REGULARITY RESULTS FOR ISOPERIMETRIC SETS WITH DENSITY

RISULTATI DI REGOLARITÀ PER INSIEMI ISOPERIMETRICI CON DENSITÀ

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Abstract. In this note, we present some recent regularity results for sets which minimize a weighted notion of perimeter under a weighted volume constraint. We focus on the case of two different densities which are merely α-Hölder continuous, and describe what are the main issues and techniques used in order to establish the optimal regularity $C^{1,\frac{\alpha}{2-\alpha}}$ for the reduced boundary of such sets.

Sunto. In questa nota, presentiamo alcuni recenti risultati di regolarità per insiemi che minimizzano una nozione pesata di perimetro sotto un vincolo di volume pesato. Ci focalizziamo sul caso di due densità diverse, che siano solo Hölderiane di ordine α e descriviamo quali sono le maggiori difficoltà e le tecniche usate per provare la regolarità ottimale $C^{1,\frac{\alpha}{2-\alpha}}$ per la frontiera ridotta di tali insiemi.

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1. Introduction

The aim of this Note is to describe some recent regularity results for isoperimetric sets with density and give some of the main ideas in their proofs.

Let us start by introducing the problem. Given two lower semi-continuous functions $f,\, h\colon \mathbb{R}^n \to (0,+\infty)$, the so-called *densities*, and an arbitrary measurable set $E \subset \mathbb{R}^n$ of locally finite perimeter, we define its weighted volume $V_f(E)$ via

$$V_f(E) := \int_E f(x)\, dx,$$

and its weighted perimeter $P_h(E)$ via

$$P_h(E) := \int_{\partial^* E} h(x)\, d\mathcal{H}^{n-1}(x),$$

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where $\partial^* E$ denotes the reduced boundary of E and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. The isoperimetric problem with density consists in searching for sets which minimize the weighted perimeter P_h under a weighted volume constraint, that is: For a fixed positive number m, we are interested in the minimization problem

$$\inf \big\{ P_h(E) \colon E \subset \mathbb{R}^n \text{ with } V_f(E) = m \big\}.$$

This is a generalization of the classical Euclidean isoperimetric problem, and it has attracted much interest in the last years. We refer the interested reader to some expository papers by F. Morgan [22, 23, 24] and the references therein.

For some specific choice of the densities, one can even hope to fully characterize the optimal sets. For instance when both densities are Gaussians, i.e. $h(x) = f(x) = e^{-|x|^2}$, it has been shown in [2, 5, 29] that isperimetric regions are half-spaces. A quantitative version of this result was established in [10]. The case of other radial densities has been also considered. When $f(x) = h(x) = |x|^p$, with $p > 0$, in [6, 15] it has been proved that isoperimetric sets are balls whose boundary contains the origin. This is a quite unexpected result, since, due to the symmetry of the problem, one could expect balls centered at the origin to be minimizers. When the perimeter density is again $h(x) = |x|^p$ but we consider Euclidean volume (i.e. $f \equiv 1$), balls centered at the origin are indeed optimal, as established in [4, 14, 7]. On the other hand, there are examples of nonradial densities (such as monomial-type densities) for which isoperimetric sets are balls centered at the origin. This was established in [8] under suitable concavity conditions on the single density $f = h$. A quantitative version of this result has been obtained recently in [11].

If, instead, we consider a generic density, it is not true in general that isoperimetric sets do exist and thus it becomes crucial to understand which assumptions on the densities ensure existence. We refer to [16, 25, 27] for some existence results and example of non-existence.

A second important issue, for generic densities, is the one related to the regularity properties of isoperimetric sets, in dependence of the regularity assumption on the density functions. Since this is the main object of this Note, let us describe more in detail, what are the main results in this regard.

A classical (and optimal) result for the case of a single density (i.e., $f = h$), under the assumption of quite high regularity, is the following ([21, Proposition 3.5 and Corollary 3.8]).

