A DENSITY RESULT ON A *BV*-TYPE SPACE ON CARNOT GROUPS

UN RISULTATO DI DENSITÀ IN UNO SPAZIO DI TIPO *BV* IN GRUPPI DI CARNOT

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Abstract. In the setting of Carnot groups (connected, simply connected and stratified Lie groups), we prove a density result for a BV-type space previously introduced in [3]. In addition, we relate the dual of this BV-type space with the dual of the well known space of functions of intrinsic bounded variation. These results extend to the setting of Carnot groups some properties studied by Phuc e Torres in [22] and [23] in the Euclidean setting.

Sunto. Si prova un risultato di densità per uno spazio di tipo BV nell'ambito dei gruppi di Carnot (gruppi di Lie connessi, semplicemente connessi e stratificati) già introdotto in [3]. Come conseguenza di questo risultato di densità si mettono in relazione lo spazio delle funzioni a variazione (intrinseca) limitata con il duale di questo spazio. Questi risultati estendono al caso dei gruppi di Carnot alcune proprietà studiate in ambito euclideo da Phuc e Torres in [22] and [23]

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1. Introduction

A Carnot group $\mathbb{G}$ of step κ is a connected, simply connected Lie group whose Lie algebra $\mathfrak{g}$ admits a step κ stratification, i.e. there exist linear subspaces $V_1, ..., V_\kappa$ such that

$$\mathfrak{g} = V_1 \oplus ... \oplus V_\kappa, \quad [V_1, V_i] = V_{i+1}, \quad V_\kappa \neq \{0\}, \quad V_i = \{0\} \text{ if } i > \kappa, \tag{1}$$

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where $[V_1, V_i]$ is the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$ with $X \in V_1$ and $Y \in V_i$. We denote by Q the *homogeneous dimension of* $\mathbb{G}$ defined by

$$Q := \sum_{i=1}^{\kappa} i \dim V_i. \tag{2}$$

The integer Q turns out to be the Hausdorff dimension of $\mathbb{G}$ when seen as a metric space (see precise definitions and properties of Carnot groups contained in Section 1).

In [3] the authors considered in the setting of Carnot groups the problem of studying distributions F for which there exists a continuous horizontal vector field Φ, vanishing at infinity, that solves the equation $\mathrm{div}_H \Phi = F$. The analogous problem for the Euclidean case has been considered by De Pauw and Torres in [10].

In [3] was introduced the space $BV^{Q/Q-1}(\mathbb{G})$, defined as the set of all functions in $L^{Q/Q-1}(\mathbb{G})$ whose distributional gradient (regarded as a measure) has finite total variation. One of the main feature of this space is that the BV-space in Carnot groups, first introduced e.g. by [14] and [16] and here denoted by $BV_H(\mathbb{G})$, is such that $BV_H(\mathbb{G}) \hookrightarrow BV_H^{Q/Q-1}(\mathbb{G}) \subset BV_{H,loc}(\mathbb{G})$ (see Section 2 below). In [3] it was also studied a closed subspace of the dual space of $BV^{Q/Q-1}(\mathbb{G})$, denoted by $\mathbf{Ch}_0(\mathbb{G})$, and it was proved that its dual is isomorphic to $BV^{Q/Q-1}(\mathbb{G})$ and that the equation $\mathrm{div}_H \Phi = F$ admits as a solution a continuous horizontal vector field Φ vanishing at infinity if and only if $F \in \mathbf{Ch}_0(\mathbb{G})$.

In Phuc-Torres [22] was shown that there is a connection between the problem of characterizing the dual of BV and solving the equation $\mathrm{div}\Phi = F$. Since in [3] the dual space of $BV^{Q/Q-1}(\mathbb{G})$ is connected with the study of the solvability of the equation $\mathrm{div}_H \Phi = F$ in this note we want study some more properties of $BV^{Q/Q-1}(\mathbb{G})$. The main results of this paper are contained in Section 3, were we prove that the spaces $BV_H(\mathbb{G})^*$ and $\big(BV_H^{Q/Q-1}(\mathbb{G})\big)^*$ are isometrically isomorphic.

