we quote the recent paper [18] by Loher, who established quantitative Schauder estimates for a general class of local hypoelliptic operators and non-local kinetic equations, in either non-divergence or divergence form.

Concerning the Taylor expansion, we recall the results due to Bonfiglioli [6] and the ones proved by Pagliarani, Pascucci and Pignotti [26] for dilation-invariant Kolmogorov operators and subsequently by Pagliarani and Pignotti [27] for the corresponding not dilation-invariant case. We emphasize that the authors of the above articles assume that the second order derivatives of the function u are Hölder continuous, while we only require that u belongs to the space $C^2_{\mathscr{L}}(\Omega)$ introduced in Definition 2.1. As the regularity of the second order derivatives of u is the very subject of this chapter, we do not assume extra conditions on them and we prove in Theorem 2.2 the Taylor approximation under the minimal requirement that $u \in C^2_{\mathscr{L}}(\Omega)$. We eventually quote the very recent preprint [23] by Manfredini, Pagliarani and Polidoro, where the authors proved an intrinsic Taylor formula for non local kinetic Kolmogorov operators.

3. Pointwise estimates

In this section, we assume the coefficients a_{ij}'s to be constant but we relax the regularity of the right-hand side, allowing it to be in L^p. Specifically, we study the pointwise regularity of solutions u belonging to the Sobolev space $S^p(\Omega)$ (see Definition 3.1) to the following Cauchy problem

$$\begin{cases} \mathscr{L}_0 u = f \quad \text{in} \quad \mathcal{Q}_1^- \\ f \in L^p(\mathcal{Q}_1^-) \quad \text{and} \quad f(0) = 0, \end{cases} \tag{3.1}$$

where $\mathscr{L}_0$ is the operator defined in (1.2), $1 < p < \infty$ and $\mathcal{Q}_r^- = B_r \times (-r^2, 0)$ is the past cylinder defined through the open ball $B_r = \{x \in \mathbb{R}^N : |x|_{\mathbb{K}} \leq r\}$, with $|\cdot|_{\mathbb{K}}$ being the semi-norm introduced in (1.11). Moreover, we suppose that the origin is a Lebesgue point of f, so that we are able to define $f(0)$ if necessary. In this setting, we show that if the modulus of L^p-mean oscillation of f at the origin is Dini in the sense of Definition 2.2, then the origin is a Lebesgue point of continuity in L^p average for the second order derivatives $\partial^2_{x_i x_j} u$, $i, j = 1, \ldots, m_0$, and the Lie derivative Yu (see Theorem 3.1 below).

In order to introduce the main results of [14], we define a class of polynomials that are homogeneous of degree 2 with respect to the dilations in (1.8). According to Definition 1.1, it is clear that the polynomials which are homogeneous of degree two with respect to dilation (1.8) are those of degree two in the first m_0 spatial variables and one in time. For this reason, it is natural to define the following class of polynomials, which will be greatly used in the sequel. Namely,

$$\begin{aligned} \widetilde{\mathcal{P}} = \{P : \; & \text{polynomials of degree less or equal to two in } \; x_1, \ldots, x_{m_0} \\ & \text{and less or equal to one in } t\}\,. \end{aligned} \tag{3.2}$$

$$\mathcal{P} : \; = \Big\{ P \in \widetilde{\mathcal{P}} : \mathscr{L}_0 P = 0 \Big\}. \tag{3.3}$$

$$\mathcal{P}_c : \; = \Big\{ P \in \widetilde{\mathcal{P}} : \mathscr{L}_0 P = c \Big\}. \tag{3.4}$$

In particular, we take P_* such that $\mathscr{L}_0 P_* = 1$ and set $\mathcal{P}_c = cP_* + \mathcal{P}$.

We now give an appropriate definition of modulus of continuity. Indeed, the previous results in literature, including the ones contained in the former section, were derived assuming a modulus of continuity defined on some open set $\mathcal{Q}^- \subset \mathbb{R}^{N+1}$ (see (2.3)).