Theorem 1.1 ([21])**.** *Let $f = h$ be of class $C^{k,\alpha}(\mathbb{R}^n)$ for some $k \geq 1$ and $\alpha \in (0,1]$. Then the boundary of any isoperimetric set is of class $C^{k+1,\alpha}$, except for a singular set of Hausdorff dimension at most $n-8$.*

The request on the density to be at least C^1 (but also Lipschitz would be enough) is crucial in several steps, for example in such a situation one can use the Euler-Lagrange equation satisfied by a minimizer. When the densities are assumed to be just Hölder continuous, we cannot even write down the associated Euler–Lagrange equation and this is a first obstacle to obtain optimal regularity.

This case of densities with low regularity was consiedered only recently. In [12, 13] the case of a single density $f = h$ was studied and a first non-optimal result was established:

Theorem 1.2 ([12, 13])**.** *Let $f = h$ be of class $C^{0,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (0,1]$. Then, if E is an isoperimetric set, we have that $\partial E = \partial^* E$ up to $\mathcal{H}^{n-1}$-negligible sets, and $\partial^* E \in C^{1,\alpha/(2n(1-\alpha)+2\alpha)}$. If $n = 2$, we have that $\partial^* E \in C^{1,\alpha/(3-2\alpha)}$.*

Observe that the 2-dimensional result, contained in [13], slightly improves the result valid in any dimension since $\alpha/(3-2\alpha) > \alpha/(4-2\alpha)$.

More recently, in [28, Theorem C] the above regularity result valid in any dimension n was generalized to the case of two different densities. As we will explain later on in Section 2, the Hölder regularity of an optimal set only depends on the Hölder regularity of the density h (weighting the perimeter), while no regularity is needed on f.

Theorem 1.3 ([28])**.** *Let h be a density of class $C^{0,\alpha}(\mathbb{R}^n)$ for some $\alpha \in (0,1]$ and f be a locally bounded function. Then, if E is an isoperimetric set, we have that $\partial E = \partial^* E$ up to $\mathcal{H}^{n-1}$-negligible sets, and $\partial^* E \in C^{1,\alpha/(2n(1-\alpha)+2\alpha)}$.*

Observe that the Hölder exponent $\alpha/(2n(1-\alpha)+2\alpha) \to 1/2$ as $\alpha \to 1$ and thus it cannot be optimal (at least for α close to 1). This lack of optimality is due to the technique used in proving the result, and we will comment on this in the next Section.

Thus, the question about optimal regularity of isoperimetric sets for merely Hölder densities remained open and was solved only recently in [3], according to the following result:

Theorem 1.4 ([3])**.** *Let h be density of class $C^{0,\alpha}(\mathbb{R}^n)$ and f be a density of class $C^{0,\gamma}(\mathbb{R}^n)$ for some α and $\gamma \in (0,1)$. Then the boundary of any isoperimetric set is of class $C^{1,\alpha/(2-\alpha)}$, except for a singular set of Hausdorff dimension at most $n-8$.*

We observe that the Hölder exponent in the previous statement does not depend on the dimension and it improves the previous regularity results, indeed:

$$\frac{\alpha}{2-\alpha} > \begin{cases} \frac{\alpha}{2n(1-\alpha)+2\alpha} & \text{for } n \geq 2, \\ \frac{\alpha}{3-2\alpha} & \text{for } n = 2 \text{ and } f = h, \end{cases}$$

for each $\alpha \in (0,1)$. In particular, the expected asymptotic behavior $\alpha/(2-\alpha) \to 1$ as $\alpha \nearrow 1$ is achieved for all dimensions $n \geq 2$. In the last Section, we will show, with an explicit example, that the regularity established in Theorem 1.4 is indeed optimal. At first glance, the fact that we can just reach $C^{1,\alpha/(2-\alpha)}$ and not $C^{1,\alpha}$, differently from the differentiable setting in which we gain one order of regularity (see Theorem 1.1) could seem surprising. However, this behaviour is well-known in the classical regularity theory for the minimization of variational functionals of the form $\mathcal{F}(w) = \int_\Omega F(x, w, Dw)\, dx$, when only an α-Hölder continuity assumption is required on the maps $u \mapsto F(x, u, z)$ and $(x, u) \mapsto D_z F(x, u, z)$. In this case, the optimal regularity of minimizers is precisely $C^{1,\alpha/(2-\alpha)}$, see [26]. As a last remark on Theorem 1.4, we stress that, even if the final optimal regulairty only depends on α, we still need to require $f \in C^{0,\gamma}$ for some $\gamma \in (0, 1)$.

