SOME ADVANCES IN ANALYTIC HYPOELLIPTICITY

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Abstract. We present a brief survey on the theory of the real analytic regularity for the solutions to sums of squares of vector fields satisfying the Hörmander condition.

Sunto. Presentiamo una breva rassegna sulla regolarità reale analitica delle soluzioni di operatori somme di quadrati di campi vettoriali che soddisfano la condizione di Hörmander.

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1. Introduction: the C^∞ hypoellipticity

The purpose of this paper is to present a brief survey to the theory of the real analytic regularity for the solutions to sums of squares type equations.

The problem of the C^∞ hypoellipticity of sums of squares has been settled by the famous paper of L. Hörmander, [35], whereas the problem of the analytic hypoellipticity is still open and seems much more involved than the latter.

The starting point for any further study is based on the results in the C^∞ category.

Consider the following second order degenerate elliptic equation

$$Q = \sum_{i,j=1}^{n} a_{i,j}(x)\partial_i\partial_j u(x) + \sum_{j=1}^{n} b_j(x)u(x) + c(x)u(x) = f(x).$$

Let us start by assuming that the coefficients of the above equation are real and smooth, i.e. C^∞ functions defined in an open subset $\Omega \subset \mathbb{R}^n$. It is well-known by Hörmander

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[35], Corollary 2.2 that if Q is $C^\infty-$hypoelliptic then the quadratic form $\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j$ corresponding to the principal symbol must be semidefinite (non negative or non positive) at any point $x \in \Omega$. In general, the quadratic form can change type moving x in Ω. To show this we can mention the Kannai example in $\mathbb{R}^2$ ([37])

$$x_1\partial_{x_2}^2 + \partial_{x_1}$$

which turns out to be $C^\infty-$hypoelliptic although its principal symbol changes type across the hypersurface $\{x_1 = 0\}$. A more detailed insight to this kind of questions can be found in Beals-Fefferman [28].

In what follows we assume that the matrix

$$A(x) = [a_{i,j}(x)]_{\substack{i=1,\dots,n\\ j=1,\dots,n}}$$

has constant rank near a point where its determinant vanishes; as a trivial consequence, if Q is $C^\infty-$hypoelliptic, its principal symbol cannot change type and, without loss of generality, we can suppose that it is non-negative

$$\sum_{i,j=1}^n a_{i,j}(x)\xi_i\xi_j \geq 0.$$

Furthermore, at least locally, we may find a finite number of vector fields

$$X_j(x, D_x) = \sum_{k=1}^n \alpha_{j,k}(x)D_k, \qquad j = 0, 1, \dots, r, \tag{1.1}$$

such that the above operator is written as

$$\sum_{j=1}^r X_j(x, D)^2 + X_0(x, D) + \alpha(x),$$

(see also the fundamental paper [35].) Here and in what follows we use the notation $D_j = i^{-1}\partial_{x_j}$.

In what follows we focus on operators of the form

$$P(x, D) = \sum_{j=1}^r X_j(x, D)^2, \tag{1.2}$$

where X_j denotes a vector field with smooth (or real analytic) coefficients, $a_{j,k}(x)$, with $a_{j,k} \in C^\infty(\Omega)$ or $a_{j,k} \in C^\omega(\Omega)$, the latter denoting the class of all real analytic functions on Ω.

A different approach consists in studying the hypoellipticity of a general second order differential operator Q replacing the sum of squares structure by some geometrical conditions on its characteristic manifold. In this respect, the lower order terms of Q play a fundamental role (see, for instance, [12], [13], [36], [34], [39], [43]).

In the paper [35] Hörmander proved his famous result on the C^∞ hypoellipticity for operators of the form (1.2)

Theorem 1.1 ([35]). *Let P be given by* (1.2)*, where the vector fields have C^∞ coefficients in the open set $\Omega \subset \mathbb{R}^n$. Assume that among the operators X_{j_1}, $[X_{j_1}, X_{j_2}]$, $\dots$, $[X_{j_1}, [X_{j_2}, [X_{j_3}, \dots X_{j_k}]]]$, $\dots$, where $j_\ell = 1, 2, \dots, r$, there exist n which are linearly independent at any given point in Ω. Then P is C^∞ hypoelliptic.*

The condition on the vector fields appearing in Theorem 1.1 has been stated literally as Hörmander stated it, but it has a deep geometric meaning. In fact by $[X, Y]$ we denote the commutator of the vector fields: $[X, Y]u = XYu - YXu$. We easily see that $[X.Y]$ is a vector field and that

$$[X, Y] = \sum_{j,k=1}^{n} \left(a_j(x)\partial_j b_k(x) - b_j(x)\partial_j a_k(x)\right)\partial_k,$$

where a_j, b_k denote the (smooth) coefficients of X and Y, respectively.

The condition in Theorem 1.1 can then be rephrased as

Hörmander's Condition (HC):

The Lie algebra over the open set Ω generated by the vector fields X_j and their brackets has dimension n, i.e. the dimension of the ambient space.

In general, the HC is only a sufficient condition in order for P to be C^∞-hypoelliptic. To see this, fix an integer $k > 0$ and let

$$f(x_1) = \begin{cases} 0 & \text{for } x_1 = 0, \\ e^{-1/|x_1|^k} & \text{for } x_1 \neq 0. \end{cases}$$

The sum of squares operator in $\mathbb{R}^2_{x_1,x_2}$

$$D^2_{x_1} + f(x_1)^2 D^2_{x_2}$$

is C^∞-hypoelliptic although P does not satisfy the Hörmander hypothesis (see Fedii[27], Thm.5).

Derridj, in [26], proved that if the coefficients of the vector fields have real analytic regularity, then the HC is also necessary.

Theorem 1.1 has received a lot of attention over the years and we would like to mention the extensions that are particularly meaningful in the discussion of the real analtic hypoellipticity.

We first remark that the proof of the hypoellipticity of the operator P is done by establishing an a priori inequality showing the loss of derivatives of the operator P. The inequality with the optimal loss of derivatives is due to Rothschild and Stein, [44].

Theorem 1.2. *Let $x_0 \in \Omega$ and denote by U a neighborhood of x_0, $U \subset \Omega$. Assume that in U the Hörmander Condition is satisfied by taking iterated brackets involving at most m vector fields. Then for every $u \in C_0^\infty(U)$ there is a positive constant C such that*

$$\|u\|^2_{\frac{1}{m}} + \sum_{j=1}^{r} \|X_j(x,D)u\|^2 \leq C\left(\langle Pu, u\rangle + \|u\|^2\right). \tag{1.3}$$

Here $\|u\|_s$ denotes the norm of u in the Sobolev space H^s and the notation $\langle u, v\rangle$ denote the L^2 scalar product.

A very important point of view when it comes to the problem of the real analytic hypoellipticity is the microlocal theory for sums of squares.

