SOME TOPICS ON THE REGULARITY OF ANALYTIC-GEVREY VECTORS

ALCUNI RISULTATI SULLA REGOLARITÀ DEI VETTORI ANALITICI-GEVREY

MAKHLOUF DERRIDJ

Abstract. My aim is to give, in this talk, some topics on the question of regularity of Analytic-Gevrey vectors of partial differential operators (p.d.o.) with analytic-Gevrey coefficients. Since the results obtained in the sixties on elliptic p.d.o's, which are both hypoelliptic (C^∞ setting), analytic-Gevrey hypoelliptic (analytic-Gevrey setting) and satisfy the so-called Kotake-Narasimhan property, a lot of works and articles were devoted to these problems in case of non elliptic p.d.o's under suitable hypotheses (for example on the degeneracy of ellipticity). I will consider the third problem on analytic-Gevrey vectors in the three cases of global (on compact manifolds), local (near a point in the base-space), microlocal (near a point in the cotangent space), situations, and say few words on the main two methods used in order to obtain positive (or negative) results. Finally I will focus on some new microlocal results on degenerate elliptic (also called sub-elliptic) p.d.o's of second order, obtained in a common work with Gregorio Chinni.

Sunto. Il mio scopo è quello di trattare, in questo intervento, alcuni argomenti sulla regolarità dei vettori analitici-Gevrey di operatori differenziali alle derivate parziali (p.d.o.) a coefficienti analitici-Gevrey. Nella parte finale mi concentrerò su alcuni nuovi risultati microlocali relativi a p.d.o. degeneri ellittici (anche detti sub-ellittici) del secondo ordine, ottenuti in un lavoro con Gregorio Chinni.

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Dipartimento di Matematica, Università di Bologna

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1. Introduction

This work follows the one we did in [15], in the same subject, but concerning the case of L. Hörmander's operators of the first kind (or commonly known as "sums of squares of vector fields"), with analytic coefficients and considered the case of analytic vectors. For that we used the method by F.B.I. transform.
In the present paper we consider second order partial differential operators, with non-negative characteristic form (first studied by O.A. Oleĭnik and E.V. Radkevič in [37]), Gevrey vectors ($s \geq 1$) and use the method of a priori estimate, (as precised in my preceeding paper, [22]). This in order to get suitable estimates of what we call microlocalized functions associated to the function under study (see details in the next sections).
Since the work T. Kotake and M. Narasimhan ([32], 1962) (where they proved the so called "Kotake-Narasimhan property", or "iterates property", for elliptic operators with analytic coefficients), an intensive investigation of this property was undertaken by many mathematicians, along with its generalizations in different directions and the use of more and more modern tools. In the case of elliptic operators, iterates property was extended to the systems and for s-Gevrey vectors($s \geq 1$, $s = 1$ corresponding to the analytic case) (see [9], [20], for surveys on this question, where there are many references).
In 1978, G. Métivier ([35]) showed that, in the case of s-Gevrey vectors with $s > 1$, the ellipticity property is necessary for "iterates property" to hold (meaning: s-Gevrey vectors are in s-Gevrey class). In the case of analytic vectors, M.S. Baouendi and G. Métivier showed Kotake-Narasimhan property for hypoelliptic partial differential operators of principal type with analytic coefficients ([3], 1982).
In the case of system of vector fields with analytic coefficients, satisfying Hörmander's condition, we mention two papers appeared in 1980, where iterates property was showed ([17] in case of analytic vectors, and [25] in the case called "reduced analytic vectors"). In the case of systems of complex vector fields R. Barostichi, P. Cordaro and G. Petronilho ([5]) studied analytic vectors in locally integrable structures in 2011.
Concerning the case of second order partial differential operators, the Hörmander operators were mostly studied, after the famous article on the hypoellipticity by L. Hörmander,

[28]. As we are interested in iterates property, we do not write in other properties like analytic or Gevrey hypoellipticity (local or microlocal). The first result, on Gevrey regularity of analytic vectors, we mention is in global context, for a subclass of "sums of squares". It appeared in 2016 ([11]) and have dealt with products of two tori. The local version of that result and for general Hörmander's operators was proved by me in two articles ([21], [19]); shortly after, for operators with non-negative characteristic form I proved an analogous result in ([22]), result for which we give in this paper the microlocal version.

Let us finish this introduction with the mention of some results using intensively the method of F.B.I. transform (and generalization of it as in [6] [7], [27], [26], [39]) and now studying mainly operators in more and more classes of ultra-differentiable functions (see [26], [23] where there are many references).

2. Some notations, definitions and preliminaries

We will be interested in the question of analytic-Gevrey regularity of partial differential operators on a manifold (analytic or Gevrey), mainly on an open set in $\mathbb{R}^n$, $n \in \mathbb{N}$.

To be more explicit, we consider here, only analytic-Gevrey vectors of those operators. Moreover we have to specify the notions of s-Gevrey vectors, $s \geq 1$, we study. I will recall some tools needed in order to attack our problem, in the different settings.

2.1. s-Gevrey vectors. ([35], [20], [10], [32], [33])

a) Global case: Let $\mathscr{M}$, compact manifold of s-Gevrey class, $s \geq 1$, and let P be a partial differential of order m with G^s coefficients on M. We say that a distribution $u \in \mathscr{D}'(\mathscr{M})$ is an s-L^2-Gevrey vector of P, in the global sense, if for every $k \in \mathbb{N}$, $P^k u$ is in $L^2(\mathscr{M})$ and there is a constant $C > 0$, independent of k, such that

$$\|P^k u\|_{L^2(\mathscr{M})} \leq C^{k+1} (mk!)^s . \tag{2.1}$$

Remark 2.2. *We recall that the case $s = 1$ corresponds to the analytic case. Moreover one may consider another norm.*

b) Local case: In this case, we take generally an open set Ω in $\mathbb{R}^n$, or if we are interested by a study near one point, say $x_0 \in \mathbb{R}^n$, we consider a small neighborhood (say Ω) of x_0. We say that $u \in \mathcal{D}'(\Omega)$ is s-Gevrey vector of P, a p.d.o. of order m on Ω, if: for every compact set in Ω, K, there exists a constant C_K such that: for every $k \in \mathbb{N}$, $P^k u$ is in $L^2(K)$, satisfying

$$\|P^k u\|_{L^2(K)} \leq C_K^{k+1} \left(mk!\right)^s . \tag{2.2}$$

c) Systems: In the case of systems, we consider just the case of systems of smooth vector fields (real or complex) on M or in Ω, say $(X_0, \ldots, X_r)$. These vector fields are also often considered as homogeneous p.d.o. of order 1. Let $s \geq 1$. Then a distribution $u \in \mathscr{D}'(\Omega)$ is an s-Gevrey vector of the system $(X) = (X_0, \ldots, X_r)$, in Ω if for every compact $K \Subset \Omega$, there exists $C_K > 0$ such that for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_N)$ one has

$$\begin{cases} X_{\alpha_1} \cdots X_{\alpha_N} u \in L^2(K), \\ \|X_{\alpha_1} \cdots X_{\alpha_N} u\|_{L^2(K)} \leq C_K^{N+1} N^{sN}, \qquad \alpha_j \in \{0, \ldots, r\}. \end{cases} \tag{2.3}$$

In the case when $(X) = (\partial_{x_1}, \ldots, \partial_{x_n})$ in $\Omega \subset \mathbb{R}^n$, (2.3) can be written, with $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\alpha_j \in \mathbb{N}$

$$\|\partial^\alpha u\|_{L^2(K)} \leq C_K^{|\alpha|+1} |\alpha|^{s|\alpha|}; \quad \partial^\alpha = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n}, \ |\alpha| = \alpha_1 + \cdots + \alpha_n. \tag{2.4}$$

So, we see that the functions satisfying (2.4) are the s-Gevrey functions in Ω, as such distributions are smooth.