On the other hand, we here introduce a *pointwise modulus of mean oscillation.* To be more precise, following [14], for $p \in (1, +\infty)$, we define the following *modulus of L^p-mean oscillation* for the function f *at the origin* as

$$\widetilde{\omega}(f;r) := \inf_{c \in \mathbb{R}} \Big(\frac{1}{|\mathcal{Q}_r^-|} \int_{\mathcal{Q}_r^-} |f(x,t) - c|^p \Big)^{\frac{1}{p}}. \tag{3.5}$$

We now set

$$\widetilde{N}(u;r) := \inf_{P \in \widetilde{\mathcal{P}}} \Big(\frac{1}{r^{Q+2+2p}} \int_{\mathcal{Q}_r^-} |u - P|^p \Big)^{\frac{1}{p}}, \tag{3.6}$$

where Q is the homogeneous dimension defined in (1.9) and $\widetilde{\mathcal{P}}$ is the class of polynomials introduced in (3.2). We observe that the exponent $Q+2+2p$ in (3.6) is the one obtained when comparing the L^p-norm of a polynomial $P \in \widetilde{\mathcal{P}}$ on a cylider of radius r and on the unit cylider (see [14, Lemma 3.1]).
Owing to (3.5) and to [17], we let c_r be the unique constant such that

$$\widetilde{\omega}(f;r) = \inf_{c \in \mathbb{R}} \Big(\frac{1}{|\mathcal{Q}_r^-|} \int_{\mathcal{Q}_r^-} |f(x,t) - c|^p \Big)^{\frac{1}{p}} = \Big(\frac{1}{|\mathcal{Q}_r^-|} \int_{\mathcal{Q}_r^-} |f(x,t) - c_r|^p \Big)^{\frac{1}{p}}.$$

If u is a solution of (3.1), we let

$$\hat{N}(u,f;r) = \inf_{P \in \mathcal{P}_{c_r}} \Big(\frac{1}{r^{Q+2+2p}} \int_{\mathcal{Q}_r^-} |u - P|^p \Big)^{\frac{1}{p}}.$$

Moreover, for $0 < a < b$, we define

$$\hat{N}(u,f;a,b) = \sup_{a \le \varrho \le b} \hat{N}(u,f;\varrho)$$

$$\widetilde{\omega}(f;a,b) = \sup_{a \le \varrho \le b} \widetilde{\omega}(f;\varrho)$$

In the sequel, we will also make use of the following notation. For a given $\lambda \in (0,1)$, we set

$$\underline{N}(r) = \hat{N}(u,f;\lambda r, r),$$

$$\underline{\omega}(r) = \widetilde{\omega}(f;\lambda^2 r, r).$$

Before stating the main result of this section, Theorem 3.1, we eventually recall the following definition.

Definition 3.1. *For Ω open set in $\mathbb{R}^{N+1}$, $p \in (1, +\infty)$, we define the Sobolev space*

$$S^p(\Omega) = \{u \in L^p(\Omega) : \partial_{x_i} u, \partial^2_{x_i x_j} u, Yu \in L^p(\Omega), \quad i, j = 1, \dots, m_0\}$$

and we set

$$\|u\|^p_{S^p(\Omega)} = \|u\|^p_{L^p(\Omega)} + \sum_{i=1}^{m_0} \|\partial_{x_i} u\|^p_{L^p(\Omega)} + \sum_{i,j=1}^{m_0} \|\partial^2_{x_i x_j} u\|^p_{L^p(\Omega)} + \|Yu\|^p_{L^p(\Omega)}.$$

We are now in a position to state our main result concerning the pointwise regularity of a solution to (3.1).

Theorem 3.1 (See Theorem 1.3 in [14])**.** *Let $p \in (1, \infty)$. Then there exist constants $\beta, r_* \in (0, 1]$, $\lambda \in (0, 1)$ and $C > 0$, such that the following holds. If $u \in L^p(\mathcal{Q}_1^-)$ satisfies* (3.1) *with the associated $\widetilde{\omega}$ defined in* (3.5)*, then we have*

i) *Pointwise BMO estimate*

$$\sup_{r \in (0,1]} \widetilde{N}(u; r) \le C \left\{ \left(\int_{\mathcal{Q}_1^-} |u|^p \right)^{\frac{1}{p}} + \left(\int_{\mathcal{Q}_1^-} |f|^p \right)^{\frac{1}{p}} + \sup_{r \in (0,1]} \widetilde{\omega}(f; r) \right\}. \tag{3.7}$$