We discuss now, very breafly, the main ideas in the proofs of Theorems 1.3 and 1.4. A more detailed description will be given in the following sections.

The proof of Theorem 1.3 uses the classical regularity theory for ω-minimal sets, and the so-called $\varepsilon - \varepsilon^\beta$ property, first established in [12, Theorem B] for the case of a single density, and then generalized to the case of double density in [28]. Section 2 is devoted to describe such results.

Sections 3 and 4 focus on the proof of the optimal regularity result Theorem 1.4. The methods employed follows the so-called direct approach from classical regularity theory for minimization problems, see e.g. [18, 19]. The main steps are the following: one defines a suitable comparison problem by keeping the original density only for the volume constraint, and by freezing it for the perimeter. Via a combination of the initial regularity result for the minimizer of the comparison problem, suitable estimates on the Lagrange multiplier (coming from the volume constraint), and classical Schauder theory (applied to the associated Euler–Lagrange equation which does now indeed exist), the minimizer is then shown to have optimal decay estimates. Finally, the decay estimates of this comparison function are transferred to the minimizer of the original constrained minimization problem, thus completing the proof, via the Campanato's characterization of Hölder continuous functions.

We conclude this Introduction by setting the notations and giving some preliminaries.

In the following, we will denote by $x = (x', x_n)$ a point in $\mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}$. Given $r > 0$ and $x'_0 \in \mathbb{R}^{n-1}$, we denote by $B_r(x'_0)$ the ball in $\mathbb{R}^{n-1}$ centered at x'_0 and with radius r, and we write $B_r := B_r(0)$ for simplicity.

Given a function w defined on $\mathbb{R}^{n-1}$, we denote its mean integral on a certain measurable set $A \subset \mathbb{R}^{n-1}$ by

$$(w)_A := \frac{1}{|A|} \int_A w(x')\, dx',$$

where $|A|$ stands for the Lebesgue measure of A. In the particular case in which $A = B_r(x'_0)$ and it is clear from the context what is the center x'_0, we simply use the abbreviation $(w)_r$ instead of $(w)_{B_r(x'_0)}$.

For proving the optimal regularity result, we will use the following Campanato's characterization of Hölder functions.

Proposition 1.5 ([9], Teorema I.2)**.** *Let B_R be a ball in $\mathbb{R}^{n-1}$, $\beta \in (0,1]$ and $p \in [1,\infty)$. A function $w \in L^1(B_R)$ is (up to the choice of a suitable representative) Hölder continuous with exponent β, i.e., $w \in C^{0,\beta}(\overline{B_R})$, if and only if there exists a constant C such that for each ball $B_\rho(y')$ centered in some point $y' \in B_R$ there holds*

$$\int_{B_R \cap B_\rho(y')} |w - (w)_{B_R \cap B_\rho(y')}|^p dx' \leq C\rho^{n-1+p\beta}.$$

Observe that the oscillation on the left-hand side is a monotone function of ρ because for any measurable set $A \subset \mathbb{R}^{n-1}$ the mapping

$$c \mapsto \int_A |w - c|^p \, dx' \quad \text{is minimized at } c = (w)_A. \tag{1}$$

Another important technical tool is the following iteration lemma, which allows to pass from ρ^{α_2} to r^{α_2} (where $r \leq \rho$), and to drop an additive non-decaying term (the one that involves ε) at the expense of lowering the order of decay.

Lemma 1.6 ([17], Lemma III.2.1)**.** *Assume that $\phi(\rho)$ is a non-negative, real-valued, non-decreasing function defined on the interval $[0, R_0]$ which satisfies*

$$\phi(r) \leq C_1\Big[\Big(\frac{r}{\rho}\Big)^{\alpha_1} + \varepsilon\Big]\phi(\rho) + C_2\rho^{\alpha_2}$$

for all $r \leq \rho \leq \rho_0$, some non-negative constants C_1, C_2, and positive exponents $\alpha_1 > \alpha_2$. Then there exists a positive number $\varepsilon_0 = \varepsilon_0(C_1, \alpha_1, \alpha_2)$ such that for $\varepsilon \leq \varepsilon_0$ and all $r \leq \rho \leq \rho_0$ we have

$$\phi(r) \leq c(C_1, \alpha_1, \alpha_2)\Big[\Big(\frac{r}{\rho}\Big)^{\alpha_2}\phi(\rho) + C_2 r^{\alpha_2}\Big].$$

Observe that if the quantity ϕ in the lemma is indeed the oscillation, the statement implies Hölder regularity of the order $(\alpha_2 - n + 1)/p$ via Proposition 1.5.

