A BRIEF NOTE ON HARNACK-TYPE ESTIMATES FOR SINGULAR PARABOLIC NONLINEAR OPERATORS

UNA BREVE NOTA SU DISUGUAGLIANZE INTEGRALI DI HARNACK PER OPERATORI PARABOLICI NON LINEARI SINGOLARI

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Abstract. In this brief note we introduce Harnack-type inequalities, which are typical in the context of singular nonlinear parabolic operators, and describe their state of art in the context of anisotropic operators.

Sunto. In questa nota breve presentiamo alcune disuguaglianze integrali di Harnack che sono tipiche di operatori parabolici nonlineari singolari, e descriviamo il loro stato dell'arte nel contesto di operatori singolari che presentano anisotropie.

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1. Harnack inequalities for isotropic equations

The history of Harnack estimates for parabolic operators goes back to the fifties when Hadamard [55] and Pini [64] extended the Harnack inequality to the heat equation, using representation formulas. We refer to [7] and [57] for an historic insight on the origins of inequalities bearing the name of Axel Von Harnack. In the 1960's several authors gave their important contributions to this topic, namely Moser [62] extended the theory to linear parabolic equations with measurable coefficients; Ivanov [56] worked within the setting of quasilinear second order parabolic equations, while Serrin [71] and Trudinger [77] gave significant contributions within the nonlinear setting.

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The extension of Harnack estimates to nonlinear evolutionary equations such as the p-Laplacian, where the prototype given by

$$u_t = \operatorname{div}\left(|Du|^{p-2}Du\right) \quad , \quad p > 1 \tag{1}$$

and to the Porous Medium equations, whose prototype is

$$u_t = \operatorname{div}\left(u^{m-1}Du\right) \quad , \quad m > 0 \tag{2}$$

turned out be be much more involved. For these equations, the behavior of solutions is very different when the equation (1) (and equation (2)) describes a *slow diffusion (degenerate case)*, i.e. for $p > 2$ ($m > 1$) or a *fast diffusion (singular case)*, i.e. $1 < p < 2$ ($0 < m < 1$). In the case of slow diffusion, the disturbances have finite speed of propagation, meaning that the support of an initial datum $u_0 \in C_o(B_r)$ evolves compactly along the flow, i.e. supp $u(\cdot, t) \subseteq B_{r(t)}$ for some radius $r(t) > 0$ depending on the initial datum; moreover, the initial mass is conserved. On the other hand in the case of fast diffusion, solutions can extinguish in finite time, meaning that there exists a time $T^* > 0$ such that for all $t > T^*$ we have $u(\cdot, t) \equiv 0$. The extinction in finite time, when taken irrespective of the size of the spatial domain, is clearly not compatible with the attainment of a pointwise Harnack inequality (see for instance [30] Chap VII) and the total mass is obliged to decay toward extinction too.

Within the fast diffusion range, equations (1) and (2) evolve in a: *subcritical* regime, for $1 < p \le 2N/(N+1)$ and $0 < m \le 1 - 2/N$, respectively; *supercritical* regime, for $2N/(N+1) < p < 2$ and $1 - 2/N < m < 1$, respectively. The aforementioned *singular equations* and related estimates are the interest of this note.

Harnack-type estimates for nonlinear isotropic equations

In what follows we present, for both equations (1) and (2), several Harnack-type estimates. We start by presenting a *(pointwise) Harnack inequality*, first obtained by DiBenedetto and Kwong [36] within the supercritical range of the slow diffusion regime (see [33] for a detailed study on this topic).

Theorem 1.1. **[(Pointwise) Harnack inequality]** *Let u be a continuous, nonnegative, local weak solution to* (1) *in $\Omega_T = \Omega \times (0,T) \subset \mathbb{R}^N \times (0,+\infty)$ for, either $p > 2$ or $2N/(N+1) < p < 2$. Consider a point $(x_o, t_o) \in \Omega_T$ such that $u(x_o,t_o) := u_o > 0$. Then there exists positive constants C and γ, depending on the data, such that for all cylinders $B_{4\rho}(x_o) \times (t_o - C(4\rho)^p\, u_o^{2-p}, t_o + C(4\rho)^p\, u_o^{2-p}) \subset \Omega_T$*

$$\gamma^{-1} \sup_{K_\rho(x_o)} u(\,\cdot\,, t_o - C\rho^p\, u_o^{2-p}) \le u_o \le \gamma \inf_{K_\rho(x_o)} u(\,\cdot\,, t_o + C\rho^p\, u_o^{2-p}). \tag{3}$$