ii) *Pointwise VMO estimate*

$$\left(\widetilde{\omega}(f; r) \to 0 \text{ as } r \to 0^+\right) \quad \Rightarrow \quad \left(\widetilde{N}(u; r) \to 0 \text{ as } r \to 0^+\right). \tag{3.8}$$

iii) *Dini continuity of $\widetilde{N}(u; \cdot)$*
If $\widetilde{\omega}(f; \cdot)$ is Dini in the sense of Definition 2.2, then $\widetilde{N}(u; \cdot)$ is Dini. In particular, for every $\varrho \in (0, \frac{\lambda}{4})$, the following holds

$$\int_0^{4\varrho} \frac{\widetilde{N}(u; r)}{r} dr \le$$
$$C \left\{ \left(\frac{4\varrho}{\lambda} \right)^{\beta} \left(\widetilde{N}(u; 1) + \widetilde{\omega}(f; 1) \right) + \int_0^{4\varrho} \frac{\widetilde{\omega}(f; r)}{r} dr + \varrho^{\beta} \int_{4\varrho}^1 \frac{\widetilde{\omega}(f; r)}{r^{1+\beta}} dr \right\}.$$

where C is a constant that does not depend on f, u and ϱ.

iv) *Pointwise control on the solution*

Let $\widetilde{\omega}(f;\cdot)$ be Dini in the sense of Definition 2.2. Then there exists a unique polynomial $P_0 \in \mathcal{P}$, namely a solution to equation $\mathscr{L}_0 P_0 = 0$, with

$$P_0(x,t) = a + \langle b, x\rangle + \frac{1}{2}\langle cx, x\rangle + d\ t,$$

where b is a vector in $\mathbb{R}^N$ such that $b_j = 0$ when $j > m$ and c is a $N \times N$ matrix such that $c_{ij} = 0$ when $i > m \vee j > m$, such that for every $r \in (0, \frac{r_*}{4}]$ there holds

$$\left(\frac{1}{|\mathcal{Q}_r^-|}\int_{\mathcal{Q}_r^-}\left|\frac{u(x,t)-P_0(x,t)}{r^2}\right|^p\right)^{\frac{1}{p}} \leq C\left\{\widetilde{M_0}\left(\frac{4r}{\lambda}\right)^{\beta} + \int_0^{4r}\frac{\widetilde{\omega}(f;s)}{s}ds + r^{\beta}\int_{4r}^{1}\frac{\widetilde{\omega}(f;s)}{s^{1+\beta}}ds\right\}, \tag{3.9}$$

with

$$\widetilde{M_0} = \int_0^1 \frac{\widetilde{\omega}(f;s)}{s}ds + \left(\int_{\mathcal{Q}_1^-}|u|^p\right)^{\frac{1}{p}} + \left(\int_{\mathcal{Q}_1^-}|f|^p\right)^{\frac{1}{p}}.$$

Moreover, we have

$$|a| + |b| + |c| + |d| \leq C\widetilde{M_0}.$$

3.1. **Idea of the proof and comparison with previous results.** The proof of Theorem 3.1 is based on decay estimates, which we achieve by contradiction, blow-up and compactness results. Local a priori estimates for functions in the Sobolev space S^p and a Caccioppoli-type estimate which we obtained ad hoc for our problem (see [14, Lemma 2.3]) are also fundamental tools in proving Theorem 3.1. However, the proof of Theorem 3.1 is very technical and therefore we here omit the details. We refer the reader to [14, Section 3] for complete computation. We just want to remark that, like in the previous section (and thus in [30]), the proof of Theorem 3.1 is based on a blow-up techique, even though the estimates we establish here are completely pointwise and we are working in a weaker setting. More precisely, the core idea in [14] is to study the behaviour as $\epsilon \to 0$ of the rescaled functions

$$v^{\epsilon}(x,t) = \frac{u(\delta_{\epsilon}(x,t))}{\epsilon^2}$$

and

$$w^{\epsilon}(x,t) = \frac{u(\delta_\epsilon(x,t)) - P(\delta_\epsilon(x,t))}{\varepsilon^2}$$

where $P \in \widetilde{\mathcal{P}}$ (see (3.2)). In particular, we focus on how operator $\mathscr{L}_0$ acts on the rescaled functions v^ϵ and w^ϵ and we study its L^p-norm as we let $\epsilon \to 0$. For a detailed description of the blow-up technique exploited in [14], we refer the reader in particular to [14, Proposition 3.3, Lemma 3.7 and Theorem 1.3].