First of all we note that the symbol of the commutator of two vector fields is the Poisson bracket of the symbols. Let $X(x,D) = \sum_{j=1}^{n} a_j(x) D_j$ then the symbol of X is

$$X(x,\xi) = \sum_{j=1}^{n} a_j(x)\xi_j.$$

Defining the Poisson bracket of two functions $f(x,\xi)$ and $g(x,\xi)$ as

$$\{f, g\} = \sum_{j=1}^{n} \left(\partial_{\xi_j} f \partial_{x_j} g - \partial_{x_j} f \partial_{\xi_j} g\right),$$

we have that

$$\sigma\left([X,Y]\right) = \frac{1}{i}\{X(x,\xi), Y(x,\xi)\}.$$

The Hörmander Condition can then be stated microlocally. In order to do this we define first the characteristic variety of the operator P in (1.2).

Definition 1.1. *Let P be as in* (1.2)*. We define the set*

$$\mathrm{Char}(P) = \{(x, \xi) \mid (x, \xi) \in T^*\Omega \setminus \{0\},\ X_j(x, \xi) = 0,\ \text{for } j = 1, \dots, r\}.$$

Here $T^*\Omega \setminus \{0\}$ denotes the cotangent bundle over Ω minus the zero section. We point out that, unless ad hoc assumptions are made this set in general is not a manifold.

The following is the microlocal statement of Hörmander's Condition; we refer to Bolley, Camus and Nourrigat, [11], and to Fefferman and Phong, [9], for a microlocal version of the results by Hörmander and Rothschild and Stein.

Microlocal Hörmander's Condition:

We may suppose that, instead of having vector fields we are dealing with (real valued) pseudodifferential operators of order 1. Let $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$. Then there exists an iterated commutator of length $r \geq 2$, i.e. an operator of the form

$$\mathrm{ad}(X_{i_1})(\mathrm{ad}(X_{i_2}(\cdots \mathrm{ad}(X_{i_{r-1}})(X_{i_r}) \cdots)),$$

where $\mathrm{ad}(X)Y = XY - YX$, whose symbol is elliptic—i.e. non zero—at (x_0, ξ_0).

As an example we state Hörmander theorem in a microlocal context.

Theorem 1.3 ([11])**.** *Let $a_j(x, D)$, $j = 1, \dots, r$, be real pseudodifferential operators of order 1 defined in Ω. Let $(x_0, \xi_0) \in T^*\Omega \setminus \{0\} \cap \mathrm{Char}(P)$, where $P(x, D) = \sum_{j=1}^{r} a_j(x, D)^2$. Assume further that the Microlocal Hörmander Condition holds at (x_0, ξ_0).*

Let U be a neighborhood of x_0 in Ω and $u, f \in \mathscr{D}'(U)$ such that $Pu = f$ in the distribution sense in U. Then if $(x_0, \xi_0) \notin WF(f)$, there is a neighborhood $U' \subset U$ of x_0 and a conic neighborhood Γ' of ξ_0, such that $WF(u) \cap U' \times \Gamma' = \varnothing$.

2. The real analytic case

As pointed out above, in [26] Derridj proved that the HC provides an "optimal" geometric characterization of $C^\infty-$hypoellipticity of a sum of squares operator with analytic coefficients. Therefore the analytic setting seems to better put in evidence the geometry underlying sums of squares operators. A natural question about the regularity of solutions to this class of operators is whether there is real analytic regularity provided the vector fields have real analytic coefficients and satisfy Hörmander Condition. More precisely,

Definition 2.1. *We say that the operator P is analytic hypoelliptic in the open subset $U \subset \Omega$ if for every $u \in \mathscr{D}'(U)$ and for every open subset $U_1 \subset U$, $Pu \in C^\omega(U_1)$ implies that $u \in C^\omega(U_1)$.*

It is well known that in the non degenerate case, i.e. the elliptic case, the answer is in the affirmative.

The first example showing that the situation might be more involved is due to Baouendi and Goulaouic, [8], but before stating and discussing it let us introduce the definition of Gevrey class of functions.

Definition 2.2. *Let Ω be an open subset of $\mathbb{R}^n$. We say that the function $u \in C^\infty(\Omega)$ is in the Gevrey class $G^s(\Omega)$, with $s \geq 1$, real number, if for every compact set $K \subset \Omega$ there is a positive constant C_K such that*

$$|\partial^\alpha u(x)| \leq C_K^{|\alpha|+1} \alpha!^s, \qquad \text{for every} \;\; x \in K,$$

and for every multiindex α.

It is straighforward that the class $G^1(\Omega) = C^\omega(\Omega)$ i.e. it coincides with the class of all real analytic functions in Ω. Roughly speaking, the order s of the class G^s measures how a smooth function is far from being analytic.

Theorem 2.1 ([8])**.** *Consider the operator in $\mathbb{R}^3$*

$$P_{BG}(x, D_x) = D_1^2 + D_2^2 + x_1^2 D_3^2. \tag{2.1}$$

It obviously satisfies Hörmander Condition, but there exist solutions of $P_{BG}u = f$, with $f \in C^\omega(\mathbb{R}^3)$, belonging to G^2 and not to G^s with $1 \leq s < 2$.

Proof. The proof is the construction of a suitable solution of the equation $P_{BG}u = 0$. Define

$$u(x) = \int_0^{+\infty} e^{ix_3\rho^2 - \frac{x_1^2}{2}\rho^2 + zx_2\rho - \rho} d\rho,$$

where $z \in \mathbb{C}$ is suitable. The integral converges provided we keep x_2 in a small neighborhood of the origin. Now

$$D_1^2 u(x) = -\int_0^{+\infty} e^{ix_3\rho^2 - \frac{x_1^2}{2}\rho^2 + zx_2\rho - \rho} \left(-\rho^2 + x_1^2\rho^4\right) d\rho.$$

Moreover

$$x_1^2 D_3^2 u(x) = -\int_0^{+\infty} e^{ix_3\rho^2 - \frac{x_1^2}{2}\rho^2 + zx_2\rho - \rho} (-x_1^2\rho^4) d\rho$$

and finally

$$D_2^2 u(x) = -\int_0^{+\infty} e^{ix_3\rho^2 - \frac{x_1^2}{2}\rho^2 + zx_2\rho - \rho} z^2\rho^2 d\rho.$$

If we choose $z = \pm 1$ we see that $P_{BG}u = 0$ in a slab where x_2 is in a sufficiently small neighborhood of 0. Setting $z = 1$ then

$$u(x) = \int_0^{+\infty} e^{ix_3\rho^2 - \frac{x_1^2}{2}\rho^2 + x_2\rho - \rho} d\rho.$$

An easy check shows that $u \in G^2$ in small neighborhood of the origin. Furthermore, if we compute $\partial_3^k u(0)$, we get:

$$\partial_3^k u(0) = \int_0^{+\infty} \rho^{2k} e^{-\rho} d\rho = (2k)! = \frac{(2k)!}{k!^2} k!^2 \geq k!^2.$$

This shows that $u \in G^2$ and that its Gevrey regularity is not better than 2. □

Moreover, in [4] it is shown that a general sum of squares operator P with analytic coefficients, satisfying the HC, can violate the analytic hypoellipticity in a "large subset" of the ambient space; more precisely, it can happen that a solution u of the homogeneous problem $Pu = f$ has a large analytic singular support even if the datum f is analytic. For instance, there exists a solution of equation

$$(D_1^2 + D_2^2 + x_1^2 D_3^2)u = 0 \quad \text{in} \quad \mathbb{R}\times]-\infty, 2[\times\mathbb{R},$$

such that $\operatorname{sing\,supp} u = \{(x_1, x_2, x_3) \in \mathbb{R}\times]-\infty, 2[\times\mathbb{R} \mid x_1 = 0\}$. Therefore, the HC is a condition too weak in order to ensure the analytic hypoellipticity.