The paper is organized as follows. In the Section 2 we recall some basic facts about Carnot groups and in Section 3 we collect the main results concerning the space BV in a Carnot group $\mathbb{G}$. In addition we remind some result presented in [3] about space $BV^{Q/Q-1}(\mathbb{G})$. The main result of this note is contained in Section 4, where we prove that the space of bounded BV functions with compact support is dense in $BV^{Q/Q-1}$. Thanks to this result we are able to prove that there exists an isometric isomorphism between the dual of BV and the dual of $BV^{Q/Q-1}(\mathbb{G})$.

2. A few facts about Carnot groups

The definition of Carnot group $\mathbb{G}$ as already given above. With the same notation, the exponential map is a one to one map from $\mathfrak{g}$ onto $\mathbb{G}$. Using *exponential coordinates*, we identify a point $p \in \mathbb{G}$ with the N-tuple $(p_1, \dots, p_N) \in \mathbb{R}^N$ and we identify $\mathbb{G}$ with $(\mathbb{R}^N, \cdot)$ where the explicit expression of the group operation $\cdot$ is determined by the Campbell-Hausdorff formula (see, e.g., [11]). In exponential coordinates the unit element e of $\mathbb{G}$ is $e = (0, \dots, 0)$.

The first layer V_1 will be called *horizontal layer*; a left-invariant vector field in V_1, identified with a differential operator, will be called an *horizontal deerivative*.

From now on, we shall denote by $\{X_1, \dots, X_m\}$ a basis of V_1.

The N-dimensional Lebesgue measure $\mathcal{L}^N$, is the Haar measure of the group $\mathbb{G}$. For any $\lambda > 0$, the *dilation* $\delta_\lambda : \mathbb{G} \to \mathbb{G}$, is defined as

$$\delta_\lambda(x_1, ..., x_N) = (\lambda^{d_1} x_1, ..., \lambda^{d_N} x_N), \tag{3}$$

where $d_i \in \mathbb{N}$ is called the *homogeneity of the variable* x_i in $\mathbb{G}$ (see [11] Chapter 1). The homogeneous dimension of $\mathbb{G}$ is defined in (2) We shall assume that $Q \geq 3$.

As customary, we also fix a smooth homogeneous norm $\|\cdot\|$ in $\mathbb{G}$ (see [25], p. 638) such that the gauge distance $d(x,y) := \|y^{-1} \cdot x\|$ turns out to be a left invariant distance in $\mathbb{G}$, which is in fact equivalent to the "Carnot-Carathéodory distance" (see [1]). We set

$$B(x,r) := \{y \in \mathbb{G};\ d(x,y) < r\}$$

to denote the open r-ball centered at $x \in \mathbb{G}$.

Following e.g. [11], we can define a group convolution in $\mathbb{G}$: if, for instance, $f \in \mathcal{D}(\mathbb{G})$ and $g \in L^1_{\mathrm{loc}}(\mathbb{G})$, we set

$$f * g(p) := \int f(q) g(q^{-1} \cdot p)\, dq \quad \text{for } q \in \mathbb{G}. \tag{4}$$

If $f:\mathbb{G}\longrightarrow\mathbb{R}$, we denote by ${}^{\mathrm{v}}f$ the function given by ${}^{\mathrm{v}}f(x):=f(x^{-1})$. We remind that, if (say) g is a smooth function and P is a left invariant differential operator, then

$$P(f*g)=f*Pg.$$

We remind also that the convolution is again well defined when $f,g\in\mathcal{D}'(\mathbb{G})$, provided at least one of them has compact support. In this case the following identities hold

$$\langle f*g|\phi\rangle=\langle g|{}^{\mathrm{v}}f*\phi\rangle \quad \text{and} \quad \langle f*g|\phi\rangle=\langle f|\phi*{}^{\mathrm{v}}g\rangle \tag{5}$$

for any test function ϕ, where we use the notation $\langle\cdot|\cdot\rangle$ for the duality between $\mathcal{D}'$ and $\mathcal{D}$ (remeber that if $T\in\mathcal{D}'(\mathbb{G})$, then ${}^{\mathrm{v}}T$ is the distribution defined by $\langle {}^{\mathrm{v}}T|\phi\rangle:=\langle T|{}^{\mathrm{v}}\phi\rangle$ for any test function ϕ).