Theorem 1.2. **[(Pointwise) Harnack inequality]** *Let u be a continuous, nonnegative, local weak solution to* (2) *in $\Omega_T = \Omega \times (0,T) \subset \mathbb{R}^N \times (0,+\infty)$ for, either $m > 1$ or $1 - 2/N < m < 1$. Consider a point $(x_o, t_o) \in \Omega_T$ such that $u(x_o,t_o) := u_o > 0$. Then there exists positive constants C and γ, depending on the data, such that for all cylinders $B_{4\rho}(x_o) \times (t_o - C(4\rho)^2\, u_o^{1-m}, t_o + C(4\rho)^2\, u_o^{1-m}) \subset \Omega_T$*

$$\gamma^{-1} \sup_{K_\rho(x_o)} u(\,\cdot\,, t_o - C\rho^2\, u_o^{1-m}) \le u_o \le \gamma \inf_{K_\rho(x_o)} u(\,\cdot\,, t_o + C\rho^2\, u_o^{1-m}). \tag{4}$$

Both estimates (3) and (4) formally recover the estimate known for the heat equation (take either $p = 2$ or $m = 1$), although being attained in a different time than t_o (see for instance [6], [62], [77]). The main difference is related to the waiting time that depends on u_o itself. Notice also that for small u_o, the time-shift reduces in the case of the singular range, while it increases in the degenerate one. This precise attribute of the estimates in the case of singular/degenerate equations is called *intrinsic scaling*. We refer to [78] for a simple introduction to this subject and to [38] for an historical and technical overview of the theory of Harnack inequalities for degenerate/singular operators in divergence form.

Adding to this, nonnegative solutions to singular operators behaving like (1) or like (2) in their supercritical regimes, also satisfy some *Harnack-type* estimates, namely $L^1 - L^1$ estimates (also known in the literature as integral Harnack-type estimates) and $L^r - L^\infty$ estimates, for $r \ge 1$.

Theorem 1.3. **[Harnack-type estimates for** (1)**]** *Let u be a nonnegative, local weak solution to* (1) *in Ω_T, for $1 < p < 2$. Let $\rho > 0$ and $\lambda = N(p-2)+p$. Then, there exists a positive constant γ, depending only on p, N, such that for any cylinder $K_{2\rho}(y) \times [s,t] \subset \Omega_T$*

we have

$$\sup_{s<\tau<t} \int_{K_\rho(y)} u(x,\tau)\, dx \le \gamma \inf_{s<\tau<t} \int_{K_{2\rho}(y)} u(x,\tau) dx + \gamma \left(\frac{t-s}{\rho^\lambda} \right)^{\frac{1}{2-p}}. \tag{5}$$

If moreover for $r \ge 1$ *we assume* $\lambda_r = N(p-2) + pr > 0$ *and* $u \in L^\infty_{loc}$*, then there exists a constant* $\gamma_r = \gamma_r(N,p,r) > 0$ *such that*

$$\sup_{K_\rho(y)\times[(s+t)/2,t]} u \le \gamma_r (t-s)^{-\frac{N}{\lambda_r}} \left(\int_{K_{2\rho}(y)} u^r(x,s)\, dx \right)^{\frac{p}{\lambda_r}} + \gamma_r \left(\frac{t-s}{\rho^p} \right)^{\frac{1}{2-p}}. \tag{6}$$

Theorem 1.4. [Harnack-type estimates for (2)] *Let* u *be a nonnegative, local weak solution to* (2) *in* Ω_T*, for* $0 < m < 1$*. Let* $\rho > 0$ *and* $\lambda = N(m-1)+2$*. Then, there exists a positive constant* γ*, depending only on* N*, such that for any cylinder* $K_{2\rho}(y)\times[s,t] \subset \Omega_T$ *we have*

$$\sup_{s<\tau<t} \int_{K_\rho(y)} u(x,\tau)\, dx \le \gamma \inf_{s<\tau<t} \int_{K_{2\rho}(y)} u(x,\tau) dx + \gamma \left(\frac{t-s}{\rho^\lambda} \right)^{\frac{1}{1-m}}. \tag{7}$$

If moreover for $r \ge 1$ *we assume* $\lambda_r = N(m-1) + 2r > 0$ *and* $u \in L^\infty_{loc}$*, then there exists a constant* $\gamma_r = \gamma_r(N,r) > 0$ *such that*

$$\sup_{K_\rho(y)\times[(s+t)/2,t]} u \le \gamma_r (t-s)^{-\frac{N}{\lambda_r}} \left(\int_{K_{2\rho}(y)} u^r(x,s)\, dx \right)^{\frac{2}{\lambda_r}} + \gamma_r \left(\frac{t-s}{\rho^p} \right)^{\frac{1}{1-m}}. \tag{8}$$