At the end of the seventies Tartakoff, [45], and Treves, [46], proved with different methods the following important result:

Theorem 2.2 ([45], [46]). *Consider a sum of squares operator*

$$P(x,D)=\sum_{j=1}^{r}X_j(x,D)^2,$$

where the vector fields X_j have real analytic coefficients defined in an open subset $\Omega\subset\mathbb{R}^n$ and satisfy Hörmander condition.

Assume further that

(a) - $\mathrm{Char}(P)$ *is a symplectic submanifold of* $T^*\mathbb{R}^n\setminus\{0\}$.

(b) - *The principal symbol of P, $p(x,\xi)=\sum_{j=1}^{r}X_j(x,\xi)^2$ vanishes exactly to the second order on* $\mathrm{Char}(P)$.

Then P is analytic hypoelliptic.

We clarify briefly what the expression "vanishes exactly to the second order" means.

Denote by $p(x,\xi)$ the (principal) symbol of P as defined above and by $\mathrm{dist}(x,\xi)$ the distance of the point $(x,\xi/|\xi|)$ to $\mathrm{Char}(P)$. We say that $p(x,\xi)$ vanishes exactly to the second order near a point $(x_0,\xi_0)\in\mathrm{Char}(P)$ if, for suitable positive constants $c_{(x_0,\xi_0)}, C_{(x_0,\xi_0)}$,

$$c_{(x_0,\xi_0)}|\xi|^2\mathrm{dist}(x,\xi)^2\le|p(x,\xi)|\le C_{(x_0,\xi_0)}|\xi|^2\mathrm{dist}(x,\xi)^2.$$

Note that the second inequality is a trivial consequence of the non negativity of the principal symbol $p(x,\xi)$, hence Hypothesis (b) is reduced to the first inequality above.

Let us list a few examples of operators satisfying the assumptions of the theorem.

(a) The quadratic Grušin operator (also called the harmonic oscillator)

$$\sum_{j=1}^{n-1}(D_j^2+x_j^2D_n^2).$$

(b) The Heisenberg Laplacian

$$(D_1-x_2D_3)^2+(D_2+x_1D_3)^2.$$

We remark that the operator P_{BG} does not satisfy the assumptions of the theorem. In fact, its characteristic variety is a real analytic submanifold of $T^*\mathbb{R}^3 \setminus \{0\}$ given by

$$\text{Char}(P_{BG}) = \{(x, \xi) \in T^*\mathbb{R}^3 \setminus \{0\} \mid \xi_1 = \xi_2 = x_1 = 0, \xi_3 \neq 0\}. \tag{2.2}$$

$\text{Char}(P_{BG})$ has codimension 3 so that it is not symplectic. This suggests that the "simplecticity" of the characteristic manifold can play an important role in the study of the analytic hypoellipticity.

3. Geometry of the characteristic variety: stratifications and the Treves conjecture

In 1996, see the paper [47], F. Treves came up with an idea for the study of the analytic hypoellipticity of sums of squares. In this section we are going to give a fairly precise description of his idea, because it is important for what follows.

Stimulated by the papers [32], [33] by N. Hanges and A. A. Himonas, who proved that the Oleĭnik and Radkevič operator for special values of p and q, is not analytic hypoelliptic, even though its characteristic manifold is a real analytic symplectic submanifold (see Thm. 4.1 below), F. Treves introduced the idea that in order to establish if there is analytic hypoellipticity or not one has to look at the strata of a stratification of the characteristic variety.

Hence he proposed a certain stratification that will be henceforth called the Poisson stratification and formulated the conjecture that an operator is analytic hypoelliptic if and only if all the strata in the stratification of its characteristic variety are symplectic real analytic submanifolds.

We now give a detailed description of the Poisson stratification as well as some examples. We shall follow the presentation in the paper [20].

Denote Σ the variety $\text{Char}(P)$, where the symbols of all the vector fields are zero.

First of all let us define what we mean by the term stratification.

Definition 3.1 (see e.g. [49]). *By an analytic stratification of Σ in $T^*\mathbb{R}^n \setminus \{0\}$ we mean a partition of Σ*

$$\Sigma = \bigcup_{i \in I} S_i,$$

where the S_i are connected analytic submanifolds of $T^*\mathbb{R}^n \setminus \{0\}$ satisfying the conditions

(i) *Every compact subset of $T^*\mathbb{R}^n \setminus \{0\}$ intersects at most finitely many submanifolds S_i.*

(ii) *For any i, i' belonging to the index family I, $S_{i'} \cap \overline{S_i} \neq \varnothing$ implies $S_{i'} \subset \partial S_i$ and $\dim S_{i'} < \dim S_i$.*

The next is the definition of a (micro)local stratification. The definition is given in general terms, the adaptation to the homogeneous-on-the-fibers situation is straghtforward.

Definition 3.2 ([49]). *By a local analytic stratification of Σ we mean a system $(U, \{S_i\}_{i\in I})$, where U is an open set in $T^*\mathbb{R}^n \setminus \{0\}$, I is a finite index family, S_i is a connected analytic submanifold of U satisfying condition (ii) above and such that*

$$\Sigma \cap U = \bigcup_{i \in I} S_i.$$

A (micro-)local analytic stratification can be accomplished in several ways; for a detailed description of this point we refer to [20].

3.1. **The analytic stratification.** We follow [20]. Let us denote by

$$X(x,\xi) = (X_1(x,\xi), \dots, X_r(x,\xi))$$

the map whose components are the symbols of the vector fields. Moreover let $\Sigma = X^{-1}(0) \cap T^*\Omega \setminus \{0\}$ be the characteristic variety. Note that, since our maps are real valued, we might have used the function $p(x,\xi) = \sum_{j=1}^{r} X_j(x,\xi)^2$ to define Σ, but since in the following steps the minors of the Jacobian matrix of X are going to play a role, keeping the consistency of the notation would have been much more complicated. Thus we stick to the vector notation.

Define $\mathfrak{R}_0(\Sigma)$ as the subset of Σ whose points $z_0 = (x_0, \xi_0)$ have a neighborhood $U_{z_0} \subset T^*\Omega \setminus 0$ such that there are indices j_α, $\alpha = 1, \dots, m$, $1 \le j_1 < \cdots < j_m \le r$, for which

$$U_{z_0} \cap \Sigma = \{z \in U_{z_0} \mid X_{j_\alpha}(x, \xi) = 0, \alpha = 1, \dots, m\},$$

and the differentials $dX_{j_\alpha}(z_0)$ are all linearly independent. The latter is equivalent to saying that the minor

$$\frac{\partial(X_{j_1}, \dots, X_{j_m})}{\partial(z_{i_1}, \dots, z_{i_m})}(z_0),$$

where $1 \le i_1 < \cdots < i_m \le 2n$, is non zero. It is evident that $\mathfrak{R}_0(\Sigma)$ is a C^ω manifold of codimension m.