The subbundle $H\mathbb{G}$ of the tangent bundle $T\mathbb{G}$ spanned by the vector fields $\{X_1,\dots,X_m\}$ is called the *horizontal bundle*.

A subriemannian structure is defined on $\mathbb{G}$ once one endows each fiber $H_x\mathbb{G}$ of the horizontal bundle $H\mathbb{G}$ with a scalar product. From now on, we shall assume that, at any $x\in\mathbb{G}$, the basis $\{X_1(x),\dots,X_m(x)\}$ is orthonormal (under the chosen scalar product).

Now, let $f:\mathbb{G}\longrightarrow\mathbb{R}$ be a smooth function, say $f\in C^\infty(\mathbb{G})$. The horizontal gradient of f is the horizontal vector field D_Hf that can be written, with respect to the the horizontal frame, as

$$D_Hf=(X_1f,...,X_mf).$$

Moreover, if $\Phi=(\phi_1,\dots,\phi_m)$ is a smooth horizontal vector field, say $\Phi\in C^\infty(\mathbb{G},H\mathbb{G})$, its horizontal divergence $\mathrm{div}_H\,\Phi$ is, by definition, the real valued function

$$\mathrm{div}_H\,\Phi:=\sum_{j=1}^m X_j\phi_j. \tag{6}$$

The same symbols D_H and div_H will be adopted later, when working with the *weak* horizontal gradient and divergence operators (intended in the sense of distributions).

Let $J:\mathbb{G}\longrightarrow\mathbb{R}$ be a mollifier (for the group structure), i.e., $J\in C_c^\infty(\mathbb{G})$, $J\geq 0$, $\mathrm{supp}\,J\subset\subset B(e,1)$, and $\int_{\mathbb{G}}J(x)\,dx=1$. Note that, if one starts from a standard mollifier J defined in $(\mathbb{R},+)$, then the function $J(\|x\|)$ turns out to be a mollifier in $\mathbb{G}$. Now, given a mollifier J, we define a family of approximations to the identity $\{J_\varepsilon\}_{\varepsilon>0}$ by setting

$$J_\varepsilon(x) := \varepsilon^{-Q} J(\delta_{1/\varepsilon} x)\,.$$

We remark explicitly that $J_\varepsilon(x) = {}^{\mathrm{v}}J_\varepsilon(x)$ for every $x \in \mathbb{G}$.

If $1 \le p < +\infty$ and $f \in L^p(\mathbb{G})$, then $J_\varepsilon * f \longrightarrow f$ in $L^p(\mathbb{G})$ as $\varepsilon \to 0$.

3. Some BV-type spaces in Carnot groups: $BV_H(\mathbb{G})$ and $BV_H^{Q/Q-1}(\mathbb{G})$

3.1. **Definitions and some properties of the space $BV_H(\mathbb{G})$.** First, we recall the definition of functions of intrinsic bounded variation, below denoted by BV_H-functions. There is a wide letterature on BV_H-functions in Carnot groups for which we refer, for instance, to [14], [16], [27], and references therein. Here we limit ourselves to recall the main results.

Let $\Omega \subseteq \mathbb{G}$ be an open set. Recall that a function $f : \Omega \longrightarrow \mathbb{R}$ is said to have *intrinsic bounded variation in* Ω, and in this case we write $f \in BV_H(\Omega)$, if $f \in L^1(\Omega)$ and

$$\|D_H f\|(\Omega) := \sup \left\{ \int_\Omega f \,\mathrm{div}_H \Phi \; dx \;:\; \Phi \in \mathcal{D}(\Omega, H\Omega),\; \|\Phi\|_\infty \le 1 \right\} < +\infty,$$

where $\|\Phi\|_\infty = \sup\{|\Phi(x)|_x \,:\, x \in \Omega\}$.

The quantity $\|D_H f\|(\Omega)$ represents the *total horizontal variation* (or, *H-variation*) of the distributional horizontal gradient $D_H f$ in Ω. Unless otherwise stated, we shall henceforth assume that $\Omega = \mathbb{G}$. In this case, the total H-variation of $D_H f$ in $\mathbb{G}$ will be simply denoted as $\|D_H f\|$.