2.2. **Notations.**

The space of distributions in Ω satisfying (2.1) or (2.2) (case $\mathscr{M}$ or Ω) are denoted $G^s(\mathscr{M} \text{ or } \Omega, P)$ (or Gevrey vectors of P in Ω).

Gevrey functions. In the case of system $(\partial_{x_1}, \ldots, \partial_{x_n})$ one gets from (2.4), using relations between L^2 and L^∞ norms,

$$|\partial^\alpha u|_K \leq \widetilde{C}_K^{|\alpha|+1} |\alpha|^{s|\alpha|}, \quad \text{for some } \widetilde{C}_K. \tag{2.5}$$

So, one has:

$$G^s(\Omega) = G^s(\Omega,\, (\partial_{x_1}, \ldots, \partial_{x_n})) = s\text{-Gevrey functions.}$$

If $s > 1$, it is also interesting to note that $G^s(\Omega)$ can be defined as follows: for every $\varphi \in \mathscr{D}(\Omega) \cap G^s = C_0^\infty(\Omega) \cap G^s$, there exists $C_\varphi > 0$:

$$\|\partial^\alpha(\varphi u)\|_{L^2} \leq \widetilde{C}_\varphi^{|\alpha|+1} |\alpha|^{s|\alpha|}; \quad \forall\, \alpha = (\alpha_1, \ldots, \alpha_n). \tag{2.6}$$

One uses in that case that, for every K, one can find $\varphi \in G^s \cap \mathscr{D}(\Omega)$ such that $\varphi \equiv 1$ on K.

Generally, for $s \geq 1$ (meaning working also for $s = 1$) one has a property replacing (2.6), but less easy to work with

$$\begin{cases} u \in G^s(\Omega) \Longleftrightarrow \text{ for every open set } \omega, \text{ with } \overline{\omega} \Subset \Omega, \\ \text{there exists an open set } \omega' \text{ with } \overline{\omega} \Subset \omega' \subset \overline{\omega'} \Subset \Omega, \\ \text{a sequence } u_N \in \mathscr{E}'(\omega'), \text{ satisfying : } \left. u_N \right|_\omega = u \text{ and :} \\ \partial^\alpha u_N \in L^2(\omega') \text{ and } \|\partial^\alpha u_N\|_{L^2(\omega')} \leq C^{|\alpha|+1} N^{s|\alpha|}, \; |\alpha| \leq N, \text{ for some } C > 0. \end{cases} \tag{2.7}$$

Remark 2.3. *As one sees the formulation* (2.7) *is interesting only when one wants to prove something for general* $s \geq 1$*. In case* $s > 1$, (2.6) *is much easier to handle. Moreover let us recall that one can replace* $|\alpha|^{s|\alpha|}$ *by* $|\alpha|!^s$ *or even by* $\alpha!^s$*,* $s \geq 1$*, in* (2.6) *or* (2.7)*, as we give it in* (2.1)*. As I spoke about in 2.5, one can replace the norm* L^2 *by the Sup-Norm* $= L^\infty$*, in the considered open sets, or also, when needed, by* L^1*-norm.*

The interest of (2.6) or (2.7) is also in the fact that they have a traduction when taking the Fourier transform of $\partial^\alpha \varphi u$ or $\partial^\alpha u_N$, which are in $\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n)$. When using L^1-norm, as said in Remark 2.3, one may obtain L^∞-norm of the functions $\widehat{\partial^\alpha \varphi u}$ or $\widehat{\partial^\alpha u_N}$, precisely bounds of it. So the estimates in (2.6) and (2.7) are replaced by:

$$\underline{\text{In (2.6)}} \text{ one has } |\xi^\alpha \widehat{\varphi u}| \leq \tilde{C}_\varphi^{|\alpha|+1} |\alpha|^{s|\alpha|}, \; \forall \alpha, \; \text{some } \tilde{C}_\varphi > 0;$$

$$\underline{\text{In (2.7)}} \text{ one has } |\xi^\alpha \widehat{u}_N| \leq \tilde{C}^{|\alpha|+1} N^{s|\alpha|}, \; |\alpha| \leq N, \; \text{some } \tilde{C} > 0.$$

Coming back, a moment, to the property (2.7), we recall that, here, one uses Ehrenpreis-sequences, say $\psi = (\psi_N)$ related to the couple of open sets in Ω, say (Ω_1, Ω_2), such that

$\Omega_1 \subset \overline{\Omega}_1 \Subset \Omega_2 \subset \overline{\Omega}_2 \Subset \Omega$, with the property:

$$\psi_N \in \mathscr{D}(\Omega_2);\ \psi_N \equiv 1 \text{ on } \overline{\Omega}_1 \text{ and } |\partial^\alpha \psi_N| \leq C^{|\alpha|+1} N^{|\alpha|},\ |\alpha| \leq N.$$

Usually the inequalities in the dual space above, are written

$$\begin{cases} |\widehat{\varphi u}(\xi)| \leq \tilde{\tilde{C}}_\varphi^{|\alpha|+1} |\alpha|^{s|\alpha|}(1+|\xi|)^{-|\alpha|},\ \forall \alpha,\ s>1,\ \text{some } \tilde{\tilde{C}}_\varphi > 0 \\ |\widehat{u}_N(\xi)| \leq \tilde{\tilde{C}}^{|\alpha|+1} N^{s|\alpha|}(1+|\xi|)^{-|\alpha|},\ |\alpha| \leq N,\ \xi \in \mathbb{R}^n,\ \text{some } \tilde{\tilde{C}} > 0. \end{cases} \tag{2.8}$$

Another interest of traduction in the dual space is that one can cut $\mathbb{R}^n$, into cones, or even take what is called a conic neighborhood of any $\xi_0 \in (\mathbb{R}^n)^* \setminus \{0\}$. For example, given $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ and $\varepsilon > 0$, one takes as a conic neighborhood $V_{\xi_0,\varepsilon}$ in $(\mathbb{R}^n)^* \setminus \{0\}$:

$$\left\{ \xi \in (\mathbb{R}^n)^* \setminus \{0\} \ : \ \left| \frac{\xi}{|\xi|} - \frac{\xi_0}{|\xi_0|} \right| < \varepsilon \right\},$$

for ε small; it is an open, convex cone around ξ_0. This opens a way to localize, not only in the base space Ω but also in the cotangent space $\Omega \times (\mathbb{R}^n)^* \setminus \{0\}$. In particular a conic neighborhood of $(x_0, \xi^0) \in \Omega \times (\mathbb{R}^n)^* \setminus \{0\}$ is a couple (ω, V), with ω neighborhood of x_0 in Ω, and V is a conic neighborhood of ξ^0 in $(\mathbb{R}^n)^* \setminus \{0\}$.