Next we define two subsets of Σ, Σ_1 and Σ_2. Let Σ_1 denote the subset of Σ in which all the $m \times m$ minors of the matrix $\frac{\partial X}{\partial z}$ vanish identically.

Define Σ_2 as the zero set in $T^*\Omega \setminus (\Sigma_1 \cup \mathfrak{R}_0(\Sigma))$ of all the $(m+1) \times (m+1)$ minors

$$\frac{\partial(X_{j_1}, \dots, X_{j_{m+1}})}{\partial(z_{i_1}, \dots, z_{i_{m+1}})},$$

$1 \le i_1 < \cdots < i_{m+1} \le 2n$.

We may now iterate for Σ_1, Σ_2 what has been done for Σ. For Σ_1 define the map

$$X^{(1)}(x, \xi) = \big(X(x, \xi), X^{j_1, \dots, j_m}_{i_1, \dots, i_m}\big) : T^*\Omega \to \mathbb{R}^{r_{1,1}}$$

with $X^{j_1, \dots, j_m}_{i_1, \dots, i_m}$ denoting the $m \times m$ minors and $r_{1,1} = r + r_1$, r_1 being the number of the $m \times m$ minors.

Analogously define

$$X^{(2)}(x, \xi) = \Big(X(x, \xi), X^{j_1, \dots, j_{m+1}}_{i_1, \dots, i_{m+1}}\Big) : T^*\Omega \to \mathbb{R}^{r_{1,2}}$$

with $X^{j_1, \dots, j_{m+1}}_{i_1, \dots, i_{m+1}}$ denoting the $(m+1) \times (m+1)$ minors and $r_{1,2} = r + r_2$, r_2 being the number of the $(m+1) \times (m+1)$ minors.

This leads to a local stratification of Σ: if V is a neighborhood of z_0 with a compact closure then

$$V \cap \Sigma = \bigcup_{\alpha=0}^{N_V} \Lambda_\alpha, \tag{3.1}$$

where the Λ_α are C^ω manifolds. The Λ_α shall be called the analytic strata of Σ.

3.2. **The symplectic stratification.** Assuming we already have a stratified variety of the form (3.1), we denote by Σ one of the strata Λ_α in (3.1), i.e. a connected C^ω submanifold defined near a point $z_0 \in \mathrm{Char}(P)$, and let σ be the symplectic form in $\mathbb{R}^{2n}$.

Then there are functions $G_j(x,\xi)$, $j = 1,\dots,s$, and an open set $\Omega' \subset \Omega$ such that $\Sigma \cap \Omega' = \{z \in \Omega' \mid G_j(z) = 0, j = 1,\dots,s\}$. Moreover we may assume that the rank of the map $G = (G_1,\dots,G_s)$ is equal to $\operatorname{codim}\Sigma$ at each point of $\Sigma \cap \Omega'$. Thus if $d = \operatorname{codim}\Sigma$, each $z_0 \in \Sigma$ has a neighborhood $U_{z_0} \subset \Omega'$ in which there are indices $1 \le i_1 < \cdots < i_d \le s$ such that

(i) The differentials $dG_{i_k}(z_0)$ are linearly independent.
(ii) $\Sigma \cap U_{z_0} = \{z \in U_{z_0} \mid G_{i_1}(z) = \cdots = G_{i_d}(z) = 0\}$.

Consider the pull back of σ to Σ and denote it by $\sigma_{|\Sigma}$. Let $\sigma_{z|\Sigma}$, $z \in \Sigma$, denote the restriction of the symplectic form to $T_z\Sigma$. The rank of the linear map corresponding to the skew symmetric bilinear form $\sigma_{z|\Sigma}$ is called the rank of the symplectic form on Σ at the point z or the symplectic rank of Σ at the point z.

Denote by μ the maximum rank of Σ. Then the set Σ_0 of all the points z where the symplectic rank is equal to μ is a dense subset of Σ. Each connected component of Σ_0 is a C^ω submanifold of U_{z_0} whose symplectic rank at every point is equal to μ.

The subset $\Sigma \setminus \Sigma_0$ is an analytic variety that can be defined by the vanishing of the functions $G_1,\dots,G_s$, as well as of all the $\nu \times \nu$ minors of the matrix $[\{G_i, G_j\}]_{1 \le i,j \le s}$, where $\nu = \mu + \operatorname{codim}\Sigma - \dim\Sigma$. Hence we can find an analytic stratification of this subset and the dimension of each analytic stratum of $\Sigma \setminus \Sigma_0$ is strictly less than the dimension of $\Sigma_0 = \dim\Sigma$.

This implies that we can decompose Σ so that

$$\Sigma \cap U = \bigcup_{\alpha=1}^{N_U} \Sigma_\alpha,$$

where each Σ_α is a connected C^ω submanifold with a constant symplectic rank.

3.3. **The Poisson stratification.** Again we start with the analytic set $\Sigma = \mathrm{Char}(P)$. For each multiindex $I = (i_1,\dots,i_\nu)$, $\nu \in \mathbb{N}$, we define

$$X_I(x,\xi) = \{X_{i_1}, \{X_{i_2}, \{\cdots\{X_{i_{\nu-1}}, X_{i_\nu}\}\cdots\}\}\}(x,\xi),$$

if $\nu \geq 2$ and $X_I = X_{i_1}$, if $I = (i_1)$. We also set $|I| = \nu$. Here $\{f, g\}$ denotes the Poisson bracket of the functions f and g:

$$\{f, g\}(x, \xi) = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right) (x, \xi).$$

Of course we *are assuming* that the vector fields X_i satisfy the microlocal Hörmander condition, i.e. that for every $(x, \xi) \in \mathrm{Char}(P)$ there exists a multiindex I such that $X_I(x, \xi) \neq 0$.

Let now U be a neighborhood of a point $z_0 = (x_0, \xi_0)$ and write as before $\Sigma = \mathrm{Char}(P)$. Then we may define a sequence of analytic subsets of U as

$$\Sigma^{(\nu)} = \{z \in U \mid \text{ for every multiindex } I, |I| \leq \nu, X_I(z) = 0\}.$$

We point out that the sequence $\Sigma^{(\nu)}$ is non increasing in ν and that in particular $\Sigma^{(1)} = \Sigma$. Furthermore, by the Hörmander condition, we have that

$$\bigcap_{\nu=1}^{\infty} \Sigma^{(\nu)} = \varnothing.$$

Now there is an increasing sequence of integers $1 = \nu_1 < \nu_2 < \cdots$ such that

(i) $\Sigma^{(\nu_{p+1})} \subsetneq \Sigma^{(\nu_p)}$.

(ii) If $\nu_p < \nu_{p+1}$, then $\Sigma^{(\nu')} = \Sigma^{(\nu_p)}$, for every ν', $\nu_p \leq \nu' < \nu_{p+1}$.