The constant γ_r blows up if either λ_r tends to zero or to infinity. From these estimates one can obtain interesting results. In fact, by letting $\rho \uparrow \infty$ inequalities (5) and (7) show that the mass is non-increasing; these same integral Harnack-type estimates allow to quantitatively understand the decay of the L^1_{loc}-norm of nonnegative solutions to (1) and (2) toward their extinction. Indeed, if in (5) one considers $T^* > 0$ a time of extinction for a nonnegative solution u of (1), then the decay of the mass toward extinction is obliged by (5) to follow the law

$$\|u(\cdot,\tau)\|_{1,K_\rho} \le \gamma \left(\frac{T^*-\tau}{\rho^\lambda} \right)^{\frac{1}{2-p}}, \quad \lambda = N(p-2)+p \tag{9}$$

for a positive constant $\gamma(N,p)$ depending only on the data. Hence the local mass $\|u(\cdot,\tau)\|_{L^1(K_\rho)}$ of the solution decays (to zero) as a power of the extinction time $(T^*-\tau)^{1/(2-p)}$. A similar reasoning can be applied to the nonnegative solutions to the fast diffusion equation (2) and then get the decay of the mass

$$\|u(\cdot,\tau)\|_{1,K_\rho} \leq \gamma \left(\frac{T^* - \tau}{\rho^\lambda} \right)^{\frac{1}{1-m}}, \quad \lambda = N(m-1) + 2 \ . \tag{10}$$

This approach, presented in [36], is only one of the possible ways to obtain the decay of the mass (and even of the supremum of u that decays toward extinction) of the solutions to (1) and (2). In [65], [66], [67], [68] Porzio shows that decay estimates of L^r-L^∞ type are indeed a consequence of evolutionary energy inequalities rather than a property of being a solution to a particular PDE. Another possibility to obtain the aforementioned decay estimates, that in nonlinear semigroup theory bear also the name of ultracontractive bounds, relates to the use of a certain logarithmic Sobolev inequality. Being this latest topic, the semigroup theory, out of the scope of this note we briefly refer to some works in which the reader can get to know more about it [10], [11], [12], [13], [29], [76], [80], [81].

The years 2008-2011 have been very fruitful in terms of obtaining Harnack-type estimates for general singular operators shaped upon (1) and (2). Indeed, in [34] DiBenedetto, Gianazza and Vespri derived the singular pointwise Harnack estimate (3) for general operators, with an approach that avoids the comparison principle; there, given the impossibility (by the extinction phenomenon) of having a pointwise Harnack inequality as in (4), Harnack-type estimates for the subcritical range $1 < p < 2N/(N+1)$ were left as an open problem. In the same year Bonforte, Iagar and Vázquez ([12] and [13]) presented, for the prototype equations (2) and (1), the correct form of Harnack-type inequality for the subcritical range; among various other interesting properties of the solutions to these very singular equations. The Harnack estimate obtained is similar to (3), but with a constant γ that depends on some ratio of the L^p-norm of u. Finally, in [35], DiBenedetto Gianazza and Vespri showed the validity of the aforementioned Harnack-type inequality for general operators, again with a technique that avoids the use of a comparison principle. Moreover, even if weaker than (3), the subcritical Harnack estimate derived implies the Hölder continuity of the solutions (see [33], [35]).

By working with a homogeneous quasi-linear equation

$$u_t = \text{div} A(x,t,u,Du), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}_0^+$$

(whose prototype is equation (1)), within the supercritical range $2N/(N+1) < p < 2$, Recalde and Vespri [69] obtained an estimate from below to its' solutions by means of the Barenblatt solution to a related Cauchy problem. In [70], the same authors improved the previous result and showed how to adapt it to the solutions to the porous medium type equation, for $(N-2)/N < m < 1$. In a similar setting but now considering $p > 2$, Bögelein, Ragnedda, Vernier and Vespri [8] obtained optimal kernel estimates and proved existence and sharp pointwise estimates from above and from below for the fundamental solutions.

Harnack-type estimates for doubly nonlinear equations

The singular equation

$$u_t - \text{div}\big(D \ln u\big) = 0, \tag{11}$$

that can be seen as the limit case of the Porous Medium type equation when $m \to 0^+$, was studied in [26] by Davis, DiBenedetto and Diller where they proved *a priori* estimates. Some years later, DiBenedetto, Gianazza and Liao [31] proved an intrinsic Harnack-type inequality to its weak solution. These same authors worked with a logarithmically singular equation, which was treated as the limit of a family of porous medium equations, for $0 < |m| < 1$ (see [32]). Equation (11) can also be seen as a special case of a wider class of nonlinear evolutionary equations, the so called *doubly nonlinear* equations, whose prototype is

$$u_t - \text{div}\big(|u|^{m-1}|Du|^{p-2}Du\big) = 0, \qquad p > 1, \tag{12}$$

once we take $p = 2$ and $m = 0$. These doubly nonlinear equations (12) were introduced by Lions [60] and are a natural bridge between (1) and (2). The range of m and p (more precisely of the sum $m+p$) brings specific properties to their solutions and characterize (12) in two major categories: degenerate, when $m+p > 3$; singular, when $2 < m+p < 3$. In the case $m+p = 3$ this equation is known as *Trudinger's equation* (it was introduced