This definition can easily be localized. To this aim, let $f \in L^1_{loc}(\Omega)$ and assume that $\|D_H f\|(V) < +\infty$ for every open subset $V \subset\subset \Omega$. In this case, we set $f \in BV_{H,loc}(\Omega)$ to denote the space of functions of locally bounded H-variation in Ω.

The (total) H-variation is lower semicontinuous with respect to the L^1_{loc}-convergence and follows because the map $f \mapsto \|D_H f\|(\cdot)$ is the supremum of a family of L^1-continuous functionals. Hence, if $\Omega \subseteq \mathbb{G}$ is an open set and $\{f_k\}_{k\in\mathbb{N}}$ be a sequence in $BV_H(\Omega)$ such that $f_k \longrightarrow f$ in $L^1_{\mathrm{loc}}(\Omega)$ as $k \to +\infty$. Then

$$\|D_H f\|(\Omega) \le \liminf_{k\to+\infty} \|D_H f_k\|(\Omega).$$

If $E \subseteq \mathbb{G}$ is a Borel set, we set $P_H(E) := \|D_H \chi_E\|$, where χ_E is the characteristic function of E. More generally, if $\Omega \subseteq \mathbb{G}$ is an open set, we set $P_H(E, \Omega) := \|D_H \chi_E\|(\Omega)$. The quantities just defined are the H-perimeter of E in $\mathbb{G}$ and in Ω, respectively.

The next result is the coarea formula for functions of bounded H-variation (see, e.g., [14], [16]).

Theorem 3.1 (Coarea formula). *Let $f \in BV_H(\Omega)$ and set $E_t := \{x \in \Omega : f(x) > t\}$. Then, E_t has finite H-perimeter in Ω for a.e. $t \in \mathbb{R}$ and the following formula holds*

$$\|D_H f\|(\Omega) = \int_{\mathbb{R}} P_H(E_t, \Omega)\, dt. \tag{7}$$

Conversely, if $f \in L^1(\Omega)$ and $\int_{\mathbb{R}} P_H(E_t, \Omega)\, dt < +\infty$, then $f \in BV_H(\Omega)$.

Remark 3.1. *Let $f \in BV_H(\Omega)$, $t \in \mathbb{R}$, and consider the function $g_t := \max\{f, t\}$. As in the Euclidean case (see, e.g., [17], p. 340), a useful consequence of the coarea formula is that $g_t \in BV_H(\Omega)$ and that $\|D_H g_t\|(\Omega) = \|D_H f\|(E_t)$.*

Finally, we have to recall the following fundamental inequality already discussed in Remark 2.11 of [3].

Remark 3.2 (Gagliardo-Nirenberg inequality). *As is well-known, the classical Gagliardo-Nirenberg inequality has been generalized to Carnot groups by many authors (and with different aims); see, e.g., [9], [12], [13], [16], [19], [21]. More precisely, if $f \in \mathcal{D}(\mathbb{G})$, the inequality states that there exists a "geometric" constant $C_{GN} = C_{GN}(Q, \mathbb{G})$ such that*

$$\|f\|_{L^{Q/Q-1}} \leq C_{GN} \|D_H f\|_{L^1}. \tag{8}$$

The inequality (8) extends to functions in $BV_H(\mathbb{G})$ having compact support.

By adapting the classical Riesz representation theorem to our setting, one can prove the following "structure theorem".

Theorem 3.2. *If $f \in BV_H(\mathbb{G})$, then $\|D_H f\|(\cdot)$ is a Radon measure on $\mathbb{G}$. In addition, there exists a bounded $\|D_H f\|$-measurable horizontal section $\sigma_f : \mathbb{G} \to H\mathbb{G}$ such that $|\sigma_f(x)|_x = 1$ for $\|D_H f\|$-a.e. $x \in \mathbb{G}$, and the following holds*

$$\int_{\mathbb{G}} f \, \mathrm{div}_H \Phi \; dx = -\int_{\mathbb{G}} \langle \Phi, \sigma_f \rangle \, d\|D_H f\| \qquad \forall\, \Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G}). \tag{9}$$