Consider now for any integer p the symplectic stratification (in the open set U) of the analytic set $\Sigma^{(\nu_p)}$:

$$\Sigma^{(\nu_p)} = \bigcup_{\alpha=1}^{N_U} \Sigma_\alpha^{(\nu_p)}.$$

In each stratum $\Sigma_\alpha^{(\nu_p)}$ the set of points $z \in \Sigma^{(\nu_p)} \setminus \Sigma^{(\nu_{p+1})}$ is either empty or else an open and dense subset of $\Sigma_\alpha^{(\nu_p)}$. If it is not empty, denote by $\Sigma_{\alpha,\beta}^{(\nu_p)}$ its connected components. Thus we get the decomposition

$$\Sigma^{(\nu_p)} = \Sigma^{(\nu_{p+1})} \cup \bigcup_{\alpha=1}^{N_U} \bigcup_{\beta=1}^{M_U} \Sigma_{\alpha,\beta}^{(\nu_p)}.$$

Finally, letting p run over the integers we obtain a decomposition of the form

$$\Sigma = \bigcup_{p} \bigcup_{j} \Sigma_j^{(\nu_p)}, \tag{3.2}$$

where p, j have a finite range (in the open set U) and

(i) The C^ω manifolds $\Sigma_j^{(\nu_p)}$ are connected and pairwise disjoint.

(ii) The symplectic rank of $\Sigma_j^{(\nu_p)}$ is constant.

(iii) At every point of $\Sigma_j^{(\nu_p)}$ the Poisson brackets X_I, with $|I| < \nu_{p+1}$ vanish, but there is at least one bracket X_I with $|I| = \nu_{p+1}$ which does not vanish.

We may then give the following

Definition 3.3. *The partition* (3.2) *of* $\mathrm{Char}(P) = \Sigma$ *is called the (local) Poisson stratification corresponding to the vector fields* $X_1, \dots, X_r$*. Each submanifold* $\Sigma_j^{(\nu_p)}$ *is a Poisson stratum, or simply just a stratum, for* Σ*. We refer to the integer* ν_p *as the depth of the stratum* $\Sigma_j^{(\nu_p)}$*.*

Remark 3.1. *It follows immediately from the definition above that the stratification of* Σ *defined by the vector fields* X_j*,* $j = 1, \dots, r$*, is invariant under nonsingular* C^ω *linear substitutions, that means if we define*

$$\tilde{X}_j(x,\xi) = \sum_{k=1}^{r} a_{jk}(x,\xi) X_k(x,\xi),$$

for $j = 1, \dots, r$ *where* $\left(a_{jk}\right)_{j,k}$ *is* C^ω *and invertible, we obtain the same stratification.*

Assume that a stratum, say Σ', of the stratification (3.2) is not symplectic. Since the symplectic rank is constant we have that Σ' is foliated by C^ω submanifolds whose tangent space is isomorphic to $T_z\Sigma' \cap (T_z\Sigma')^\sigma$.

We may then state the

Conjecture 1 (Treves conjecture, [47], [48], [20])**.** *The operator* P *is analytic hypoelliptic if and only if each stratum in its Poisson stratification is (microlocally) a symplectic* C^ω *submanifold.*

4. Examples and counterexamples

In this section we discuss some model operators and examine their Poisson stratification as well as—when known—their hypoellipticity properties. To do so, it is useful to introduce the following definition.

Definition 4.1. *We say that the operator P is Gevrey s $(s \geq 1)$ hypoelliptic in the open subset $U \subset \Omega$ if for every $u \in \mathscr{D}'(U)$ and for every open subset $U_1 \subset U$, $Pu \in G^s(U_1)$ implies that $u \in G^s(U_1)$.*

Obviously, if $s = 1$ P is analytic hypoelliptic.
In view of [29] (see also [19]) every sum of squares operator P is G^m-hypoelliptic where the integer m is the maximal length of the Poisson brackets required to span the ambient space.
Therefore, finding the optimal Gevrey regularity of P can provide information about the geometric obstruction to be analytic hypoelliptic. Let us show this by introducing the following example.

4.1. **The Oleĭnik and Radkevič example.** An important example was singled out by Oleĭnik, Oleĭnik and Radkevič in [41], [42]. Let p, q be positive integers and consider in $\mathbb{R}^3$ the following sum of squares

$$P_{OR}(x,\xi) = D_1^2 + x_1^{2(p-1)} D_2^2 + x_1^{2(q-1)} D_3^2, \tag{4.1}$$

where $1 < p \leq q$. Then

Theorem 4.1 ([41], [42], [24])**.** *The operator in* (4.1) *is Gevrey hypoelliptic of order q/p. This threshold is optimal.*

This result shows that P_{OR} is analytic hypoelliptic if and only if $q/p = 1$, that is $q = p$. Let us discuss this fact in view of the Treves conjecture 1. Consider the operator in (4.1), with $1 < p < q$. Then

$$\mathrm{Char}(P_{OR}) = \{(0, x_2, x_3; 0, \xi_2, \xi_3) \mid \xi_2^2 + \xi_3^2 > 0\}.$$

This is obviously a symplectic submanifold, so that the rank of the symplectic form restricted to $\mathrm{Char}(P_{OR})$ is constant and equal to 4.
All Poisson brackets of length $k+1$ of the form $\mathrm{ad}(X_1)^k X_j$ are zero for $k < p-1$. It is evident that X_1 is the only field contributing to this computation since both X_2 and X_3 carry vanishing coefficients. The first non-vanishing Poisson bracket is

$$\mathrm{ad}(X_1)^{p-1} X_2 = (p-1)!\xi_2. \tag{4.2}$$

Therefore the first Poisson strata (of depth p) are then

$$\Sigma_{p,\pm} = \{(x,\xi) \mid \xi_1 = x_1 = 0, \xi_2 \gtrless 0\}.$$

The Poisson bracket in (4.2) is zero if $\xi_2 = 0$, which is possible, provided $\xi_3 \neq 0$. Hence the strata of depth q are

$$\Sigma_{q,\pm} = \{(x,\xi) \mid \xi_1 = x_1 = 0 = \xi_2, \xi_3 \gtrless 0\}$$

since

$$\mathrm{ad}(X_1)^{q-1} X_3 = (q-1)! \xi_3 \neq 0.$$

The latter is not symplectic since it has codimension 3. According to the Treves conjecture P_{OR} is not analytic hypoelliptic.
Furthermore, if $q = p$ we have only a Poisson stratum coinciding with the characteristic manifold

$$\{(0, x_2, x_3; 0, \xi_2, \xi_3) \mid \xi_2^2 + \xi_3^2 > 0\},$$

which is symplectic; again, according to the Treves conjecture P_{OR} turns out to be analytic hypoelliptic.

4.2. **Counterexamples.** Let r, p, $q \in \mathbb{N}$, $1 < r < p < q$, and $x \in \mathbb{R}^4$. Consider the operator

$$P_1(x,D) = D_1^2 + D_2^2 + x_1^{2(r-1)}\left(D_3^2 + D_4^2\right) + x_2^{2(p-1)} D_3^2 + x_2^{2(q-1)} D_4^2. \tag{4.3}$$

Evidently P_1 is a sum of squares operator verifying Hörmander condition, since $\mathrm{ad}(D_1)^{r-1} x_1^{r-1} D_i$ yields D_i, $i = 3, 4$.