We finally emphasize that, although we consider the regularity problem for weak solutions to Kolmogorov operators in the framework of Sobolev spaces, our procedure is basically pointwise. Indeed, we consider some L^p norm of the function $u - P_0$ on a past cylinder of radius r and we obtain our result by letting r going to zero. Thus, this approach follows the lines of regularity theory for classical solutions rather than the ones for weak solutions, which does not seem to be usual when dealing with Kolmogorov-type operators.

A straightforward consequence of Theorem 3.1 *iv)* (inequality (3.9)) is the following corollary.

Corollary 3.1. *If the modulus of L^p-mean oscillation of f at the origin is Dini in the sense of Definition 2.2, then the origin is a Lebesgue point of continuity in L^p average for the second order derivatives $\partial^2_{x_i x_j} u$, $i,j = 1,\dots,m_0$, and the Lie derivative Yu.*

As with Theorem 2.1, it follows straightforwardly from Theorem 3.1 that the second order derivatives $\partial^2_{x_i x_j} u$, $i,j = 1,\dots,m_0$, and the Lie derivative Yu are Hölder continuous in some open set $\Omega \subset \mathbb{R}^{N+1}$, when f is Hölder continuous with respect to the distance introduced in (1.12). Moreover, let us remark that Theorem 3.1 provides us with a Taylor-type expansion up to second order with an estimate of the rest in L^p norm and therefore can be seen as a generelization of Theorem 2.2 in this weaker setting.

The results contained in Theorem 3.1 may be seen as a generalization of [24] and [17], where this kind of results are obtained respectively for elliptic and parabolic equations. However, up to our knowledge, the case of Kolmogorov-type operators was investigated for the first time in [14].

The main difficulty with respect to the previous literature lies in the fact that the regularity properties of the Kolmogorov equations on $\mathbb{R}^{N+1}$ depend strongly on the geometric Lie group structure introduced in (1.6). In particular, this reflects on the family of dilations we consider in the blow-up argument. Furthermore, according to (1.1), we here take into account also the case where $m_0 < N$ and therefore $\mathscr{L}_0$ is strongly degenerate. We emphasize that when $m_0 = N$ and $B \equiv \mathbb{O}$, our result recovers the one contained in [17].

3.2. **Further developments: the obstacle problem.** The method we follow in [14] has the advantage of being quite flexible, as shown in [24, 17], where it was applied to study new regularity results for the obtacle problem for the Laplace equation and the heat equation. Thus, it would be of interest to study the obstacle problem associated to (1.1), namely

$$\begin{cases} \mathscr{L}u = f(x) \cdot \mathbb{1}_{\{u>0\}} \quad \text{in} \quad \mathcal{Q}_1^- \\ u \geq 0 \quad \text{in} \quad \mathcal{Q}_1^- \\ u, f \in L^p(\mathcal{Q}_1^-) \quad \text{and} \quad f(0) = 1 \\ 0 \in \partial\{u > 0\}, \end{cases} \tag{3.10}$$

where $\mathbb{1}_{\{u>0\}}$ is the characteristic function of the set $\{u > 0\}$. The obstacle problem in (3.10) is not only fascinating for theoretical purposes but also for multiple applications. For example, this comes as an interest in mathematical finance to determine the arbitrage free price of options of American-type (see [28]). In recent years, many attempts have been made to study the existence and regularity of solutions to the obstacle problem in the framework of PDE (see [8, 10, 25] and the references therein). However, in the promising aforementioned results, they could only deal with classical solutions and continuous obstacles. For this reason, the results established in [14] aim at constituting an important step towards developing the weak regularity theory for solutions to the obstacle problem associated to Kolmogorov-type equations. We eventually recall that a very recent development is contained in the paper [3], which deals with the existence of the solution of an obstacle problem possibly equivalent to (3.10) in a functional setting.

Acknowledgements

The author would like to thank Annamaria Montanari and Sergio Polidoro for the opportunity of giving a talk in the ”Seminari di Analisi Matematica Bruno Pini”.

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