by Trudinger in [77]) while finally, when $m + p = 2$, we are in the *logarithmic* case.
Regarding Trudinger's equation, and apart from his work [77], Gianazza and Vespri in [50] proved a Harnack inequality, for $p > 2$, by combining DeGiorgi's methods with Moser's logarithmic estimates; in [58] Kinnunnen and Kuusi, by means of Moser's approach and for $p > 1$, obtained a Harnack inequality considering a more general measure setting (general Borel measure).
Both pointwise Harnack estimates and Harnack-type estimates to the prototype doubly nonlinear equation (12) were obtained by Vespri [79] working within the singular setting $3 - p/N < m + p < 3$; Fornaro and Sosio [45] considered the degenerate range (corresponding to $m + p \geq 3$, $p \geq 2$ and $m \geq 1$); DiBenedetto, Gianazza and Vespri [33] presented a new way to approach the topic; Fornaro, Sosio and Vespri working for $2 < m + p < 3$, obtained an integral Harnack estimate [46] and a pointwise Harnackype estimate [47] (see also [48]). More recently, in [9] Bögelein, Heran, Schätzler and Singer proved a Harnack inequality for the full range of the slow diffusion case, *i.e.* $m > 0$, $p > 1$ and $m(p-1) > 1$.
As for Harnack estimates for doubly nonlinear logarithmic equations, corresponding to $m + p = 2$ and $p > 1$ (aside from [31]), we refer to the works of Fornaro, Henriques and Vespri [44]. These same authors (see [42] and [43]) went further on the study of Harnack estimates by working within the *very singular* range $3 - p < m + p < 2$; the logarithmic case $m + p = 2$ seems not to be a threshold anymore.

2. Harnack (type) inequalities for anisotropic equations

All the works presented in the previous section shared a common feature: the parabolic equations under study were considered in an isotropic framework. By isotropic framework we address the property of the energy to be homogeneous. When considering (1), and roughly speaking, this amounts to consider this formulation as the gradient flow of an energy $\mathcal{E} : W^{1,p}(\Omega) \to \mathbb{R}^+$, such as

$$\dot{u} + \nabla \mathcal{E}(u) = 0, \qquad \mathcal{E}(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx, \qquad p > 1, \quad \Omega \subset\subset \mathbb{R}^N,$$

then, $\mathcal{E}$ is homogeneous with respect of some power, as in this case $\mathcal{E}(\lambda u) = \lambda^p \mathcal{E}(u)$ can be easily checked. The formulation of the porous medium diffusion (given by equation (2)) as a gradient flow requires a more abstract approach (we refer to [1] or [63] for another interpretation) that uses a different underlying differential structure.

When describing the mathematical physics of a flow through nonhomogeneous media (see for instance the last chapters of [2] and [16]), the energy $\mathcal{E}$ usually loses the above property: the nonlinearity tangles in a competitive, mixed behavior of the solutions, and much of the simplest local regularity theory fades into darkness (see for instance [14], [23], [40], [41]). As an example, consider the sum of energies

$$\mathcal{E}^*(u) = \sum_{i=1}^{N} \frac{1}{p_i} \int_{\Omega} |u_{x_i}|^{p_i} \, dx, \qquad p_i > 1.$$

Even for stationary solutions, the nonhomogeneity of $\mathcal{E}^*$ results in minimizers that are unbounded if the gap between $\min\{p_i : i = 1, \dots, N\}$ and $\max\{p_i : i = 1, \dots, N\}$ is too wide (see for instance [49], [61]). To our knowledge, the first to introduce $\mathcal{E}^*$ as sum of monotone operators was Lions in [60], in the context of well-posedness for abstract evolution equations.

The study of evolutionary anisotropic equations - equations modeling diffusion processes which take different forms within each single space direction (e.g. water motion in an anisotropic porous medium) - has known recent developments in several different topics and under different settings (such as the definition of solution under taken). For a comprehensive insight on the topic of existence of solutions to partial differential equations in anisotropic Musielak-Orlicz spaces (and some applications) we invite the reader to consult the recent book [16]. Regarding existence and uniqueness we refer to [3], [5], [72] and [73]; as for qualitative (and quantitative) properties of the solutions apart for what is also presented in the previous works we refer to [17], [18], [23], [25], [37], [39], [51], [52], [53], [54], [74], [75], [28] and the references therein.

In what follows we consider two prototypes of these nonhomogeneous equations: the *anisotropic p-Laplacian* equation

$$u_t = \sum_{i=1}^{N} \left(|u_{x_i}|^{p_i - 2} u_{x_i} \right)_{x_i} , \quad p_i > 1 \tag{13}$$

and the *anisotropic porous medium equation*

$$u_t = \sum_{i=1}^{N} \left(u^{m_i-1} u_{x_i}\right)_{x_i} \ , \quad m_i > 0, \tag{14}$$