2.3. **Microlocal Gevrey functions - Gevrey wave front sets.**

In the research of G^s-regularity of a distribution, one now can ask for G^s-regularity in open sets of the form $\omega \times V$, with ω and V as above. So in the general case $s \geq 1$, the inequality (2.8) is written as

$$|\widehat{u}_N(\xi)| \leq C^{|\alpha|+1} N^{s|\alpha|}(1+|\xi|)^{-|\alpha|},\ |\alpha| \leq N,\ \xi \in V, \tag{2.9}$$

of course, when working with $s > 1$, it is simpler to use φ in $G_0^s = G^s(\Omega) \cap \mathscr{D}(\Omega)$, and rewrite here:

$$|\widehat{\varphi u}(\xi)| \leq C_\varphi^{|\alpha|+1} |\alpha|^{s|\alpha|}(1+|\xi|)^{-|\alpha|},\ \forall\, \alpha,\ \xi \in V. \tag{2.10}$$

In order to study the set in $\Omega \times (\mathbb{R}^n)^* \setminus \{0\}$ where u is G^s-regular, it is more suitable to consider the set of points (x, ξ) which will be called G^s-singular or wave front set of u and denoted by $WF_s(u)$:

(2.11)

$$WF_s(u) = \complement \big\{ (x,\xi);\ \exists \text{ conic neighborhood of } (x,\xi) \text{ satisfyng (2.9) or (2.10) when } s > 1 \big\}.$$

So one sees that $WF_s(u)$ is closed in $\omega \times (\mathbb{R}^n)^* \setminus \{0\}$, as it is defined as complementary of an open set in $\omega \times (\mathbb{R}^n)^* \setminus \{0\}$.

Remark 2.4. *In case $s > 1$, (2.10) may be rewritten:*

$$|\widehat{\varphi u}(\xi)| \le C e^{-c|\xi|^{1/s}},\ \xi \in V,\ \text{for some } C, c > 0. \tag{2.12}$$

2.4. **Ehrenpreis sequence-Andersson sequence.**

We told in subsection 2.2, in inequality (2.7), that one can take the sequence $u_N \in \mathscr{E}'(\Omega)$, as $u_N = \psi_N u$, u given in $\mathscr{D}'(\Omega)$ where ψ_N is an Ehrenpreis sequence. Let us give precisions:

Proposition 2.1. *([24], [29], [41]) Let Ω_1 and Ω_2 two open sets in $\mathbb{R}^n$ such that $\overline{\Omega_1} \Subset \Omega_2$ (compact in Ω_2). There is a constant $C > 0$ a sequence (ψ_N), $N \in \mathbb{N}$, $N \ge 1$, such that $\psi_N \in \mathscr{D}(\Omega_2)$, $\psi_N = 1$ on Ω_1, $|\psi_N^{(\alpha)}| \le C^{|\alpha|+1} N^{|\alpha|}$, $|\alpha| \le N$.*
The sequence (ψ_N) is an Ehrenpreis sequence relative to (Ω_1, Ω_2).

When working microlocally, one needs an analogue of Ehrenpreis sequence which plays a role of tool similar to that of Ehrenpreis sequence in local case. As we saw, we recall that one uses Ehrenpreis sequence (ψ_N) in order to truncate a given distribution $u \in \mathscr{D}'(\Omega)$, precisely if $\overline{\Omega_1} \Subset \Omega_2 \subset \overline{\Omega_2} \Subset \Omega$ and (ψ_N) in Proposition 2.1, then $u_N = \psi_N u \in \mathscr{E}'(\Omega_2)$ with $u_N = u$ on Ω_1.
In order to localize on cones in the dual space, there are Andersson sequences (also called Andersson-Hörmander sequences). We first precise some terms here. Let V_1 and V_2 two open cones in $(\mathbb{R}^n)^*$. We say that V_1 is relatively compact in V_2, if $\overline{V}_1 \cap \mathbb{S}^n$ is compact in $V_2 \cap \mathbb{S}^n$, and we note $V_1 \Subset V_1$. So

Proposition 2.2. *([29], [10]) Let V_1 and V_2 two open cones in $(\mathbb{R}^n)^* \setminus \{0\}$, $V_1 \Subset V_2$. There exist $C > 0$ and a sequence (Θ_N) of smooth functions in $(\mathbb{R}^n)^*$ such that:*

$$\begin{cases} \text{supp } \Theta_N \subset V_2 \cap \{\xi \in (\mathbb{R}^n)^* \,:\, |\xi| \ge \frac{N}{2}\}, \\ \Theta_N(\xi) = 1 \text{ for } \xi \in V_1 \cap \{\xi \in (\mathbb{R}^n)^* \,:\, |\xi| \ge N\}, \\ |\Theta_N^{(\alpha)}(\xi)| \le C^{|\alpha|+1} N^{|\alpha|} (1 + |\xi|)^{-|\alpha|},\ \ |\alpha| \le N,\ N \in \mathbb{N}^*. \end{cases} \tag{2.13}$$

The sequence (Θ_N) is an Andersson sequence.

Remark 2.5. *There is a refined version of Ehrenpreis or Andersson sequences; we used heavily in [16], that version. The refinement consists on the precision on estimates of derivatives:*

$$
(2.14)\qquad \begin{cases} \bullet \textit{Case of Ehrenpreis sequence: One gives } \Omega_1 \Subset \Omega_2, \textit{ and } M \in \mathbb{N}. \\ \textit{Then one has, for } (\psi_N) : \\ \psi_N = 1 \textit{ on } \Omega_1 \textit{ and } \ |\psi_N^{(\alpha)}| \leq C^{|\alpha|+1} N^{(|\alpha|-M)^+}, \, |\alpha| \leq N. \\ \bullet \textit{Case of Andersson sequence: One gives } V_1 \Subset V_2, \textit{ and } M \in \mathbb{N}. \\ \textit{Then one has, for } (\Theta_N) : \\ \Theta_N(\xi) = 1 \textit{ for } \xi \in V_1 \cap \{\xi \in (\mathbb{R}^n)^* \, : \, |\xi| \geq N\}, \\ \textit{and } |\Theta_N^{(\alpha)}| \leq C^{|\alpha|+1} N^{(|\alpha|-M)^+} \left(1+|\xi|\right)^{-|\alpha|}, \, |\alpha| \leq N, \\ \textit{where } \ (|\alpha|-M)^+ = 0 \textit{ if } |\alpha| \leq M, = |\alpha| - M \textit{ if } |\alpha| > M. \end{cases}
$$

Now, some definitions are in order to introduce properties on partial differential operators (p.d.o.) on an open set Ω in $\mathbb{R}^n$ (or even on compact manifold $\mathscr{M}$). Let P be a p.d.o.:

$$P = \sum_{|\alpha| \leq m} a_\alpha D^\alpha, \quad a_\alpha \in C^\infty(\Omega), \ m \text{ being the order of } P; \ D = \frac{1}{i}\partial.$$

The symbol $p(x,\xi)$ of P is a function on $\Omega \times (\mathbb{R}^n)^*$, defined by:

$$p(x,\xi) = \sum_{|\alpha| \leq m} a_\alpha \xi^\alpha. \text{ The principal symbol is } \ p_m(x,\xi) = \sum_{|\alpha| = m} a_\alpha \xi^\alpha.$$

Definition 2.1. *Let* $(x,\xi) \in \Omega \times (\mathbb{R}^n)^* \setminus \{0\}$. *$P$ is said elliptic in (x,ξ) if $p_m(x,\xi) \neq 0$. The characteristic set of P is:*

$$Char(P) = \{(x,\xi) \in \Omega \times (\mathbb{R}^n)^* \setminus \{0\} \ : \ p_m(x,\xi) = 0\}.$$

The operator P is said elliptic at the point $x_0 \in \Omega$, if P is elliptic at all points (x_0,ξ), $\xi \in (\mathbb{R}^n)^* \setminus \{0\}$. P is said elliptic in $\omega \subset \Omega$, if P is elliptic at any point in ω.

Now in order to study the characteristic set of P, we recall the notion of bracket or Poisson bracket.