Moreover, $\mu = \sigma_f \, \|D_H f\|$ is a vector measure in $H\mathbb{G}$ (see Section 2 in [3] for more details). Writing σ_f with respect to the horizontal frame as $\sigma_f = \sum_{i=1}^m \sigma_{f,i} X_i$, where the components $\sigma_{f,i} : \mathbb{G} \longrightarrow \mathbb{R}$ $(i = 1, \dots, m)$ are bounded measurable functions, we have $\mu = (\sigma_{f,1}, \dots, \sigma_{f,m})\|D_H f\|$. We shall set $[D_H f] := \mu$. Thus, (9) becomes

$$\int_{\mathbb{G}} f \, \mathrm{div}_H \Phi \, dx = - \int_{\mathbb{G}} \langle \Phi, d\,[D_H f] \rangle. \tag{10}$$

Remark 3.3 (product rule: a particular case)**.** *Let $f \in BV_{H,loc}(\mathbb{G})$ and $\phi \in \mathcal{D}(\mathbb{G})$. Then, we claim that*

$$D_H(\phi f) = \phi D_H f + f D_H \phi \tag{11}$$

as measures.

Proof. We first show that the equality holds in the sense of distribution. To prove this claim, we argue exactly as in [28], Proposition 5.3.2. We consider the mollifier J_ε and we set $f_\varepsilon = J_\varepsilon * f$. Since $\phi \, f_\varepsilon \in \mathcal{D}(\mathbb{G})$ it holds

$$D_H(\phi f_\varepsilon) = \phi D_H f_\varepsilon + f_\varepsilon D_H \phi \, .$$

Since $f_\varepsilon \to f$ in L^1, in particular f_ε tends to f as distributions therefore also $D_H f_\varepsilon \to D_H f$ in $\mathcal{D}'(\mathbb{G})$. Since $\phi \in \mathcal{D}(\mathbb{G})$, also $\phi \, D_H f_\varepsilon \to \phi \, D_H f$ in $\mathcal{D}'(\mathbb{G})$ and $D_H \phi \, f_\varepsilon \to D_H \phi \, f$ in $\mathcal{D}'(\mathbb{G})$. Finally, with the observation that $\phi f_\varepsilon \to \phi f$ in $\mathcal{D}'(\mathbb{G})$ which implies that $D_H(\phi f_\varepsilon) \to D_H(\phi f)$ in $\mathcal{D}'(\mathbb{G})$, the conclusion of the first claim follows. To conclude the proof we need just to notice that the right and the left hand side of (11) are two distribuition of order 0 that concide on $\mathcal{D}(\mathbb{G})$, hence they concide as measures as well.

□

3.2. The space $BV_H^{Q/Q-1}(\mathbb{G})$.

We introduce another intrinsic BV_H-type space, which is a subspace of $L^{Q/Q-1}(\mathbb{G})$. In the Euclidean setting this space was introduced and studied in [10] and in the setting of Carnot groups it has been introduced in [3].

Definition 3.1. *The space $BV_H^{Q/Q-1}(\mathbb{G})$ is the set of functions $f \in L^{Q/Q-1}(\mathbb{G})$ whose distributional gradient $D_H f$ is a finite vector measure, i.e.,*

$$\|D_H f\| := \|D_H f\|(\mathbb{G}) = \sup \left\{ \int_{\mathbb{G}} f \, \mathrm{div}_H \Phi \, dx \; : \; \Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G}), \; \|\Phi\|_\infty \le 1 \right\} < +\infty.$$

The space $BV_H^{Q/Q-1}(\mathbb{G})$ is a Banach space when endowed with the norm

$$\|f\|_{L^{Q/Q-1}} + \|D_H f\|.$$

Note also that $BV_H^{Q/Q-1}(\mathbb{G}) \subset BV_{H,loc}(\mathbb{G})$.