The characteristic variety of P_1 is

$$\mathrm{Char}(P_1) = \left\{(x,\xi) \mid \xi_1 = \xi_2 = 0, x_1 = x_2 = 0, \xi_3^2 + \xi_4^2 > 0\right\}.$$

The stratification associated with P_1 is made up of a symplectic single stratum

$$\Sigma_1 = \left\{(0,0,x_3,x_4;0,0,\xi_3,\xi_4) \mid \xi_3^2 + \xi_4^2 > 0\right\} = \mathrm{Char}(P_1).$$

In this framework, we refer also to [16] for a multi-strata case. Then we have

Theorem 4.2 ([3]). *Let*

$$\frac{1}{s_0} = \frac{1}{r} + \frac{r-1}{r}\frac{p-1}{q-1}.$$

Then P_1 in a neighborhood of the origin is locally Gevrey s_0 hypoelliptic and not better.

It is not difficult to show that Theorem 4.2 implies the following

Corollary 4.1. *The Conjecture 1 does not hold in dimension n for $n \geq 4$.*

We refer to [3] for a complete proof of Theorem 4.2. The first step consists in using the subelliptic inequality to show that a distribution solution of $P_1 u = f$, with f real analytic is in G^{s_0} near a characteristic point (in [15] it is provided an approach that does not use the subelliptic inequality at all.)
The second step (which is actually the crucial point) is the converse statement: there is a real analytic function f and a G^{s_0} function, u, such that $P_1 u = f$ and moreover u is not better than G^{s_0}. To this end we must construct such a function u, basically doing the same as in Theorem 2.1, i.e. constructing some sort of inverse Fourier transform whose exponential decay at infinity prevents analyticity. Of course both the (complex) phase and the amplitude are more involved in this case. In particular the amplitude is obtained by studying the semiclassical eigenfunctions and eigenvalues of a certain Schrödinger operator with a double well potential with non degenerate minima blowing up at infinity.

We emphasize that in a global (or semiglobal) setting the operator P_1 may be analytic hypoelliptic, suggesting that analytic hypoellipticity might be a consequence of the spectral behavior of some operator. Concerning this we cite the following theorem by Chinni [22]:

Theorem 4.3 ([22]). *Let*

$$P_1(x, D) = D_1^2 + D_2^2 + a^2(x_1)\left(D_3^2 + D_4^2\right) + b_1^2(x_2) D_3^2 + b_2^2(x_2) D_4^2,$$

defined on $\mathbb{T}^4$, where a, b_1, b_2 are real valued real analytic functions not identically zero. Then, given any subinterval $I \subset \mathbb{T}^2_{x'}$, $x' = (x_1, x_2)$, and given any $u \in \mathscr{D}'(I \times \mathbb{T}^2_{x''})$, $x'' = (x_3, x_4)$, the condition $P_1 u \in C^\omega(I \times \mathbb{T}^2_{x''})$ implies $u \in C^\omega(I \times \mathbb{T}^2_{x''})$.

We also would like to mention the following result: let r, p, q and k be positive integers such that $r < p < q$. Consider the sum of squares operator in $\mathbb{R}^4$, obtained adding the square of the vector field $x_2^{p-1}x_3^k D_4$ to the operator in (4.3),

$$\begin{aligned} P(x,D) = D_1^2 + D_2^2 + x_1^{2(r-1)}D_3^2 + x_1^{2(r-1)}D_4^2 + x_2^{2(p-1)}D_3^2 \\ + x_2^{2(p-1)}x_3^{2k}D_4^2 + x_2^{2(q-1)}D_4^2 \end{aligned} \tag{4.4}$$

The characteristic variety of P is actually the real analytic manifold

$$\mathrm{Char}(P) = \{(x,\xi) \mid x_i = \xi_i = 0, i = 1,2, \xi_3^2 + \xi_4^2 > 0\},$$

which is a symplectic manifold. Actually $\mathrm{Char}(P) = \mathrm{Char}(P_1)$.

We have

Theorem 4.4 ([17])**.** *The operator P in* (4.4) *is analytic hypoelliptic.*

The theorem above as well as the choice of the operator P are worth some explanation.

The operator P_1 in (4.3) is a counterexample to Treves conjecture. Actually the stratification associated to P_1 in the statement of the conjecture is made of the sole stratum

$$\mathrm{Char}(P_1) = \{(x,\xi) \mid x_i = \xi_i = 0, i = 1,2, \xi_3^2 + \xi_4^2 > 0\} = \mathrm{Char}(P).$$

An inspection of the proof though, shows that the real analytic submanifold

$$\Sigma_1 = \{(x,\xi) \mid x_i = \xi_i = 0, i = 1,2, \xi_3 = 0, \xi_4 \neq 0\}$$

is important for the Gevrey regularity of P_1 because of the presence of the vector field $x_2^{p-1}D_3$. This remark would lead us to consider the characteristic set $\mathrm{Char}(P_1)$ as the disjoint union of the following two analytic strata

$$\Sigma_0 = \{(x,\xi) \mid x_i = \xi_i = 0, i = 1,2, \xi_3 \neq 0\},$$

$$\Sigma_1 = \{(x,\xi) \mid x_i = \xi_i = 0, i = 1,2, \xi_3 = 0, \xi_4 \neq 0\}.$$

Actually Σ_1 is non symplectic and has Hamilton leaves which are the x_3 lines where the propagation of the Gevrey–s_0 wave front set occurs. Hence we might think of Σ_1 as a "non Treves stratum" where the existence of Hamilton leaves implies non analytic regularity.

We must make it clear though that, to our knowledge, there is neither a replacement conjecture nor an alternative definition of stratification.

The model operator P is such that, even though almost all the properties of P_1, as far as the Treves stratification is concerned, are retained, the manifold Σ_1 is replaced by

$$\Sigma_1 = \{(x,\xi) \mid x_i = \xi_i = 0, i = 1,2,3, \xi_4 \neq 0\}, \tag{4.5}$$

due to the presence in P of both vector fields $x_2^{p-1}D_3$ and $x_2^{p-1}x_3^k D_3$. We point out that in this case Σ_1 is a symplectic submanifold and hence has no Hamilton leaves.

In other words it seems that the analytic regularity of a sum of squares should depend on a suitable stratification of the characteristic variety of the operator and on the fact that its strata are analytic symplectic manifolds.

However, the following question has, to our knowledge, received no answer yet:

Problem 1. *Define a stratification of the characteristic variety in real analytic manifolds such that when each stratum is a symplectic manifold then the operator in analytic hypoelliptic.*

This would allow to reformulate, regardless of the local or microlocal aspect of the question, the Treves conjecture as

Conjecture 2. *A sum of squares operator with real analytic coefficients is analytic hypoelliptic if and only if every stratum of the stratification is a symplectic real analytic manifold.*

5. Open problems

5.1. **The 2 dimensional case.** Let us consider a sum of squares operator in $\mathbb{R}^2$. Denote by (x, y) the variables in $\mathbb{R}^2$:

$$P(x, y, D_x, D_y) = \sum_{j=1}^{N} X_j^2(x, y, D_x, D_y). \tag{5.1}$$

Without loss of generality we may suppose we are working in a neighborhood of the origin, Ω, and that $X_1 = D_x$.