Definition 2.2. *The Poisson bracket of two functions, smooth on* $\Omega\times(\mathbb{R}^n)^*\setminus\{0\}$*,* $f(x,\xi)$ *and* $g(x,\xi)$ *is defined by*

$$\{f,g\}(x,\xi)=\sum_{j=1}^{n}\left(\frac{\partial f}{\partial \xi_j}\frac{\partial g}{\partial x_j}-\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial \xi_j}\right). \tag{2.15}$$

The following properties are easy to see:

a) if X_1 and X_2 are two smooth vector fields, with symbols $\widetilde{X}_1(x,\xi)=\sum_{j=1}^{n}a_j(x)\xi_j$, $\widetilde{X}_2(x,\xi)=\sum_{j=1}^{n}b_j(x)\xi_j$, then $\{\widetilde{X}_1,\widetilde{X}_2\}(x,\xi)$ is also a symbol of degree one and it is the symbol of the bracket of X_1 and X_2 as p.d.o. $[X_1,X_2]=X_1X_2-X_2X_1$;

b) from a), one deduces that the linear space of vector fields with smooth coefficients is an algebra, with the Poisson bracket as inner product on that space.

It is not our subject to recall the theory of pseudodifferential operators, but in order to explain some notions like subellipticity, it is sufficient to look at homogeneous symbols $p(x,\xi)$ of real degree σ; meaning

$$p(x,\lambda\xi)=\lambda^{\sigma}p(x,\xi),\quad \lambda>0,\ x\in\Omega,\ \xi\in(\mathbb{R}^n)^*\setminus\{0\}.$$

To such a symbol, it is associated a linear operator on $\mathscr{D}(\Omega)$

$$Pv(x)=(2\pi)^{-n}\int p(x,\xi)\,e^{ix\xi}\widehat{v}(\xi)\,d\xi$$

where $p(x,\xi)\in C^{\infty}\left(\Omega\times(\mathbb{R}^n)^*\setminus\{0\}\right)$ and when the integral exists (under some bounds for p). When p and q are homogeneous of order 1, then $\{p,q\}$ is homogeneous of order 1. So one can consider all Poisson brackets in the space of such symbols. So if one considers a finite family of such symbols $p'=\{p_1,\,\dots,\,p_k\}$, then for any multi-index $I=(i_1,\,\dots,\,i_\ell)$, the symbol p_I defined by: $p_I=\{p_{i_1},\{\dots\{p_{i_{\ell-1}},p_{i_\ell}\}\dots\}\}$, is homogeneous of order 1.

Let now $(x,\xi)\in\Omega\times(\mathbb{R}^n)^*\setminus\{0\}$ and call $|I|=\ell$, the length of I, we say that the system is subelliptic at (x,ξ) if there exists a multi-index I such that $p_I(x,\xi)\neq 0$. We call index of subellipticity of p' at (x,ξ): $\sigma=\sup\{|I|^{-1}\,;\,p_I(x,\xi)\neq 0\}$ (this is in [30], [37], see in the following section).

3. The case of elliptic operators

a) The case with analytic coefficients: The iterates property for elliptic operators is due to T. Kotake and M. Narasimhan, [32]. So such property is also called "*Kotake-Narasimhan property*".

Theorem 3.1. *if P is an elliptic operator with analytic coefficients in Ω, then $G^1(\Omega, P) \subset G^1(\Omega)$.*

There is also another proof of this theorem by K. Komatsu [33]. The above theorem was generalized for systems of elliptic operators by P. Bolley, J. Camus: $G^s(\Omega, P_1, \ldots, P_\ell) \subset G^s(\Omega)$, with s-Gevrey coefficients hypothesis ([10]).
After the celebrated theorem of L. Hörmander on hypoellipticity of second order p.d,o, with real coefficients (particularly sums of squares), appeared in 1967 ([28]), the question of further regularity properties of such operators raised and many mathematicians worked on. There are many papers which appeared in the seventies and eighties about analytic-Gevrey regularity of solutions or of analytic-Gevrey vectors (see survey [10]). There was a result of G. Métivier (1978) opening a new window of research about non-elliptic operators [35]:

Theorem 3.2. *Let P be a p.d.o. with s-Gevrey coefficients in ω, $s > 1$, then there is $u \in G^s(\omega, P)$, ($x_0 \in \omega$ where P is not elliptic), such that $u \notin G^s(\omega)$.*

This result gave rise to two kinds of questions: what about the case $s = 1$?, and, in the case $s > 1$, what is the best s' such that $G^s(\omega, P) \subset G^{s'}(\omega)$?
Researching p.d.o's, non elliptic, with analytic coefficients which satisfy the Kotake-Narasimhan property, M. S. Baouendi and G. Métivier proved, in 1982, the following theorem:

Theorem 3.3. *([3]) Let P a p.d.o. in Ω, open set in $\mathbb{R}^n$, with analytic coefficients,of principal type, meaning:*

$$|p_m(x,\xi)| + |d_\xi p_m(x,\xi)| \neq 0, \quad (x,\xi) \in \Omega \times (\mathbb{R}^n)^* \setminus \{0\}.$$

Then $G^1(\Omega, P) = G^1(\Omega)$, if P is hypoelliptic.

Remark 3.2. *The inclusion $G^1(\Omega) \subset G^1(\Omega, P)$ is the easy part, as it is common with many other operators. The inclusion $G^1(\Omega, P) \subset G^1(\Omega)$ is a consequence of the two hypotheses above (principal type and hypoellipticity) We recall that for a differential operator P of principal type to be hypoelliptic in Ω, it is necessary and sufficient that the following condition holds for all $(x_0, \xi_0) \in T^*\Omega \setminus \{0\}$: if either $p_m(x_0, \xi_0) \neq 0$ or $p_m(x_0, \xi_0) = 0$ and for all $z \in \mathbb{C}$ such that $d_\xi \Re(zp_m)(x_0, \xi_0) \neq 0$, the function $\Im(zp_m)$, restricted to the bicharacteristic strip of $\Re(zp_m)$ through (x_0, ξ_0) has a zero of finite order at that point.*

In fact they gave, in the case $s > 1$, a precise s' such that $G^s(\Omega, P) \subset G^{s'}(\Omega)$, with s' related to s, via a relation depending on the order m, and the even order of vanishing evoked above, relation giving the equality for $s = 1$ as known from Theorem 3.3 above, but $s' > s$, which is in line with G. Métivier result (Theorem 3.2).

Of course, when one works on, say a compact manifold, analytic or Gevrey, $\mathcal{M}$, with a p.d.o. elliptic with analytic-Gevrey coefficients, one may ask for global regularity or for a local regularity, for an analytic or Gevrey vector. It is immediate from the definitions that the local regularity implies the global one. But there are operators, in case of non ellipticity, where it is much easier to prove global analytic or Gevrey regularity but very hard to do the same in the local case.

Now, let us look at a finer notion, the microlocal one; it is finer as we look the regularity on conic neighborhoods of points in $\Omega \times (\mathbb{R}^n)^* \setminus \{0\}$ of the form $\omega \times V$, $x_0 \in \omega$, $\xi_0 \in V$, where V can be assumed, moreover, convex (a useful property). More precisely we examine the behavior, the rapidly decreasing, of the Fourier transform of the Gevrey vector u in V, i.e. we ask if the inequalities in (2.8) are true only for $\xi \in V$ and not for any $\xi \in \mathbb{R}^n$ (we recall that in the case $s > 1$ we consider $\widehat{\varphi u}$, with $\varphi \in G_0^s(\omega)$, and in the case $s = 1$ we consider the suitable sequence $\widehat{u}_N$, where u_N are in $\mathscr{D}(\widetilde{\omega})$, $u_N = u$ in ω, $\overline{\omega} \Subset \widetilde{\omega} \Subset \Omega$.)