2. An initial (non-optimal) regularity result

In this Section, we describe the main ideas in the proof of the initial regularity result stated in Theorem 1.3, which was established in [12] for the case of a single density, and in [28] for the case of a double density.

The main ingredients in the proof of such (non-optimal) regularity result are the so called $\varepsilon{-}\varepsilon^\beta$ *property* and the classical regularity theory for ω-*minimal* (or *almost minimal*) sets. More precisely, using the $\varepsilon - \varepsilon^\beta$ property (see Definition 2.3 below), one can show that an isoperimetric set with density is in fact an ω-minimal set in the classical sense (for a suitable modulus of continuity ω) and hence the standard regularity theory for this class of sets applies.

We now recall the notion of ω-minimality.

Definition 2.1. *Let $E \subseteq \mathbb{R}^n$ be a set of locally finite perimeter. We say that E is* ω-minimal*, for some continuous and increasing function $\omega\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $\omega(0) = 0$, if, for every ball $B_r = B_r(x) \subset \mathbb{R}^n$ and every set F such that $F \triangle E \subset\subset B_r$, one has*

$$P_{\rm Eucl}\big(E, B_r\big) \leq P_{\rm Eucl}\big(F, B_r\big) + \omega(r)\, r^{n-1}. \tag{2}$$

Here, $P_{\rm Eucl}\big(E, B_r\big)$ denotes the Euclidean perimeter of E in B_r.

The following classical regularity result for ω-minimal sets can be found for instance in [30].

Proposition 2.2. *If a set E is ω-minimal with $\omega(r) = Cr^\eta$ for some $0 < \eta \leq 1$, then $\partial^* E$ is of class $C^{1,\eta/2}$. Moreover, the singular set $\partial E \setminus \partial^* E$ has Hausdorff dimension at most $n - 8$.*

We now define the $\varepsilon - \varepsilon^\beta$ property.

Definition 2.3. *Let $F \subseteq \mathbb{R}^n$ be a set of locally finite perimeter and finite volume, $0 \leq \beta \leq 1$, and $C > 0$. We say that F* fulfills the $\varepsilon - \varepsilon^\beta$ property with constant C *if for any ball B s.t. $\mathcal{H}^{n-1}(B \cap \partial^* F) > 0$, there exist constants $C > 0$ and $\bar\varepsilon > 0$ such that, for every $|\varepsilon| < \bar\varepsilon$, there is a set $G \subseteq \mathbb{R}^n$ satisfying*

$$G \triangle F \subset\subset B\,, \qquad V_f(G) - V_f(F) = \varepsilon\,, \qquad P_h(G) \leq P_h(F) + C|\varepsilon|^\beta\,.$$

This property basically means that a certain set can be locally modified in order to increase or decrease its weighted volume by ε, while the weighted perimeter increases at most by $C|\varepsilon|^\beta$. It is easy to see that if the densities and the set are supposed to be regular enough, the $\varepsilon - \varepsilon$ property (i.e. the above property with $\beta = 1$) holds true. Its validity is also known if the set is even just of locally finite perimeter in the case of a single density $f = h$ which is Lipschitz continuous (see [21]); on the other hand, it is very easy to observe that the validity (for $\beta = 1$) may fail as soon as the density is not Lipschitz. This is more or less the reason why most of the regularity results in this context use at least a Lipschitz assumption on the density.

When f is a merely α-Hölder density, the $\varepsilon - \varepsilon$ property does not hold anymore, but a weaker $\varepsilon - \varepsilon^\beta$ property holds true for some $\frac{n-1}{n} \leq \beta < 1$ depending on α. This was established in [12, Theorem B] for the case of a single density and then generalized to the case of two different densities in [28, Theorem A]. We report here below the general statement.