The quest of deriving Harnack estimates to these particular equations modelling anisotropic processes proved itself to be very demanding and challenging, thereby several questions remain unanswered. For instance, to the extent of our knowledge, a pointwise Harnack inequality as (3) is known only for nonnegative solutions to (13) in the degenerate case that allows finite propagation of disturbances ($2 < p_i < p(1+1/N)$, for all $i = 1, \dots, N$ and $p = N(\sum_{i=1}^{N} 1/p_i)^{-1}$, see [23]), and takes the following form

$$\frac{1}{\gamma} \sup_{\mathcal{K}_\rho(x_o,M)} u(\,\cdot\,, t_o - M^{2-p}\,(C_2\,\rho)^p) \le u(x_o,t_o) \le \gamma \inf_{\mathcal{K}_\rho(x_o,M)} u(\,\cdot\,, t_o + M^{2-p}\,(C_2\,\rho)^p) \tag{15}$$

being $M = (u(x_o,t_o)/C_1)$ and γ, C_1, C_2 positive constants depending only on $\{N, p_i\}$ and where the space geometry considered reflects the anisotropy of the operator (*intrinsic scaling*)

$$\mathcal{K}_\rho(x_o, M) = \prod_{i=1}^{N} \left\{ |x_i - x_{oi}| < \rho^{\frac{p}{p_i}} M^{\frac{p_i - p}{p_i}} \right\}.$$

Observe that in (15) when $u(x_o,t_o)$ is close to zero also the space geometry shrinks in some directions and expands in the other ones. In the anisotropic case, the aforementioned intrinsic scaling plays a role (a distinct one) in every single space direction (see for instances [19], [22], [40]). Although typically parabolic, this method of intrinsic scaling has also found interests in elliptic problems, see for instance [24] and [59] for a singular case and [37] for a degenerate case. This same method allowed to prove that solutions to (13), within the finite propagation range, are Hölder continuous and, when solving the equation in $\mathbb{R}^N \times (-\infty, T)$, enjoy some interesting properties of rigidity that recall the theorem of Liouville in the case of harmonic functions ([18]). Therefore, and in accordance to the available literature for the isotropic case of (3) and (4), it seems reasonable to expect that solutions to the singular equations (13) and (14) satisfy a Harnack estimate. This, however, is still a major open problem. Nevertheless some steps were already taken on that direction. In [27], Degtyarev and Tedeev proved L^1-L^∞ estimates for the solutions of a Cauchy problem evolving the doubly nonlinear degenerate anisotropic equations, for

$\beta \in (0,1]$ and $p_i > 1+\beta$,

$$\begin{cases} \partial_t \left(|u|^{\beta-1}u\right) - \displaystyle\sum_{i=1}^{N} \left(|u_{x_i}|^{p_i-2}u_{x_i}\right)_{x_i} = 0, & x \in \mathbb{R}^N,\ t>0 \\ |u|^{\beta-1}u(x,0) = |u_o|^{\beta-1}u_o(x), & x \in \mathbb{R}^N\ . \end{cases} \tag{16}$$

More recently, in [25] the authors worked with a more general setting than the one presented above (but whose prototype is precisely the one given above) and, within a standard geometry, determined (among other estimates) ultra contractive bounds. Feo, Volzone and Vazquez [40], among other topics such as existence and uniqueness of self-similar fundamental solutions, proved symmetrization results (together with a comparison principle) by which they derived L^1-L^∞ estimates for the solution of the Cauchy problem (16), with $\beta = 1$, considering all $1 < p_i < 2$ (fast diffusion). The same authors obtained analogous results [41] for the solutions of the Cauchy problem

$$\begin{cases} u_t - \displaystyle\sum_{i=1}^{N} \left(u^{m_i-1}u_{x_i}\right)_{x_i} = 0, & x \in \mathbb{R}^N,\ t>0, \\ u(x,0) = u_o(x), & x \in \mathbb{R}^N\ . \end{cases} \tag{17}$$

within the fast diffusion range $0 < m_i < 1$, for all $i = 1, \cdots, N$. In [20] the present authors proved several results related to Harnack-type inequalities, namely integral Harnack type inequalities and L^1-L^∞ estimates, together with extinction profile toward singular anisotropic porous medium type equations, for anisotropic evolution operators of the kind

$$u_t - \text{div}\ A(x,t,u,Du) = B(x,t,u,Du)\ , \quad \Omega \times (0,T] \subset \mathbb{R}^N \times \mathbb{R}^+, \quad N>2 \tag{18}$$

being $A = (A_1, \cdots, A_N)$ and B measurable functions satisfying the structure conditions (for given constants $C_o, C_1 > 0$ and $C \geq 0$), for $0 < m_1 \leq \cdots \leq m_N < 1$,

$$\begin{cases} A(x,t,u,Du)\cdot Du \geq C_o \displaystyle\sum_{i=1}^{N} m_i u^{m_i-1}|u_{x_i}|^2 - C^2 \sum_{i=1}^{N} u^{m_i+1}; \\ |A(x,t,u,Du)| \leq C_1 \displaystyle\sum_{i=1}^{N} m_i u^{m_i-1}|u_{x_i}| + C \sum_{i=1}^{N} u^{m_i}; \\ |B(x,t,u,Du)| \leq C \displaystyle\sum_{i=1}^{N} m_i u^{m_i-1}|u_{x_i}| + C^2 \sum_{i=1}^{N} u^{m_i}\ . \end{cases} \tag{19}$$