In [3] several properties of $BV_H^{Q/Q-1}(\mathbb{G})$ have been proved. Among them, the lower semicontinuity of the H-variation with respect to the weak convergence in $L^{Q/Q-1}(\mathbb{G})$ (see [3], Theorem 3.2): if $\{f_k\}_{k\in\mathbb{N}}$ is a sequence in $BV_H^{Q/Q-1}(\mathbb{G})$ such that $f_k \rightharpoonup f$ in $L^{Q/Q-1}(\mathbb{G})$ as $k \to +\infty$, then

$$\|D_H f\| \leq \liminf_{k\to+\infty} \|D_H f_k\|. \tag{12}$$

Also an approximation result for $BV_H^{Q/Q-1}(\mathbb{G})$ is proved in [3] (Theorem 3.3 threin) which enable to obtain as a consequence the following inequality

Proposition 3.1 (see Corollary 3.4 in [3])**.** *Let* $f \in BV_H^{Q/Q-1}(\mathbb{G})$*. Then*

$$\|f\|_{L^{Q/Q-1}} \leq C_{GN} \|D_H f\|. \tag{13}$$

By (13), if $f \in BV_H^{Q/Q-1}(\mathbb{G})$ it follows that the H-variation $\|D_H f\|$ is an equivalent norm to $\|f\|_{L^{Q/Q-1}} + \|D_H f\|$. For this reason, in the sequel the H-variation will be taken as a norm and we shall set

$$\|f\|_{BV_H^{Q/Q-1}} := \|D_H f\|.$$

Note also that (13) immediatly implies the continuous embedding

$$BV_H(\mathbb{G}) \hookrightarrow BV_H^{Q/Q-1}(\mathbb{G}). \tag{14}$$

4. Main results

This section contains the main result of this note which is a new density result related to the space $BV_H^{Q/Q-1}$. As a corollary, we deduce a duality property of the space $BV_H(\mathbb{G})$: we prove that the spaces $BV_H(\mathbb{G})^*$ and $\left(BV_H^{Q/Q-1}(\mathbb{G})\right)^*$ are isometrically isomorphic.

First, we consider the space of bounded functions with compact support that are in $BV_H(\mathbb{G})$, namely

$$BV_{H,c}^{\infty}(\mathbb{G}) := BV_{H,c}(\mathbb{G}) \cap L^{\infty}(\mathbb{G}),$$

where $BV_{H,c}(\mathbb{G})$ denotes the set of functions in $L^1_c(\mathbb{G})$ (i.e., the space of functions in $L^1(\mathbb{G})$ with compact support) with bounded H-variation.

The result below extends Theorem 3.1 in [23] (compare also with [22], Lemma 3.4).

Theorem 4.1. *The space $BV^{\infty}_{H,c}(\mathbb{G})$ is dense in $BV_H^{Q/Q-1}(\mathbb{G})$.*

Proof. We will show that for any $f \in BV_H^{Q/Q-1}(\mathbb{G})$ there is a sequence of functions $\{f_k\}_{k\in\mathbb{N}} \subset BV^{\infty}_{H,c}(\mathbb{G})$ such that

$$\lim_{k\to+\infty} \|f_k - f\|_{BV_H^{Q/Q-1}} = 0.$$

Step 1. *We claim that the space $BV_{H,c}(\mathbb{G})$ is dense in $BV_H^{Q/Q-1}(\mathbb{G})$, with respect to the topology induced by the norm $\|\cdot\|_{BV_H^{Q/Q-1}}$.*

Let $\{g_k\}_{k\in\mathbb{N}} \subset \mathcal{D}(\mathbb{G})$ be a sequence of cut-off functions such that:

$$\chi_{B(e,k)} \leq g_k \leq \chi_{B(e,2k)}, \qquad |D_H g_k| \leq \frac{C}{k} \qquad \forall\, k \in \mathbb{N}. \tag{15}$$

Clearly g_k has compact support and $g_k(x) \longrightarrow 1$ as $k \to +\infty$ for every $x \in \mathbb{G}$.

Let $f \in BV_H^{Q/Q-1}(\mathbb{G})$ (and note that $fg_k \longrightarrow f$ in $L^{Q/Q-1}(\mathbb{G})$ as $k \to +\infty$). Since $BV_H^{Q/Q-1}(\mathbb{G}) \subset BV_{H,loc}(\mathbb{G})$, by applying the formula in Remark 3.3 we have $D_H(\phi f) = \phi D_H f + f D_H \phi$ (in the distributional sense and as measures). Thus, we get that $fg_k \in BV_H(\mathbb{G}) \subset BV_H^{Q/Q-1}(\mathbb{G})$. Now, if $\Phi \in \mathcal{D}(\mathbb{G}, H\mathbb{G})$, $\|\Phi\|_\infty \leq 1$, we can estimate the functional $\int_{\mathbb{G}} \langle \Phi, d[D_H(fg_k - f)]\rangle$ as follows:

$$\begin{aligned}
\left|\int_{\mathbb{G}} \langle \Phi, d[D_H(fg_k - f)]\rangle\right| &\leq \int_{\mathbb{G}} |g_k - 1||\Phi| d\|D_H f\| + \int_{\mathrm{supp}(D_H g_k)} |\Phi||f||D_H g_k| dx \\
\leq\; & \int_{\mathbb{G}} |g_k - 1| d\|D_H f\| + \frac{C}{k}\int_{B(e,2k)\setminus B(e,k)} |f| dx \qquad \text{(by (15))} \\
\leq\; & \int_{\mathbb{G}} |g_k - 1| d\|D_H f\| + \frac{C}{k}\left(\int_{B(e,2k)\setminus B(e,k)} |f|^{Q/Q-1} dx\right)^{Q-1/Q} |B(e,2k)\setminus B(e,k)|^{1/Q} \\
\leq\; & \int_{\mathbb{G}} |g_k - 1| d\|D_H f\| + C\left(\int_{B(e,2k)\setminus B(e,k)} |f|^{Q/Q-1} dx\right)^{Q-1/Q}.
\end{aligned}$$

By the arbitrariness of Φ, taking the supremum on the left-hand side we obtain

$$\|D_H(fg_k - f)\| \leq \int_{\mathbb{G}} |g_k - 1| d\|D_H f\| + C\left(\int_{B(e,2k)\setminus B(e,k)} |f|^{Q/Q-1} dx\right)^{Q-1/Q}.$$

Finally, since $f \in L^{Q/Q-1}(\mathbb{G})$, by the dominated convergence theorem both terms on the right-hand side vanish as $k \to +\infty$. Hence

$$\lim_{k\to+\infty} \|D_H(fg_k - f)\| = 0, \tag{16}$$

which shows the initial claim.

Step 2. *We claim that the space $BV^{\infty}_{H,c}(\mathbb{G})$ is dense in $BV_{H,c}(\mathbb{G})$, with respect to the topology induced by the norm $\|\cdot\|_{BV_H^{Q/Q-1}}$.*

Let $h \in BV_{H,c}(\mathbb{G})$ and let us first assume that $h \geq 0$. In order to prove the claim, we consider the truncation of h defined, for any $x \in \mathbb{G}$, by

$$h_j(x) := \begin{cases} j & \text{if} \quad h(x) > j \\ h(x) & \text{if} \quad 0 \leq h(x) \leq j \end{cases} \qquad \forall\, j \in \mathbb{N}.$$

By the coarea formula (7), we have

$$\begin{aligned} \|D_H(h - h_j)\| &= \int_0^{+\infty} P_H(\{x \in \mathbb{G} \,:\, h(x) - h_j(x) > t\})\, dt \\ &= \int_0^{+\infty} P_H(\{x \in \mathbb{G} \,:\, h(x) - j > t\})\, dt \\ &= \int_0^{+\infty} P_H(\{x \in \mathbb{G} \,:\, h(x) > j + t\})\, dt \\ &= \int_j^{+\infty} P_H(\{x \in \mathbb{G} \,:\, h(x) > s\})\, ds. \end{aligned}$$

But since $h \in BV_{H,c}(\mathbb{G})$, we have $\int_{\mathbb{R}} P_H(\{x \in \mathbb{G} : h(x) > s\})\, ds < +\infty$. Hence, by the dominated convergence theorem, we infer that

$$\lim_{j\to+\infty} \|D_H(h - h_j)\| = 0. \tag{17}$$

In other words, if $h \geq 0$, we have shown that there exists $\{h_j\}_{j\in\mathbb{N}} \subset BV^{\infty}_{H,c}(\mathbb{G})$ approximating h in the topology induced by the norm $\|\cdot\|_{BV_H^{Q/Q-1}}$.