Proposition 2.4 ([12, 28])**.** *Assume that f and h are positive and locally bounded, and that h is of class $C^{0,\alpha}$, for some $\alpha \in [0,1]$. Then every set F of locally finite perimeter and finite volume fulfills the $\varepsilon - \varepsilon^\beta$ property, where β is defined by*

$$\beta = \beta(\alpha, n) = \frac{\alpha + (n-1)(1-\alpha)}{\alpha + n(1-\alpha)}\,. \tag{3}$$

Remark 2.5.

- *Observe that the above result covers also the case $\alpha = 0$, that is the situation in which both densities are just locally bounded. In this case the $\varepsilon - \varepsilon^\beta$ property holds true with $\beta = \frac{n-1}{n}$.*
- *If f is locally bounded and h is continuous, one can show that, not only the $\varepsilon - \varepsilon^{\frac{n-1}{n}}$ property is valid but also it holds true with* any *constant $C > 0$. This was an important tool for giving optimal results on the boundedness of isoperimetric sets. See* [12, 28] *for details.*

For the sake of completeness, we reproduce here a sketch of the proof of Proposition 2.4; for the details we refer to [12, 28].

Proof of Proposition 2.4. Most of the technical difficulties of the proof of [12, Theorem B] arise from the fact that one needs to work with a set of locally finite perimeter F, for which a lot of problems may arise (since the boundary of F can be of low regularity). In this technical part of the proof, the densities do not play any role and the most important ingredients are standard results in geometric measure theory, such as the Blow-up Theorem and Vol'pert Theorem (see [12], Theorems 1.10 and 1.12; for standard properties of sets of finite perimeter we refer to [1]). Hence, for the sake of simplicity, we may assume here that F is smooth and we perform the same computations as in [28].

Take a point $x = (x', x_n) \in \partial F$, and a (possibly negative) ε with small modulus. In order to show the claim of the proposition we need to modify F locally around x in such a way that the volume of the resulting set G equals $V_f(F) + \varepsilon$, while the perimeter does not increase too much. Without loss of generality, we may think that the normal vector to ∂F at x points in the x_n-direction. Take then all the points $(y', y_n) \in \partial F$ such that y' have distance less than a small quantity $a > 0$ from x', and push them vertically by a small distance δ, either positive or negative. Figure 1 shows a sketch of this procedure: In the case $\delta > 0$, the original set F lies below the boundary ∂F and the new set G is given by the union of F with the "new part". It is obvious that the Euclidean volume has been increased by a quantity of order $a^{n-1}\delta$.

We choose now a and δ depending on ε in such a way that the Euclidean volume of the new part is of order ε. More precisely, for some $\gamma > 0$ to be chosen later, we set

$$a = \varepsilon^\gamma \quad \text{and} \quad \delta \approx \frac{\varepsilon}{\varepsilon^{\gamma(n-1)}},$$

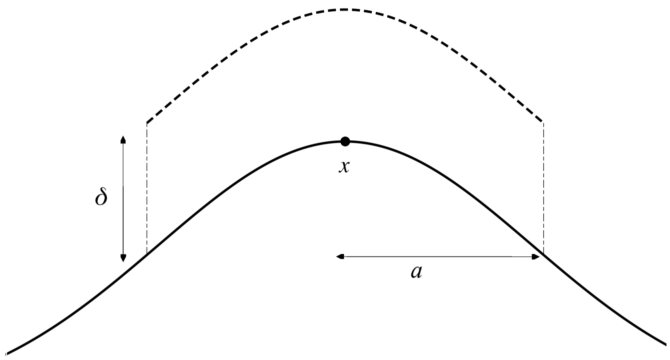

FIGURE 1. The construction of G.

where we assumed $\varepsilon, \delta > 0$ for notational convenience.