Thus one of the equations of the characteristic variety is $\xi = 0$. For $j \geq 2$ we may then write $X_j(x, y, \xi, \eta) = a_j(x, y)\xi + b_j(x, y)\eta$. Since $\eta \neq 0$ we find that the other relations describing the characteristic variety are $b_j(x, y) = 0$, where the b_j are real analytic functions defined in Ω.

Since we are assuming that Hörmander condition is satisfied, we may suppose that $(0, 0; 0, \eta \neq 0)$ is a point of the characteristic variety and that, possibly shrinking Ω, there is an index j, $2 \leq j \leq N$, such that $\partial_x^m b_j(0, 0) \neq 0$; here m is minimal, i.e. $\partial_x^k b_j(0, 0) = 0$ when $2 \leq j \leq N$ and $k < m$. It is also evident that $X_1 = D_x$ is the only field that we can meaningfully use to form brackets of vector fields, i.e. we have to consider only brackets of the form $\mathrm{ad}(X_1)^k X_j$, since any other vector field has a vanishing coefficient in front.

Set

$$f(x, y) = \sum_{j=2}^{N} b_j(x, y)^2.$$

The characteristic variety of P is then given by

$$\mathrm{Char}(P) = \{(x, y; 0, \eta) \mid \eta \neq 0, f(x, y) = 0\}.$$

We apply the Weierstrass preparation theorem to f and write

$$f(x, y) = e(x, y) \left(x^{2m} + \sum_{\ell=1}^{2m} a_\ell(y) x^{2m-\ell} \right),$$

where $e(x, y)$ is a C^ω function such that $e(0, 0) \neq 0$, $a_\ell(0) = 0$ for every $\ell = 1, \dots, 2m$. Since e is different from zero, we may replace f by the Weierstrass polynomial above, because they define the same variety. Let us denote it by $q(x, y)$.

Definition 5.1 ([38], [49]). *We say that a polynomial of the form*

$$q(z', z_n) = z_n^m + \sum_{k=1}^{m} a_k(z') z_n^{m-k},$$

$z = (z', z_n) \in U$ *open subset of* $\mathbb{C}^n$, $0 \in U$, $a_k \in \mathscr{O}(U)$, *holomorphic functions on* U *such that* $a_k(0) = 0$ *for every* k *is a Weierstrass type polynomial of degree* m.

We have the following theorem

Theorem 5.1 ([38], [49]). *Let f be a holomorphic function defined in a neighborhood of the origin, $U \subset \mathbb{C}^n$. Suppose that $f(0,\dots,0,z_n) \not\equiv 0$ in U. Then there exists a Weierstraß type polynomial, $q^\#$, whose discriminant is not identically zero in U and such that $f = 0$ iff $q^\# = 0$.*

Same statement for a real analytic case.

Denote by $D_\#(y) = \operatorname{discr} q^\#$. We have that $D_\# \in C^\omega(\pi_2(U))$, where π_2 is the projection onto the y–axis.

As a consequence $D_\#^{-1}(0) = \{y_1,\dots,y_\nu\}$, for a certain $\nu \in \mathbb{N}$. Let $m^\# = \deg q^\#$ and denote by $\rho_1(y),\dots,\rho_{m^\#}(y)$ the roots (real or complex) of $q^\#$. For every $j \in \{1,\dots,\nu\}$, there are at least two indices, i_1, i_2 in the range $\{1,\dots,m^\#\}$ such that $\rho_{i_1}(y_j) = \rho_{i_2}(y_j)$. We set

$$\tilde{\rho}_j = (x_{i_1}, y_j), \quad x_{i_1} = \rho_{i_1}(y_j), \quad j = 1,\dots,\nu. \tag{5.2}$$

See, for instance, Fig.1 where two of such points are drawn.

Definition 5.2. *We call $\tilde{\rho}_j$ a branching point of $f^{-1}(0)$. Denote by $\mathscr{B}(U)$ the set of branching points in U.*

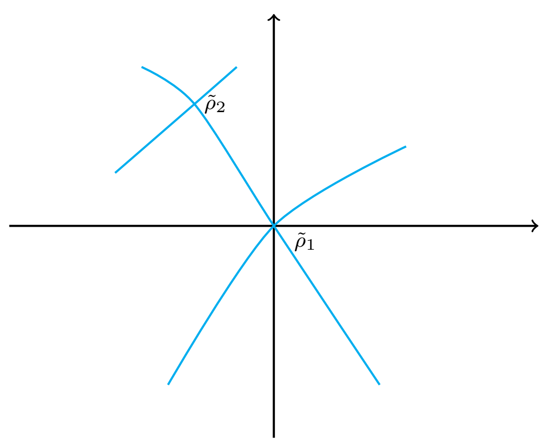

FIGURE 1. An example of $f^{-1}(0)$ near $(0,0) = \tilde{\rho}_1$

The above described facts determine the stratification. There are two cases:

(a) The set $\mathscr{B}(U)$ is empty. This means that the roots of $q^\#$ are simple and have the form $x = \rho_k(y)$, $k = 1,\dots,m^\#$. Since, according to our assumption, $(0,0) \in$

$f^{-1}(0,0)$, we deduce that there is only one $k \in \{1, \dots, m^{\#}\}$ such that $\rho_k(0) = 0$. Possibly shrinking U we obtain that f has the form

$$f(x, y) = \tilde{e}(x, y)(x - \rho(y))^{2m'}, \qquad \tilde{e}(0, 0) \neq 0, \qquad m' \leq m.$$

The characteristic variety of P is then symplectic and P is analytic hypoelliptic. This has been proved by Ōkaji, [40], and Cordaro and Hanges, [25], for operators where f has the above form.

(b) The set $\mathscr{B}(U)$ is not empty. Then we may always shrink the neighborhood U so that the origin—or $\tilde{\rho}_1$ is the only branching point in U. Then f has the form

$$f(x, y) = \tilde{e}(x, y) \prod_{j=1}^{m'} (x - \rho_j(y))^{m_j},$$

and $\rho_j(y) \neq \rho_k(y)$ if $y \neq 0$, but $\rho_j(0) = 0$ for every j, $m' \leq m^{\#}$, $\tilde{e}(0, 0) \neq 0$.

The deeper stratum is

$$\Sigma_1 = \{(0, 0; 0, \eta) \mid \eta \neq 0\},$$

as we can see by taking derivatives of f with respect to x. $\mathrm{Char}(P) \setminus \Sigma_1$ is a union of disjoint arcs of C^ω curves of the form

$$\{(x, y, 0, \eta) \mid \eta \neq 0, (x, y) \neq (0, 0), x = \rho_j(y)\},$$

which gives simplectic strata at each point of which we get real analyticity.

Thus it seems that the Treves stratification completely describes all possible situations in two dimensions. The problem of the non analytic hypoellipticity of P in case (b) as well that about its Gevrey regularity are open. In this setting, we refer to [23] for the study of a meaningful model.

We explicitly note that proving that in case (b) there is no analytic hypoellipticity amounts to proving that the Treves conjecture holds in dimension two.

5.2. **The 3 dimensional case.** There are no known counterexamples to the Treves conjecture in dimension 3. However in [18] some examples have been proposed that should violate the conjecture. We briefly describe those model in this section.