In [21] the authors and Skrypnik worked with singular anisotropic equations of the form (18) but now considering a generalization of the anisotropic p-Laplacian equation whose coefficients satisfy, for all $1 < p_i < 2$ and given constants $C_o, C_1 > 0$ and $C \geq 0$,

$$
\begin{cases}
A_i(x,t,s,\xi)\,\xi_i \geq C_o|\xi_i|^{p_i} - C^{p_i}\ ; \\
|A_i(x,t,s,\xi)| \leq C_1|\xi_i|^{p_i-1} + C^{p_i-1}; \\
|B(x,t,s,\xi)| \leq \displaystyle\sum_{i=1}^{N} C(|\xi_i|^{p_i-1} + C^{p_i-1})\ .
\end{cases}
\tag{20}
$$

Although different, and for that requiring distinct approaches, these two general cases (18)-(19) and (18)-(20), treated in [20] and [21] respectively, enjoy the common feature of being inspired by the techniques and the procedures presented in [33]. The derived estimates were obtained in three different topological settings: $L^1_{loc}(\Omega)$, $L^1_{loc}(\Omega)$-$L^\infty_{loc}(\Omega)$ and $L^r_{loc}(\Omega)$-$L^r_{loc}(\Omega)$ backwards in time, and the choice of working within a standard or an intrinsic geometry, meaning a geometry for which time tangles within the cube's radius, played a significant role along the proofs (with an inherent price to be paid). In order to describe the results, a specification of the geometry is in force: let ρ and t be fixed positive numbers and define

$$
\mathbb{K}_\rho = \prod_{i=1}^{N}\left\{|x_i| < \rho^{\frac{p}{p_i}}\right\} \qquad \text{and} \qquad \mathcal{K}_\rho(t) = \prod_{i=1}^{N}\left\{|x_i| < \rho^{\frac{p}{p_i}}\left(\frac{t}{\rho^p}\right)^{\frac{p-p_i}{(2-p_i)p_i}}\right\}.
\tag{21}
$$

The set $\mathbb{K}_\rho$ is referred to the *standard anisotropic* cube, while $\mathcal{K}_\rho(t)$ is the *intrinsic anisotropic* cube. Notice that when $p_i \equiv p$ both cubes correspond to the classical cube $K_\rho = \{|x| < \rho\}$ of edge 2ρ; all the cubes, regardless their geometry, has volume $(2\rho)^N$.

With these two definitions at hand, the following L^1-L^1 Harnack-type inequalities were found, respectively in the two geometries described in (21).

Theorem 2.1. [**L^1-L^1 Harnack-type estimates**] *Let u be a nonnegative local weak solution to* (1) *in Ω_T and let ρ, t be positive fixed numbers. Then, the following two estimates hold true in their respective space configurations.*

1 *Let* $\mathcal{K}_\rho(t)$ *be defined as in* (21). *Then, there exists a constant* $\gamma(N, p_i) > 1$ *such that*

$$\sup_{0<\tau<t} \int_{\mathcal{K}_\rho(t)} u(x,\tau)\,dx \le \gamma \inf_{0<\tau<t} \int_{2\mathcal{K}_\rho(t)} u(x,\tau)\,dx + \gamma \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{2-p}}.$$

2 *Let* $\mathbb{K}_\rho$ *be defined as in* (21). *Then, there exists a constant* $\gamma(N, p_i) > 1$ *such that*

$$\sup_{0<\tau<t} \int_{\mathbb{K}_\rho} u(x,\tau)\,dx \le \gamma \inf_{0<\tau<t} \int_{2\mathbb{K}_\rho} u(x,\tau)\,dx + \sum_{i=1}^{N} \left(\frac{t}{\rho^{\lambda_i}}\right)^{\frac{1}{2-p_i}}.$$

The main interesting feature is the trade-off between geometry and estimates: if the standard anisotropic geometry $\mathbb{K}_\rho$ is taken into account, then the estimate is affected by a nonhomogeneous right-hand side; if the intrinsic anisotropic geometry $\mathcal{K}_\rho(t)$ is considered, the estimate is the same as the p-Laplacean one (5). The price to be paid here is the fact that as soon as $t \downarrow 0$ the cube $\mathcal{K}_\rho(t)$ becomes unbounded along some directions, while shrinking to its centre along the other ones. One consequence of these estimates relates to the decay of the L^1_{loc}-norm of u: being T^* an extinction time for nonnegative solutions u to (1), within the standard anisotropic geometry one gets

$$\int_{\mathbb{K}_\rho} u(x,\tau)\,dx \le \gamma \sum_{i=1}^{N} \left(\frac{T^* - \tau}{\rho^{\lambda_i}}\right)^{\frac{1}{2-p_i}},$$

while, within the intrinsic anisotropic geometry, the decay is given by

$$\int_{\mathcal{K}_\rho(T^*-\tau)} u(x,\tau)\,dx \le \gamma \left(\frac{T^* - \tau}{\rho^{\lambda}}\right)^{\frac{1}{2-p}}.$$

From the point of view of L^1-L^∞ Harnack-type estimates one has

Theorem 2.2. [**L^1-L^∞ Harnack-type inequalities**] *Let u be a nonnegative, locally bounded, local weak solution to* (1) *in* Ω_T, *and suppose p is in the supercritical range, i.e.*

$$\lambda = N(p-2) + p > 0.$$

Then, the following two estimates hold true in their respective space configurations.