It is clear that the Euclidean volume of the new part is of order ε since the density f is bounded above and below (away from zero). By this construction (up to adjusting the constants in the definition of a and δ) we have

$$V_f(G) = V_f(F) + \varepsilon.$$

Let us compute now the change in perimeter. We observe that there are two new contributions in the perimeter of G with respect to the one of F. On one hand, we have added a "lateral boundary" of area

$$a^{n-2}\delta \approx \varepsilon^{\gamma(n-2)}\varepsilon^{1-\gamma(n-1)} = \varepsilon^{1-\gamma}.$$

This contribution leads to an increase of the *weighted* perimeter by a quantity of order $\varepsilon^{1-\gamma}$, due to the fact that h is bounded above and below. On the other hand, there is a piece of boundary of area of order a^{n-1} which has been moved by a distance δ. To bound the weighted perimeter associated to this boundary, the Hölder regularity of h plays a crucial role. Indeed, the difference in the values of h between any point on the translated boundary and its former position is at most of order δ^α, and then this causes a change of the weighted perimeter at most of order

$$a^{n-1}\delta^\alpha \approx \varepsilon^{\gamma(n-1)}\varepsilon^{\alpha(1-(\gamma(n-1)))} = \varepsilon^{\alpha+\gamma(n-1)(1-\alpha)}.$$

As a consequence, while the volume is changed by a quantity of order ε, the difference in perimeter between F and G is estimated by

$$P_h(G) - P_h(E) \lesssim \varepsilon^{1-\gamma} + \varepsilon^{\alpha+\gamma(n-1)(1-\alpha)}.$$

Optimizing in γ, i.e, choosing

$$\gamma = \frac{1-\alpha}{\alpha + n(1-\alpha)},$$

we deduce that the perimeter has changed by order ε^β with β as in (3).

□

With the $\varepsilon - \varepsilon^\beta$ property at hand, we can prove our initial regularity result. We give here just a sketch of the proof, highlighting the main ideas (see [12, 28] for the details).

Proof of Theorem 1.3. By Proposition 2.2, it is enough to show that, given f and h as in the statement, any isoperimetric set E is $\bar\omega$-minimal with $\bar\omega(r) = \overline{C}\, r^{\alpha/(n(1-\alpha)+\alpha)}$. We divide the proof into two steps.

Step 1. First, we show that E is a *quasi-minimizer* of the classical perimeter, that is

$$P_{\mathrm{Eucl}}(E, B_r) \le C_1 r^{n-1} \qquad \text{for some } C_1 > 0 \text{ and every ball } B_r \subset \mathbb{R}^n. \tag{4}$$

In this step, it is enough to assume that f and h are locally bounded (from above and below away from 0). Indeed, the crucial ingredient in the proof is the $\varepsilon - \varepsilon^{\frac{n-1}{n}}$ property, which is valid under this assumption (see Remark 2.5). The argument is by contradiciton: we assume that E is not quasi-minimal and, thanks to the $\varepsilon - \varepsilon^{\frac{n-1}{n}}$ property, we construct a new set F which has the same weighted volume of E but lower weighted perimeter. This contradicts the minimality of E, thus concluding the proof of Step 1.

Step 2. In the second step, we proof that E is $\bar\omega$-minimal with $\bar\omega(r) = \overline{C}\, r^{\alpha/(n(1-\alpha)+\alpha)}$. Here all the assumption in the statement are needed, in particular we use that $h \in C^{0,\alpha}$. Similarly as in the previous step, the proof is by contradiction. We assume that E is not $\bar\omega$-minimal: using that E is quasi-minimal by the previous step, and the $\varepsilon - \varepsilon^\beta$ property, we can construct a competitor F with the same weighted volume of E but lower weighted perimeter thus contradicting, again, the minimality of E. This concludes the proof. □

Remark 2.6. *We observe that, since we have proven that any isoperimetric set E is ω-minimal, then by applying Theorem 2.2, we also obtain automatically an optimal bound on the size of the singular set. More precisely, we have that $\partial E \setminus \partial^* E$ has at most Hausdorff dimension $n-8$. This was not stated in* [12, 28] *but it comes automatically from Theorem 2.2.*

3. Optimal regularity for the case of a single density

In this Section, we describe the main ideas in the proof of the optimal regularity result.

For the convenience of the reader, we recall that our density functions satisfy the following assumption:

$$\begin{gathered} f,\, h : \mathbb{R}^n \to (0, +\infty), \\ f \in C^{0,\gamma}(\mathbb{R}^n), \quad h \in C^{0,\alpha}(\mathbb{R}^n), \quad \text{for some } \alpha,\, \gamma \in (0,1), \end{gathered} \tag{5}$$

where f denotes the density on the volume and h the density on the perimeter.