Definition 3.1. *Let $u \in \mathscr{D}'(\Omega)$, Ω open subset of $\mathbb{R}^n$, we shall say that $u \in G^s(\omega, V)$, $s \geq 1$, ω open subset strictly in Ω, $\overline{\omega} \Subset \Omega$ and V convex cone in $(\mathbb{R}^n)^* \setminus \{0\}$, if and only if (2.8) is valid for the sequence $u_N \mathscr{E}'(\Omega)$, which is equal to u in ω, for some constant C and for all $\xi \in V$.*

Equivalently we shall say that $u \in G^s(\omega, V)$, $s \geq 1$, if and only if there are (ψ_N), an

Ehrenpreis sequence relative to the couple (ω, Ω)*, Proposition 2.1, and* (Θ_N)*, an Andersson sequence relative to the couple* (V, V_1)*,* $\overline{V} \Subset V_1$*, Proposition 2.2, such that*

$$|\Theta_N(D) D^\alpha \psi_N(x) u(x)| \leq C^{|\alpha|+1} N^{s|\alpha|}, \qquad \forall \alpha \in \mathbb{N}^n \text{ with } |\alpha| \leq N.$$

Then the Theorem 3.1 is true, microlocally as follows:

Theorem 3.4. *Let* $(x_0, \xi_0) \in \Omega \times (\mathbb{R}^n)^* \setminus \{0\}$ *and* P *elliptic at* (x_0, ξ_0) *with* G^s *coefficients,* $s \geq 1$*, near* x_0*. Then there exists a conic neighborhood of* (x_0, ξ_0)*,* $\omega \times V$*, such that* $G^s(\omega, P) \subset G^s(\omega, V)$*.*

Let us just mention that there is, moreover, a finer notion of Gevrey vectors microlocally defined, leading to the notion of G^s-wave front set of a distribution u, with respect to a p.d.o. P (see [10]), and some results are given in that setting. Before going further we recall some operators defined with association to functions given on $\Omega \times (\mathbb{R}^n)^* \setminus \{0\}$ by:

$$(Pv)(x) = (2\pi)^{-n} \int e^{ix\xi} p(x, \xi) \widehat{v}(\xi)\, d\xi, \quad v \in \mathscr{D}(\Omega),$$

if say $|p(x, \xi)| \leq C|\xi|^M$ (in that case , $Pv(x)$ is well defined as $v \in \mathscr{S}(\mathbb{R}^n)$). In the case $p = \xi_j$ is the derivative of v with respect to x_j. More generally, if $p(x, \xi)$ is the symbol of a partial differential operator: $p(x, \xi) = \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha$, then

$$(2\pi)^{-n} \int e^{ix\xi} p(x, \xi) \widehat{v}(\xi)\, d\xi = (2\pi)^{-n} \int e^{ix\xi} \sum_{|\alpha| \leq m} a_\alpha(x) \xi^\alpha \widehat{v}(\xi)\, d\xi$$
$$= \sum_{|\alpha| \leq m} a_\alpha(x) (2\pi)^{-n} \int e^{ix\xi} \widehat{D^\alpha v}(\xi)\, d\xi = \sum_{|\alpha| \leq m} a_\alpha(x) \left(\mathscr{F}^{-1} \mathscr{F} D^\alpha v\right)(x) = Pv(x).$$

4. The case of systems of smooth vector fields

There are two cases to distinguish, the real and the complex case.

a) **The case of real vector fields:** we introduce that case in the subsection **2.1.c)** i.e. $(X_0, X_1, \ldots, X_r)$ with X_j real, $j = 0, \ldots, r$, defined on $\Omega \subset \mathbb{R}^n$. We saw that in case $(X) = (\partial_{x_1}, \ldots, \partial_{x_n})$ in Ω then, clearly: $G^s(\Omega, (X)) = G^s(\Omega)$, $s \geq 1$. The first less clear fact is that if the smooth vector fields X_j, $j = 0, \ldots, r$, span all tangent space at every point of $\omega \subset \Omega$ and if the coefficients are in G^s, $s \in [1, +\infty]$, then one has $G^s(\omega,(X)) = G^s(\omega)$: this is a direct consequence of the fact that any ∂_{x_k} is a linear combination of the X_j's in a neighborhood U at any given point in ω, with $G^s(U)$-coefficients. Then one uses that $G^s(\omega) = \bigcup_U G^s(U)$.

We recall that, in this last case (X) is clearly elliptic. We saw in subsection **2.4**, Definition 2.2, the Poisson bracket of two functions (smooth) in $\Omega \times (\mathbb{R}^n)^* \setminus \{0\}$. To any vector field X_j, $j = 0, \ldots, r$, is associated the function $\widetilde{X}_j(x,\xi)$, defined by replacing ∂_{x_ℓ} by ξ_ℓ in the expression of $X_j = \sum a_{j\ell}\partial_{x_\ell}$, so $\widetilde{X}_j(x,\xi) = \sum_\ell a_{j\ell}(x)\xi_\ell$. But it is easy to see here that $\{\widetilde{X}_j, \widetilde{X}_k\}(x,\xi) = \widetilde{[X_j, X_k]}(x,\xi)$. So, in view in what we saw in Subsection **2.4**, about the functions $\widetilde{X}_I$, $I = (i_1, \ldots, i_p)$ can be expressed by the repeated brackets of the vector fields X_j, $[X_j, X_k]$, ...

Definition 4.1. *The system (X) is of finite type at x_0 if, for every $\xi \neq 0$ there is I_ξ, such that $\widetilde{X}_{I_\xi}(x_0,\xi) \neq 0$, with $|I_\xi|$ minimal.*

As all $\widetilde{X}_I$ are homogeneous of degree 1 with respect to ξ, and using the compactness of the unit sphere $\mathbb{S}^n$, for any point x at which (X) is of finite type, one may define the type at x by $\tau_x(X) = \sup_{\|\xi\|=1} |I_\xi|$. The type of $\omega \subset \Omega$, with respect to (X) is $\tau(\omega,(X)) = \sup\{\tau_x(X)\,;\, x \in \omega\}$, finite or infinite.

Theorem 4.1. *([28]) Assume $\tau(\omega,(X))$ finite, (X) as above, then*

$$\|v\|_\sigma \leq C\left(\sum_{j=0}^{r} \|X_j v\| + \|v\|\right), \quad v \in \mathscr{D}(\omega), \tag{4.1}$$

with $\sigma = (\tau(\omega,(X)))^{-1}$, $C = C(\omega,(X))$, $\overline{\omega} \Subset \Omega$.

This is a subelliptic estimate for the system (X). The Theorem was proved by J.J. Kohn for σ greater in ([31]).

Such estimate gives, after some work the following corollary

Corollary 4.1. *Assume moreover that the coefficients of the system (X) are in $G^s(\omega)$. Then $G^s(\omega,(X)) \subset G^{\sigma s}(\omega)$.*

But the question is also: $G^s(\omega,(X)) = G^s(\omega)$?