The general case can be achieved as follows. Let $h \in BV_{H,c}(\mathbb{G})$ and let us write $h = h^+ - h^-$, where $h^{\pm} \geq 0$ denote the positive/negative parts of h. Using Remark 3.1 it follows that $h^{\pm} \in BV_{H,loc}(\mathbb{G})$ and $\|D_H h^{\pm}\|(\Omega) \leq \|D_H h\|(\Omega)$ for every open set $\Omega \subset\subset \mathbb{G}$. Therefore $\|D_H h^{\pm}\| \le \|D_H h\| < +\infty$, which implies also that $h^{\pm} \in BV_{H,c}(\mathbb{G})$. Moreover, we set

$$h_j^+ := (h^+)_j, \qquad h_j^- := (h^-)_j \qquad \forall\, j \in \mathbb{N}.$$

From what we have seen above, $h_j^+ - h_j^- \in BV^{\infty}_{H,c}(\mathbb{G})$ and we have

$$\begin{aligned} \left\|D_H[h-(h_j^+ - h_j^-)]\right\| &= \left\|D_H h^+ - D_H h^- - D_H h_j^+ + D_H h_j^-\right\| \\ &\le \left\|D_H(h^+ - h_j^+)\right\| + \left\|D_H(h^- - h_j^-)\right\| \xrightarrow[j\to+\infty]{} 0, \end{aligned}$$

where we have used (17). This shows the initial claim.

Combining Step 1 and Step 2, the proof is complete. □

The following corollary extends to our setting an interesting isomorphism result contained in [23].

Corollary 4.1. *Let*

$$S : \left(BV_H^{Q/Q-1}(\mathbb{G})\right)^* \longrightarrow BV_H(\mathbb{G})^*, \qquad S(T) := T|_{BV_H(\mathbb{G})},$$

where $\cdot|_{BV_H(\mathbb{G})}$ *denotes the restriction to* $BV_H(\mathbb{G}) \subset BV_H^{Q/Q-1}(\mathbb{G})$.

Then, the map S is an isometric isomorphism.

Proof. We start by proving that S is injective. Let $T \in \left(BV_H^{Q/Q-1}(\mathbb{G})\right)^*$ be such that $S(T) = 0$. Then, by definition, $T|_{BV_H(\mathbb{G})} = 0$. Since $BV^{\infty}_{H,c}(\mathbb{G}) \subset BV_H(\mathbb{G})$, it follows that $T|_{BV^{\infty}_{H,c}(\mathbb{G})} = 0$. But since the space $BV^{\infty}_{H,c}(\mathbb{G})$ is dense in $BV_H^{Q/Q-1}(\mathbb{G})$ and T is continuous, it follows that for any $f \in BV_H^{Q/Q-1}(\mathbb{G})$ there exists $\{f_k\}_{k\in\mathbb{N}} \subset BV^{\infty}_{H,c}(\mathbb{G})$ such that $\|f_k - f\|_{BV_H^{Q/Q-1}} \longrightarrow 0$ as $k \to +\infty$. Hence $0 = T(f_k) \xrightarrow[k\longrightarrow+\infty]{} T(f)$ and this shows that $T(f) = 0$ for any $f \in BV_H^{Q/Q-1}(\mathbb{G}) = 0$, which means that $T = 0$.

It remains us to show that S is surjective. To this aim, let us take $T \in BV_H(\mathbb{G})^*$. Note, in particular, that $T|_{BV^{\infty}_{H,c}(\mathbb{G})}$ is a continuous linear functional. Now we use that $BV^{\infty}_{H,c}(\mathbb{G})$ is dense in $BV_H^{Q/Q-1}(\mathbb{G})$. More precisely, by the classical Continuous Linear Extension Theorem (see, e.g., [24]), there exists a continuous linear functional $\hat{T}$ defined on the whole $BV_H^{Q/Q-1}(\mathbb{G})$ that extends T. By its very definition, $S(\hat{T}) = \hat{T}|_{BV_H(\mathbb{G})} = T|_{BV_H(\mathbb{G})}$, and hence $S(\hat{T}) = T$. Moreover, the extended functional $\hat{T}$ preserves the norm, i.e., $\|\hat{T}\|_{\left(BV_H^{Q/Q-1}\right)^*} = \|T\|_{BV_H^*}$. Since $S(\hat{T}) = T$, we get that

$$\|\hat{T}\|_{\left(BV_H^{Q/Q-1}\right)^*} = \|S(\hat{T})\|_{BV_H^*},$$

which shows that S is an isometry.

□

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