We start by observing that, using the initial regularity Theorem 1.3, we already know that the reduced boundary of isoperimetric sets (at some given regular point) can be locally written as the graph of a C^1-function. Hence, in order to study the local regularity of isoperimetric sets, it is enough to consider minimizers u of the functional

$$w \mapsto \int_{B_R(0)} h(x', w)\big(1 + |Dw|^2\big)^{\frac{1}{2}} dx' \tag{6}$$

among all functions w satisfying the constraint

$$\int_{B_R(0)} \int_0^{w(x')} f(x', t) dt\, dx' = m \tag{7}$$

for a given constant m and with prescribed boundary values on $\partial B_R(0)$.

A second observation concerns local bounds for the density functions. Since we are assuming the densities f and h are positive continuous functions, they are in particular locally bounded both from above and below away from zero. In particular, for any $T > 0$, there exists a constant $M > 0$ that can be chosen independently from our localization scale $R \leq R_0$ introduced in (6) and (7) such that

$$\frac{1}{M} \leq f(x', t) \leq M \quad \text{and} \quad \frac{1}{M} \leq h(x', t) \leq M \quad \text{for any } (x', t) \in B_R \times (-T, T). \tag{8}$$

For the fixed local representation of the reduced boundary as the graph of a C^1-function u introduced before, we will always choose T large enough so that $\|u\|_{L^\infty(B_R)} \leq T$ uniformly in $R \leq R_0$, which allows for choosing $t = u(x')$. Moreover, as a consequence of the initial regularity statement of Theorem 1.3, the gradient of u is locally bounded by a constant $K \geq 1$ that is independent of $R \leq R_0$, i.e.,

$$\|Du\|_{L^\infty(B_R)} \leq K. \tag{9}$$

In the rest of the paper, when we write $A \lesssim B$, we mean that there exists a constant c which depends only on n, K, M, $[f]_{C^{0,\gamma}}$ and $[h]_{C^{0,\alpha}}$ (hence, in particular, it is independent of R), such that $A \leq cB$.

As already explained in the Introduction, in order to prove the main result, thanks to the Campanato's characterization of Hölder functions, it is enough to prove that there exists a constant $C > 0$ such that for any ball $B_\rho(x') \subset B_R$ we have

$$\int_{B_\rho(x')} |\partial_i u - (\partial_i u)_r|^2 \leq C \rho^{n-1+\frac{2\alpha}{2-\alpha}}.$$

To obtain such decay estimates, we proceed in two steps:

- **Step 1.** We consider a (more regular) comparison problem and show Campanato-type estimates for the solution v of such problem;

- **Step 2.** We transfer such estimates from v to the solution u of our original minimization problem.

Let us now introduce our comparison problem, in which the density $h(x', w)$ in the surface area functional is frozen to a constant so that it is possible to write the Euler–Lagrange equations. A further simplification is achieved by modifying the surface function $a(z)$ for large values of z, in such a way that the minimization problem remains unchanged but now it has quadratic (and not linear) growth (this is possible thanks to (9)). We set

$$a_K(z) = \begin{cases} (1+|z|^2)^{\frac{1}{2}} & \text{if } |z| \le K, \\ c_K(1+|z|^2) & \text{if } |z| \ge 2K, \end{cases}$$

for some constant c_K.

Our *comparison problem* is thus the following: We study the problem of minimizing the functional

$$w \mapsto \int_{B_R} a_K(Dw)dx' \tag{10}$$

among all functions w with $w = u$ on the boundary ∂B_R which satisfy the weighted volume constraint

$$\int_{B_R} \int_0^{w(x')} f(x', t)dt\, dx' = \int_{B_R} \int_0^{u(x')} f(x', t)dt\, dx'. \tag{11}$$

Using standard compactness arguments, we get existence of minimizers of the comparison problem and show that they satisfies the associated Euler–Lagrange equation (See [3, Lemma 3.1]).