We saw, when we began this section that if the system (X) span $\mathbb{R}^n$, at every point of ω, it is true. Otherwise, the known result is the following proved by M. Damlakhi-B. Helffer and B. Helffer-C. Mattera ([17], [25]):

Theorem 4.2. *Assume that (X) is with analytic coefficients in ω and that $\tau(\omega,(X))$ is finite. Then $G^1(\omega,(X)) = G^1(\omega)$.*

Remark 4.2. *One of the two above references gave a finer version where the hypothesis on the notion of Gevrey vectors of (X) is refined.*

b) **The case of complex vector fields:** Generally in that case the vector fields are denoted L_j's. Here we cite two papers, concerning those dealing with involutive structure. As we are interested mainly on the p.d.o. side of that structures we give the form of system $(L_1,\dots,L_r)$ defining the structure locally in $\mathbb{R}^n$, we cite precisely a simple case, (studied in [13]) of tubular structure of corank 1, meaning that there are coordinates near 0, say (x,t), with $x=(x_1,\dots,x_{n-1})$ such that

$$\begin{cases} L_j = \frac{\partial}{\partial x_j} - i\frac{\partial\varphi}{\partial x_j}(x)\frac{\partial}{\partial t}, \qquad j=1,\dots,n-1,\, t\in\mathbb{R},\\ \varphi:\omega\to\mathbb{R},\, \varphi\not\equiv 0;\, \varphi(0)=0;\, d\varphi(0)=0,\\ \omega,\ \text{neighborhood of 0 in } \mathbb{R}^{n-1}_x;\, \varphi:\omega\to\mathbb{R} \text{ is open map.}\end{cases} \tag{4.2}$$

Let us remark that when $d\varphi(0)\neq 0$, the system is elliptic. So the interesting case, is when $d\varphi(0)=0$, as we assumed. Moreover if $\varphi\equiv 0$ in a neighborhood ω of 0, (L) is not even hypoelliptic, as $L_j u = 0$, for every function of t, $u=u(t)$ in ω. In the case φ real analytic, we know (S. Łojasiewicz, [34]) that

$$\begin{cases} \text{there exists } \theta\in[\tfrac12,1[,\ \text{and a small neighborhood } \omega \text{ of 0 such that:}\\ |\varphi(x)|^\theta \le C|d\varphi(x)|,\ \text{for some } C>0,\,\forall\, x\in\omega.\end{cases} \tag{4.3}$$

Then in [13], the authors proved the following, but here the used norm is not the L^2-norm; the one used is $L^\infty_t(L^1_x)$: $|||v||| = \sup_t \|v(x,t)\|_{L^1_x}$, $v\in\mathscr{D}$.

Theorem 4.3. *Let (L) as in (4.2) and (4.3). Then, given $s\ge 1$, there exists a neighborhood $\omega\times I$ of $(0,0)$ in $\mathbb{R}^n=\mathbb{R}^{n-1}_x\times\mathbb{R}_t$ such that*

$$G^s(\omega\times I,(L)) \subset G^{s/(1-\theta)}(\omega\times I).$$

For general involutive structures see [13] and [41].

5. Second order partial differential operators

The first result, at my knowledge, in this question of analytic-Gevrey vectors, I know concerning p.d.o.'s of second order, outside elliptic ones, was in the case of global result by N. Braun Rodrigues, G. Chinni, P. D. Cordaro, and M. R. Jahnke, result which is a part of a paper they published in [11]. They considered on a product of two tori $\mathbb{T}^m \times \mathbb{T}^n$, a special subclass of the class of Hörmander's operators.

5.1. **The local case.**

We recall that in an open set $\Omega \subset \mathbb{R}^n$, a Hörmander operator is associated to the smooth real vector fields (X_j):

$$P = \sum_{j=1}^{r} X_j^2 + X_0 + c, \tag{5.1}$$

where (X_j), $j = 0, \dots, r$, given in Section **4, a)**, and c is a smooth complex function on Ω. Then

Theorem 5.1. *(L. Hörmander, [28]) For any $\omega \subset \Omega$, if $\tau(\omega, X)$ is finite, then P is hypoelliptic in ω.*

There was, in the years following the publication of L. Hörmander, an explosion of articles in many fields in Mathematics concerning the operators (5.1). Let us come back to the asking for analytic-Gevrey vectors of such operators, more precisely the result I spoke about for a subclass of operators (5.1) defined on the product $\mathbb{T}^m_x \times \mathbb{T}^n_t = \mathbb{T}_{(x,t)}$ of dimension $m + n$. In [11], the authors considered (globally) on $\mathbb{T}_{(x,t)}$ the following real vector fields, with analytic coefficients:

$$\begin{cases} X_j = \displaystyle\sum_{\ell=1}^{n} a_{j\ell}(t) \frac{\partial}{\partial t_\ell} + \sum_{p=1}^{m} b_{jp}(t) \frac{\partial}{\partial x_p}, \quad j = 1, \dots, r, \ a_{j\ell}, \, b_{jp} \in G^1\left(\mathbb{T}^n_t\right); \\ P = \displaystyle\sum_{j=1}^{r} X_j^2. \end{cases} \tag{5.2}$$

Theorem 5.2. *([11]) Let P as in (5.2). Assume that the vector fields $\sum_{\ell=1}^{n} a_{j\ell}(t)\frac{\partial}{\partial t_\ell}$, $j=1,\dots,r$, span $T_t(\mathbb{T}^n_t)$, $\forall\, t\in\mathbb{T}^n$ and $\tau\left(\mathbb{T}_{(x,t)};(X)\right)$ is finite, denoted by ρ, $(X)=(X_1,\dots,X_r)$. Then, for any $s\geq 1$, $G^s\left(\mathbb{T}_{(x,t)},P\right)\subset G^{\rho s}\left(\mathbb{T}_{(x,t)}\right)$.*

After that, I worked on the G^s-vectors, $s\geq 1$, for general Hörmander's operators, in local context, in two articles, the second one [19] giving precisions in the case of what I called Hörmander's operators of first kind, and [21] in which I proved the following theorem

Theorem 5.3. *Let P as in (5.1) such that $X_0=\sum_{j=1}^{r} a_jX_j$, and $\tau\left(\omega;\,(X_1,\dots,X_r)\right)=\rho$, with X_j and a_j in $G^s(\omega)$, $s\geq 1$. Then $G^s(\omega,P)\subset G^{\rho s}(\omega)$.*

Remark 5.2. *So Theorem 5.3 generalizes Theorem 5.2 in two directions: first for a class of Hörmander's operators containing in particular "Sums of squares", and secondly as the result is local (we saw before that the local case implies the global one).*

5.2. **The microlocal case for operators as in Theorem 5.1.**

By the microlocal case here, we mean that if P is such that $\tau\left((x_0,\xi_0);\,(X)\right)=\rho$, then what about $G^s(\omega,(X))$, with ω small neighborhood of x_0?