Let $x \in \mathbb{R}^2$, $y \in \mathbb{R}$, a, p, q, r be positive integers. We shall specify later the relation among these integers. Define

$$Q(x, y, D_x, D_y) = D_1^2 + D_2^2 + x_2^{2(r-1)} D_y^2 + x_1^{2(q-1)} D_y^2 + x_1^{2(p-1)} y^{2a} D_y^2. \tag{5.3}$$

If we assume that $1 < p < q < r$, the Lie algebra is generated with brackets of length $m = q$. The characteristic manifold is $\{(0, 0, y; 0, 0, \eta) \mid \eta \neq 0\}$.

If we look at the powers of the monomials in x, we can draw a (convex) Newton polygon in the x-plane. When the powers of x having a possibly degenerate coefficient are added to the picture we obtain

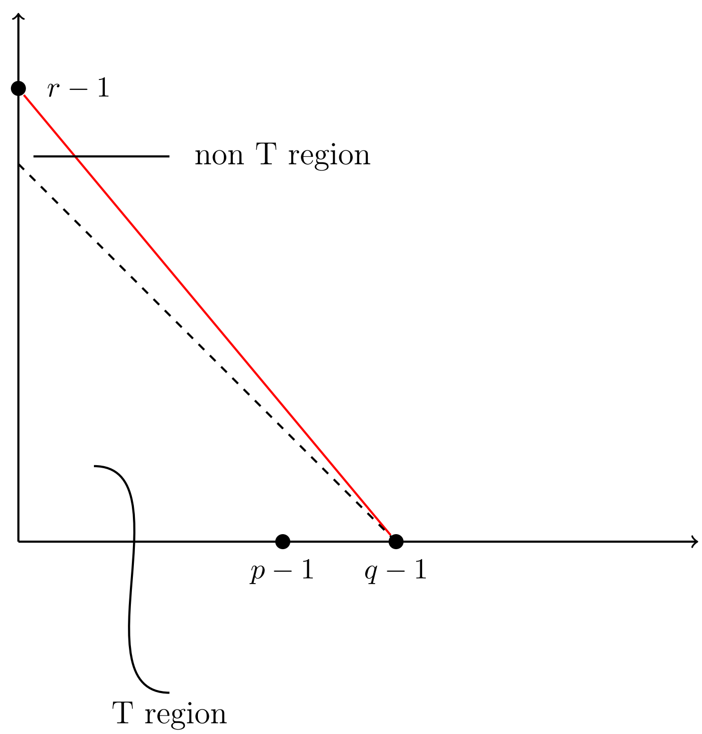

FIGURE 2. The Newton polygon for Q in (5.3) when $1 < p < q < r$

where the dashed line has slope -1 and starts from the vertex closest to the origin, the triangle underneath the dashed line has points corresponding to monomials where the Treves stratification identifies a non symplectic stratum (T region).

In [18] it is proved that

Theorem 5.2. *The operator Q in (5.3) is Gevrey s hypoelliptic for*

$$s \geq \left(1 - \frac{1}{a}\frac{p-1}{q}\right)^{-1}.$$

There is no proof of the optimality of the above index; we believe that it is optimal, due to the fact that Theorem 5.2 is a particular case of a result proved in [18], which, in the known cases, gives optimal values.

Let us now consider the operator Q in (5.3) when $1 < r < p < q$. An important tool in the analysis of Q is the associated Newton polygon (see also [14] in case of one dimensional case). If we draw the Newton polygon for Q and add to the picture the dots corresponding to degenerate monomials (i.e. monomials having coefficients containing powers of y) we obtain

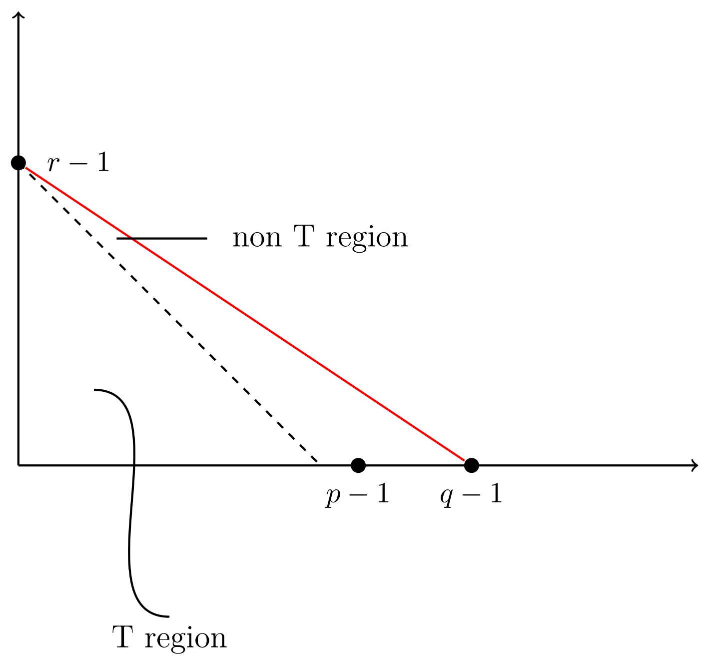

FIGURE 3. The Newton polygon for Q when $1 < r < p < q$

In [18] it is proved that, in the latter case, Q is Gevrey s hypoelliptic for

$$s \geq \left(1 - \frac{1}{a} \cdot \frac{q-p}{q-1} \cdot \frac{r-1}{r}\right)^{-1}. \tag{5.4}$$

On the other hand Q has a symplectic characteristic manifold: $\mathrm{Char}(Q) = \{x = \xi = 0, \eta \neq 0\}$ and no strata are found using the Poisson brackets of the fields, so that according to the conjecture it should be analytic hypoelliptic. We believe that the Gevrey regularity in (5.4) is optimal, based on the striking similarity of Q with the operator discussed in [3] which violates the conjecture. Actually the main difference between Q and the operator in [3] consists in the fact that the putative stratum is a non symplectic "stratum" whose Hamilton leaf lies on the fiber of the cotangent bundle.

At the moment we have no optimality proof for the Gevrey regularity (5.4) of Q both in the case of Figure 2 and of Figure 3. We also remark that the optimality of (5.4) would imply that the Treves conjecture does not hold in dimension 3.

Even though for the case considered in [18] the Newton polygon helps in identifying a (non symplectic) stratum in the three variables case, we would like to point out that this is not the case when the vector fields are not monomials. Here are two examples:

$$Q_1 = D_1^2 + D_2^2 + (x_1 - x_2^2)^2 D_y^2 + (y^2 x_1^3 + x_2^4)^2 D_y^2 \tag{5.5}$$

and

$$Q_2 = D_1^2 + D_2^2 + (x_1 - x_2^2)^2 D_y^2 + (x_1^3 + y^2 x_2^4)^2 D_y^2. \tag{5.6}$$

It is easy to show that

$$\mathrm{Char}(Q_j) = \{(0, 0, y; 0, 0, \eta) \mid \eta \neq 0\},$$

i.e. a symplectic manifold.

One can prove, using the L^2 estimate , that Q_1 is analytic hypoelliptic. Unfortunately the same proof does not work for Q_2. We believe that Q_2 has a non symplectic non Treves stratum, and hence is not analytic hypoelliptic. No proof is known.

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