1 *Let* $\mathbb{K}_\rho$ *be defined as in* (21). *Then, there exists a constant* $\gamma(N,p_i)>0$ *such that*

$$\sup_{\mathbb{K}_{\rho/2}\times[t/2\,,\,t]} u \le \gamma t^{\frac{-N}{\lambda}} \left(\inf_{0\le\tau\le t} \int_{\mathbb{K}_{2\rho}} u(x,\tau)\,dx \right)^{\frac{p}{\lambda}} + \\ + \gamma \sum_{i=1}^{N} \left(\frac{t}{\rho^p}\right)^{\frac{\lambda_i}{(2-p_i)\lambda}} + \gamma \sum_{i=1}^{N} \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p_i}},$$

for $\lambda_i = N(p_i-2)+p$ *that can be of either sign.*

2 *Let* $\mathcal{K}_\rho$ *be defined as in* (21). *Then, there exists a constant* $\gamma(N,p_i)>1$ *such that*

$$\sup_{\mathcal{K}_{\rho/2}(t)\times[t/2\,,\,t]} u \le \gamma\, t^{\frac{-N}{\lambda}} \left(\inf_{0\le\tau\le t} \int_{\mathcal{K}_{2\rho(t)}} u(x,\tau)\,dx \right)^{\frac{p}{\lambda}} + \gamma \left(\frac{t}{\rho^p}\right)^{\frac{1}{2-p}}.$$

Here, the first estimate has a more evolved expression and can have both negative or positive powers on its right hand side. Once again, one can derive a decay of the solution toward extinction: consider $T^*>0$ to be a time of extinction of a nonnegative solution u and p in the supercritical range. Then we can evaluate the decay toward extinction of the whole solution: in the standard anisotropic geometry the decay rate has the more complex form, being $\lambda_i = N(p_i-2)+p$ and $\lambda = N(p-2)+p$,

$$\|u(\cdot,t)\|_{\infty,\mathbb{K}_\rho} \le \gamma \sum_{i=1}^{N} \left(\frac{T^*-t}{\rho^p}\right)^{\frac{\lambda_i}{(2-p_i)\lambda}} + \gamma \sum_{i=1}^{N} \left(\frac{T^*-\tau}{\rho^p}\right)^{\frac{1}{2-p_i}},$$

being clear that the extinction rate depends on the smallness of T^*-t and the maximum of the exponents in the sum. On the other hand, in the intrinsic anisotropic geometry the decay profile of extinction is the same as the one to the p-Laplacian,

$$\|u(\cdot,t)\|_{\infty,\mathcal{K}_\rho(T^*-t)} \le \gamma \left(\frac{T^*-t}{\rho^p}\right)^{\frac{1}{2-p}}.$$

A similar path can be followed for the class of anisotropic fast diffusion equations (18)-(19). We start by fixing two positive number ρ and t for which we consider two distinct space geometries ($a>0$):

- intrinsic anisotropic geometry

$$\mathcal{K}_{a\rho} = \prod_{i=1}^{N} \left\{ |x_i| < \left(\frac{t}{\rho^2}\right)^{\frac{m_i-m}{2(1-m)}} a\rho \right\}, \qquad m = \frac{\sum_{i=1}^{N} m_i}{N},$$

- standard geometry $\quad K_{a\rho} = \{|x| < a\rho\}$

Theorem 2.3. [Harnack-type estimates: intrinsic anisotropic geometry] *Let u be a nonnegative, local weak solution to* (18)-(19) *in* Ω_T *and* $\lambda = N(m-1)+2$.

(i) L^1-L^1 **estimate.** *There exists a positive constant γ, depending on N, C_o, C_1, m_N, such that*

$$\sup_{0\leq\tau\leq t} \int_{\mathcal{K}_\rho} u(x,\tau)\ dx \leq \gamma \left\{ \inf_{0\leq\tau\leq t} \int_{\mathcal{K}_{2\rho}} u(x,\tau)\ dx + \left(\frac{t}{\rho^\lambda}\right)^{\frac{1}{1-m}} \right\}. \tag{22}$$

(ii) L^1-L^∞ **estimate.** *Consider in addition u to be locally bounded and $m > N-2/N$. Then, there exists a positive constant γ, depending on N, C_o, C_1, m_i, such that*

$$\sup_{\mathcal{K}_{\rho/2}\times[t/2,t]} u \leq \gamma\ t^{-\frac{N}{\lambda}} \left(\inf_{0\leq\tau\leq t} \int_{\mathcal{K}_{2\rho}} u(x,\tau)\ dx \right)^{2/\lambda} + \gamma \left(\frac{t}{\rho^2}\right)^{\frac{1}{1-m}}. \tag{23}$$