Now, as the type-hypothesis is microlocal, the hope is to obtain s'-Gevrey regularity in a conic neighborhood of (x_0,ξ_0). In a work published in 2022, [15], G. Chinni and I obtained the following

Theorem 5.4. *Let P as in (5.1) such that $X_0=\sum_{j=1}^{r} a_jX_j$, with X_j's and a_j's in $G^1(\omega)$. Assume $\tau\left((x_0,\xi_0);\,(X)\right)=\rho$, $x_0\in\omega$, $\xi_0\neq 0$. Then there exists a conic neighborhood $\omega_0\times V_0$ of (x_0,ξ_0) such that $G^1(\omega_0,P)\subset G^\rho(\omega_0,V_0)$.*

For the proof of Theorem 5.4, we used heavily the F.B.I. transform, introduced by J. Bros and D. Iagolnitzer ([12]) and generalized in many papers ([6], [7], [8], [41]). Moreover we used the trick relating analytic hypoellipticity with regularity of analytic vectors: more precisely, let $P=\sum X_j^2$ for example and assume that $u=u(x)$ is an analytic vector of P in $\omega\subset\mathbb{R}^n_x$. Introduce $Q=D_t^2+P$ on $I\times\omega$, $I_\varepsilon=(-\varepsilon,\varepsilon)$. Define U on $I\times\omega$ by:

$$U(t,x)=\sum_{k\geq 0}\frac{t^{2k}}{(2k)!}P^ku(x).$$

Using the fact that $\|P^k u\|_{\overline{\omega}} \leq C^{k+1}(2k)!$, one sees that the series converges in $I_\varepsilon \times \omega$ for ε small enough. The distribution U satisfies $QU = 0$ and $U(0,x) = u(x)$, as U is continuous in t. Assuming analytic hypoellipticity of Q, we get U analytic in $I \times \omega$. So we get analyticity of u. But this assumption is not always true: in general this is false [2] (For example $P = \partial_{x_1}^2 + x_1^2 \partial_{x_2}^2$, P is analytic hypoelliptic). Our proof consists to relate the two properties $(x_0, \xi_0) \notin WF_\rho(u)$ and $(0, x_0; 0, , \xi_0) \notin WF_\rho(U)$.

6. Second order p.d.o. with non negative characteristic form

These operators are more general and are as follows:

$$(6.1)\quad \begin{cases} P(x,D) = \displaystyle\sum_{j,\ell=1}^{n} a_{j,\ell} D_j D_\ell + \sum_{j=1}^{n} i b_j D_j + c, \text{ with smooth and real coefficients} \\ p(x,\xi) = \displaystyle\sum_{j,\ell=1}^{n} a_{j,\ell}(x)\xi_j\xi_\ell + \sum_{j=1}^{n} i b_j(x)\xi_j + c(x), \qquad D_j = \frac{1}{i}\frac{\partial}{\partial x_j}. \\ \text{The matrix } (a_{j\ell}) \geq 0 \left(\text{ i.e. } \displaystyle\sum_{j,\ell=1}^{n} a_{j,\ell}(x)\xi_j\xi_\ell \geq 0 \quad \forall\, \xi \in \mathbb{R}^n \right). \end{cases}$$

The operator of the form (6.1) cannot always be written of the form (5.1), in the preceding section. In order to give a similar hypothesis to one given in the case of Hörmander operators, O.A. Olĕinik and E.V. Radkevič who introduced and studied hypoellipticity of such operators, defined it as follows. Let $p_0(x,\xi)$ be the principal symbol, and define

$$(6.2)\quad \begin{cases} p_0(x,\xi) = \displaystyle\sum_{j,\ell=1}^{n} a_{j,\ell}(x)\xi_j\xi_\ell, \quad q(x,\xi) = \sum_{j=1}^{n} i b_j(x)\xi_j, \\ p^j(x,\xi) = \frac{\partial}{\partial \xi_j} p_0(x,\xi), \quad p_j(x,\xi) = D_j p_0(x,\xi), \quad j = 1,\dots,n. \end{cases}$$

The functions p^j are homogeneous of order 1 with respect to ξ. The functions p_j are homogeneous of order 2 with respect to ξ. We consider now the following new system of pseudodifferential operators homogeneous of order 1 with respect to ξ.

$$(6.3)\quad p^{n+j} \text{ is the principal symbol of } \Lambda^{-1} p_j,\ j = 1,\dots,n.$$

So we have the following system of $2n+1$ pseudodifferential operators of order 1

$$(6.4)\quad \{p^1,\dots,p^{2n},q\}.$$

Our hypothesis less general than that Oleĭnik-Radkevič, will be based only on the symbols $\{p^1, \dots, p^{2n}\}$.

$$\text{If } (p) = \{p^1, \dots, p^{2n}\}, \text{ we assume that } \tau\left((x_0, \xi_0), (p)\right) = \rho < +\infty. \tag{6.5}$$

In [22] I proved the local result assuming that

$$\tau(\omega) = \sup\{\tau\left((x_0, \xi_0), (p)\right); (x_0, \xi_0) \in \omega \times (\mathbb{R}^n)^* \setminus \{0\}\} = \rho < +\infty.$$

Recently G. Chinni and I proved the following microlocal version of the Gevrey regularity for the Gevrey vectors of P given in (6.1):

Theorem 6.1. *Let P as in (6.1) with coefficients in $G^s(\omega)$. We assume that $\tau\left((x_0, \xi_0), (p)\right) = \rho$, where (p) is defined in (6.5) and $(x_0, \xi_0) \in \omega \times (\mathbb{R}^n)^* \setminus \{0\}$, then $G^s(\omega, P) \subset G^{\rho s}(\widetilde{\omega}, V)$, where $\widetilde{\omega} \times V$ is some conic neighborhood of (x_0, ξ_0).*

Sketch of the proof: it is composed in several parts:

i) Adapted basic estimate for operators in (6.1), I gave in [22]

Proposition 6.1. *Let P as in (6.1) on Ω. Let Ω_1, with $\overline{\Omega}_1 \Subset \Omega$, Λ_{-1} elementary pseudodifferential operators with symbol $(1 + |\xi|^2)^{-1/2}$, $\psi \in \mathscr{D}(\Omega)$, $\psi = 1$ on Ω_1 and $E_m = D_m \psi \Lambda_{-1}$, $E_0 = I$, $m = 1, \dots, n$. Then there exists a constant $C > 0$ such that*

$$\sum_{j=1}^{n} \left(\|P^j v\|^2 + \|P_j v\|_{-1}^2\right) \leq C \left(\sum_{m=0}^{n} |(E_m P v, E_m v)| + \|v\|^2\right), \ \forall v \in \mathscr{D}(\Omega_1), \tag{6.6}$$

where the norms $\|\cdot\|_t$ are norms in Sobolev spaces H^t, $t \in \mathbb{R}$.

ii) Microlocal subelliptic estimate of P. Bolley, J. Camus and J. Nourrigat

Proposition 6.2. *([9]) Let P as in (6.1). Assume $\tau\left((x_0, \xi_0), (p)\right) = \rho \in \mathbb{N}^*$. Let $\widetilde{\omega}_0$ be an open neighborhood of ξ_0, $\widetilde{\omega}_0 \subset \mathbb{R}^n$, then there exist a pseudodifferential operator of order 0 elliptic in (x_0, ξ_0), $q(x, D)$ with $q(x, D) \equiv 0$ for $x \notin \omega_0$, ω_0 neighborhood of x_0 such that $\overline{\omega}_0 \Subset \widetilde{\omega}_0$, and $C > 0$, such that:*

$$\|qv\|_{\frac{1}{\rho}} \leq C \left(\sum_{j=1}^{2n} \|p^j v\| + \|v\|\right) \quad \forall v \in \mathscr{D}(\widetilde{\omega}_0), \ \ |q| \geq c_0 \ \ on \ \omega_0 \times V_0, \tag{6.7}$$

where V_0 conic neighborhood of ξ_0.

One remarks that it is sufficient to say: there exists q, with $q(x_0,\xi_0)\neq 0$, which implies the existence of $\widetilde{\omega}_0\times V_0$, conic around (x_0,ξ_0). We also recall here that the pseudodifferential $q(x,D)$ is defined by

$$q(x,D)v(x)=(2\pi)^{-n}\int e^{ix\xi}q(x,\xi)\widehat{v}(\xi)\,d\xi,\quad v\in\mathscr{D}\left(\omega_0\right). \tag{6.8}$$

In order to use (6.6), (6.7), we first proved the following proposition:

Proposition 6.3. *Let P as in Proposition 6.2, with all notations $\widetilde{\omega}_0$, V_0. Let now $\psi\in\mathscr{D}(\omega_0)$ and $\Theta=\Theta(\xi)$, with support in V_1, with $\overline{V}_1\Subset V_0$, $0\le\Theta\le 1$. Then, there exist C_0, C_1 such that*

$$\begin{aligned}\|\psi\Theta(D)v\|^2_{\frac{1}{\rho}}&\le C_0\left(\sum_{j=1}^{2n}\|p^jv\|^2+\|v\|^2\right)\\&\le C_1\left(\sum_{m=0}^{n}|(E_mPv,E_mv)|+\|v\|^2\right),\quad\forall\, v\in\mathscr{D}\left(\widetilde{\omega}_0\right).\end{aligned}\tag{6.9}$$

iii) Microlocal sequences associated to an analytic-Gevrey vector:

Given s-Gevrey vector u, $s\ge 1$, in $\omega\Subset\Omega$, $x_0\in\omega$, our goal is to show that it is in $G^{\rho s}(\widetilde{\omega},\widetilde{V})$, where $\widetilde{\omega}\times\widetilde{V}$ is some conic neighborhood of (x_0,ξ_0), with $\tau\left((x_0,\xi_0),(p)\right)=\rho\in\mathbb{N}^*$.

In that way, we consider two Ehrenpreis sequences (φ_N) and (ψ_N), $N\in\mathbb{N}^*$, associated to couples (ω_0,ω_1), (ω_1,ω_2), with $\overline{\omega}_0\Subset\omega_1$, $\overline{\omega}_1\Subset\omega_2$, $\overline{\omega}_2\Subset\widetilde{\omega}_0$, $\widetilde{\omega}_0$ as in Proposition 6.2, and an Andersson sequence (Θ_N) associated to the couple (V_1,V_2), $\overline{V}_1\Subset V_2$, $\overline{V}_2\Subset V_0$, where V_0 is as in (6.7), and the couples seen in Proposition 2.1, (2.13), with the precision in (2.14).

Our microlocal sequences are defined as follows. For any multiple $(\alpha\,\beta,\gamma,\delta,k,N)\in(\mathbb{N}^n)^4\times\mathbb{N}\times\mathbb{N}^*$, we consider $\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^ku$, u given above; of course the interesting case is when $|\alpha|\le N$, $|\beta|\le N$, $|\gamma|\le N$, $|\delta|\le N$, as the estimates in (2.14) are valid in that case it will be the case in all the paper other more precise relations between the parameters α, β , γ, δ, k and N will be given later. Now

$(\alpha\,\beta,\gamma,\delta,k)\in(\mathbb{N}^n)^4\times\mathbb{N}$, we consider the sequence:

$$N\in\mathbb{N}^*\to\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^k u.$$

The goal is to give suitable estimates of the L^2-norms of the terms of the sequence, which will imply the needed result.

Remark 6.1. *The operator* $\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha$ *is a pseudodifferential operator of order* $|\alpha|-|\gamma|$, *whose symbol is* $\psi_N^{(\beta)}(x)\Theta_N^{(\gamma)}(\xi)\xi^\alpha$. *Moreover, from the choice of* $(\omega_0,\omega_1,\omega_2)$, $\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^k u\in\mathscr{D}(\widetilde{\omega}_0)$.

iv) Application of the estimate (6.9) to the functions $\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^k u=v$ (From the last line in Remark 6.1, as $v\in\mathscr{D}(\widetilde{\omega}_0)$ we can apply (6.9)):

$$\|\psi\Theta(D)\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^k u\|_{\frac{1}{\rho}}^2 \leq C_1\left(\sum_{m=0}^{n}\left|\left(E_mP\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^k u,E_m\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^k u\right)\right| +\|\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^k u\|^2\right). \tag{6.10}$$

Now if we know a bound for the last term, and one for the sum in the second member we get a bound of the norm in $H^{1/\rho}$. So we gain $\frac{1}{\rho}$ step. The idea is to make a suitable manipulation of the term $E_mP\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^k u$ in order to get the term $E_m\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^{k+1}u$, modulo good terms, precisely terms in $[P,\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}]$, which is a bracket of two pseudodifferential operators.

a) As E_m's are $n+1$ pseudodifferential operators of order 0, then the L^2-norms of E_mv are dominated by $\|\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^{k+1}u\|$, which is quite a good term in the induction process we will use.

b) The last term, the bracket, will be handled by using expansions of brackets of pseudodifferential operators, at a suitable order, which depends on the order $|\alpha|-|\gamma|$ of the pseudodifferential operator in Remark 6.1: so there are $A(\alpha,\gamma)$ terms in the sum in the expansion, and a remainder of order $|\alpha|-|\gamma|+1$ (as the bracket is of order $|\alpha|-|\gamma|+1$). If $R_{|\alpha|-|\gamma|+1}$ is such a remainder, then

$$\|R_{|\alpha|-|\gamma|+1}g\|\leq C(\alpha,\beta,\gamma,\delta)\|g\|_{L^2(\overline{\omega}_1)},\ \text{where } g=P^k u.$$

The goal is to estimate all L^2-norms in the sum, the number $A(\alpha,\gamma)$ and the constant $C(\alpha,\beta,\gamma,\delta)$.

Once this (quite long) step is done, we do the same for the other steps, gaining $\frac{1}{\rho}$ in the Sobolev norms at each step. We apply estimate (6.9) to suitable function at each of the ρ steps, finally getting a bound of H^1-norm. From this process we get estimates of $\|\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^k u\|_{\frac{\ell}{\rho}}$ for $\ell = 1,\dots,\rho$. A crucial theorem we obtain is

Theorem 6.2. *Under all above notations and hypotheses, there exist two constants A and B such that if:*

$$(1)_0 \begin{cases} \|\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^k u\| \le A_1^{|\sigma|+1}B_1^{2\rho m+|\gamma|+1}N^{s[\rho m+|\gamma|+\sigma]}, \\ \text{for } 2\rho|\alpha| - (2\rho-1)|\gamma| + \sigma \le N, \ \text{where } \sigma = |\beta|+|\delta|+2k, \\ m = |\alpha| - |\gamma| \ \text{and } |\gamma| \le |\alpha|. \end{cases} \tag{6.11}$$

Then one has, for $1 \le \ell \le \rho$

$$(1)_p \begin{cases} \|\psi_N^{(\beta)}\Theta_N^{(\gamma)}D^\alpha\varphi_N^{(\delta)}P^k u\|_{\frac{\ell}{r}} \le A_1^{|\sigma|+\ell+1}B_1^{2\rho m+|\gamma|+\ell+1}N^{s[\rho m+|\gamma|+\sigma+\ell]}, \\ \text{for } 2\rho|\alpha| - (2\rho-1)|\gamma| + \sigma \le N - 2\ell \ \text{and } |\gamma| \le |\alpha|. \end{cases} \tag{6.12}$$

Corollary 6.1. *There exist two constants A and B such that the property* (6.11) *is true.*

Finally, using the Corollary we deduce the following:

Theorem 6.3. *Given (φ_N), (Θ_N) as above, there exists a constant A such that:*

$$\|\Theta_N D^\alpha \varphi_N u\|v \le A^{|\alpha|+1}N^{\rho s|\alpha|} \qquad |\alpha| \le N. \tag{6.13}$$

This implies that $u \in G^{\rho s}(\widetilde{\omega},\widetilde{V})$ and then $(x_0,\xi_0) \notin WF_{\rho s}(u)$.

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5, Rue de la Juviniére, 78350 Les Loges en Josas, France

Email address: `maklouf.derridj@numericable.fr`