This last Harnack-type inequality leads to the decay toward extinction. In fact, let $m > (N-2)/N$ and $T^\star$ be the finite time of extinction (proved to exist in [20]); from (23) one gets

$$||u(\cdot,t)||_{L^\infty(\mathcal{K}_\rho)} \leq \gamma \left(\frac{T^\star - t}{\rho^2}\right)^{\frac{1}{1-m}}, \qquad \text{for all} \quad T^\star/2 < t < T^\star. \tag{24}$$

Although estimate (24) is analogous to the one obtained for the (isotropic) porous medium type equation, which *per se* is a quite interesting and (at first) important feature, there is again a setback. When t grows closer to the extinction time $T^\star$ the intrinsic anisotropic cube changes its shape: in some directions it flattens while in the other directions it becomes larger. Similar results were derived when working with standard cubes.

Theorem 2.4. [Harnack-type estimates: standard geometry] *Let u be a nonnegative, local weak solution to* (18)-(19) *in* Ω_T, $\lambda = N(m-1)+2$ *and* $\lambda_i = N(m_i-1)+2$, $i = 1, \cdots, N$.

(i) L^1-L^1 **estimate.** *There exists a positive constant γ, depending on N, C_o, C_1, m_1, such that*

$$\sup_{0\leq\tau\leq t} \int_{K_\rho} u(x,\tau)\ dx \leq \gamma \left\{ \inf_{0\leq\tau\leq t} \int_{K_{2\rho}} u(x,\tau)\ dx + \sum_{i=1}^{N} \left(\frac{t}{\rho^{\lambda_i}}\right)^{\frac{1}{1-m_i}} \right\} \tag{25}$$

(ii) L^1-L^∞ **estimate.** *Consider in addition* u *to be locally bounded and* $m > N - 2/N$. *There exists a positive constant* γ*, depending on* N, C_o, C_1, m_i*, such that*

(26)
$$\sup_{K_{\rho/2}\times[t/2,t]} u \leq \gamma\, t^{-\frac{N}{\lambda}} \left(\inf_{0\leq\tau\leq\rho} \int_{K_{2\rho}} u(x,\tau)\ dx \right)^{2/\lambda} + \gamma \sum_{i=1}^{N} \left(\frac{t}{\rho^2}\right)^{\frac{1}{1-m_i}} + \gamma \sum_{i=1}^{N} \left(\frac{t}{\rho^2}\right)^{\frac{\lambda_i}{\lambda(1-m_i)}} .$$

From this L^1-L^∞ estimate and when considering $m_1 > (N-2)/N$, we get the decay rate of extinction (for all $T^\star/2 < t < T^\star$), distinguishing the cases

(a) when $(T^\star - t)/\rho^2 \leq 1$,

$$||u(\cdot,t)||_{L^\infty(K_\rho)} \leq \gamma \left(\frac{T^\star - t}{\rho^2}\right)^{\frac{\lambda_1}{\lambda(1-m_1)}} ;$$

(b) if otherwise $(T^\star - t)/\rho^2 \geq 1$,

$$||u(\cdot,t)||_{L^\infty(K_\rho)} \leq \gamma \left(\frac{T^\star - t}{\rho^2}\right)^{\frac{\lambda_N}{\lambda(1-m_N)}} .$$

3. Final Remarks and Open problems

The curtains close on the scenario of the basic regularity (such as Hölder continuity and Harnack inequality) for anisotropic operators as (13)-(14) with many unsolved questions. The Hölder continuity for the bounded solutions and the pointwise Harnack inequality for nonnegative solutions are still open problems.

The solutions to equations such as (13)-(14), even if possessing a modest degree of regularity, describe very interesting phenomena. For instance, consider the aforementioned phenomenon of the finite speed of propagation. The novelty related to the anisotropic phenomenon (13) is that, when the initial datum u_0 is such that its support is compact only along some coordinates, i.e. $\operatorname{supp} u_0 \subset \mathbb{R}^{N-M} \times \{(x_{N-1}, \ldots, x_N) : |x_i| < R_0\}$, with $M < N$, the evolution of u_0 along the flow keeps the property of being compactly supported along theses M components (see [39] for more details). Moreover, as claimed in [4], a very interesting feature is that the phenomenon of extinction in finite time pertains to the whole range $1 < p < 2$, meaning that some p_is may be in the degenerate range. As specified there, the alternative *finite speed of propagation/ vanishing in a finite time* as linked to *degenerate/singular* equations is no longer valid, while being replaced by conditions more similar to those known for diffusion–absorption evolution. This property is also enjoyed by the solutions to equations like (14), where the existence of a finite time of extinction is verified in the full range $0 < m < 1$, being $m = \sum_{i=1}^{N} m_i/N$ (see [20]). Another interesting feature related to (14) is that bounded solutions are continuous (as shown in [52]), however an estimate on the modulus of continuity is missing, leaving again an open door to future investigation.

One possible approach to cope with these (and even other) difficulties and constraints might be to look for new methods in the theory of regularity that dispense with the dichotomy *singular/degenerate*, as in the case of elliptic equations.

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