Lemma 3.1 ([3])**.** *In the above setting, a minimizer $v \in u + W_0^{1,2}(B_R)$ to the functional* (10) *under the constraint* (11) *always exists. Moreover, every minimizer v satisfies an Euler–Lagrange equation with Lagrange multiplier $\lambda \in \mathbb{R}$: for every $\varphi \in W_0^{1,2}(B_R)$, there holds*

$$\int_{B_R} D_z a_K(Dv(x')) \cdot D\varphi(x')dx' + \lambda \int_{B_R} f(x', v)\varphi(x')dx' = 0. \tag{12}$$

For proving the needed decays estimates for the solution v of the comparison problem introduced above (the core of Step 1), we combine the initial regularity property of the original minimizer u and classical Schauder theory for the Euler-Lagrange equation (12) satisfied by v. Even though this part is quite standard, one main issue is the presence of the Lagrange multiplier λ (which comes from the volume constraint). Indeed, for having good decay estimates, one needs the precise dependence of all constants (and thus also of λ) in terms of R and ρ. Finding a good upper bound for λ as a function of R is thus a crucial point.

As we will see, dealing with the case of a single density is a bit easier for both Steps. For the convenience of the reader, we first describe our strategy in this simpler situation and we will then explain how to treat the case of two different densities in the next Section.

Actually, as it will become clear later on, the argument that we present here works not just for the case $f = h$ but for the case of two possibly different densities f and h such that $f \in C^{0,\gamma}$, $h \in C^{0,\alpha}$, with $\gamma \geq \alpha$ (i.e. the volume density is regular, at least, as the perimeter density).

Let us start with the first easy estimate on the Lagrange multiplier λ.

Lemma 3.2 (First bound on λ, [3]). *Let f, g satisfy* (5). *There exists a constant $R_* = R_*(n, f)$ such that if $R \leq R_0 \leq R_*$ then we have the following estimate for the Lagrange multiplier λ:*

$$|\lambda| \lesssim R^{-1}. \tag{13}$$

The proof follows easily by testing the equation (12) satisfied by v with a standard cut-off function. For the detailed computation, see the proof of [3, Lemma 3.3].

The previous estimate on λ plays a crucial role both in Step 1 (that is, for establishing good decay Campanato-type estimates for the solution v of the comparison problem) and Step 2 (for the error estimate which allows to pass from v to u). Let us see more in detail, without giving all the computations, the main ideas in each of these steps.

Step 1. The following Proposition contains the decay estimates for the solution v of the comparison problem.

Proposition 3.3 ([3]). *Let f, g satisfy* (5). *Let $u \in C^{1,\sigma}(B_R)$ and let $v \in u + W_0^{1,2}(B_R)$ be a solution of* (12)*, under the volume constraint* (11). *Then there exists $\rho_0 > 0$ of the form $\rho_0 = \varepsilon_0 R$ (with ε_0 depending only on n, K, M, α, γ, $[f]_{C^{0,\gamma}}$ and $[h]_{C^{0,\alpha}}$) such that $B_{\rho_0}(x_0') \subset B_{R/4}$, and for all $0 < r < \rho \leq \rho_0$, we have*

$$\int_{B_r(x_0')} |\partial_i v - (\partial_i v)_r|^2 \, dx' \lesssim \left(\frac{r}{\rho}\right)^{n-1+2\gamma} \int_{B_\rho(x_0')} |\partial_i v - (\partial_i v)_\rho|^2 \, dx' + r^{n-1+2\gamma}, \tag{14}$$

for any $i = 1, \ldots, n-1$.

In particular, if $\gamma \geq \alpha$, we have

$$\int_{B_r(x_0')} |\partial_i v - (\partial_i v)_r|^2 \, dx' \lesssim \left(\frac{r}{\rho}\right)^{n-1+2\alpha} \int_{B_\rho(x_0')} |\partial_i v - (\partial_i v)_\rho|^2 \, dx' + r^{n-1+2\alpha}.$$

We omit here the detailed proof, which can be found in [3, Sect. 4], and just make some comments. As already discussed above, the result follows by classical Schauder theory for the Euler-Lagrange equation (12) and the crucial estimate on λ established in Lemma 3.2. Observe that, in (14), only the exponent γ appears since the equation satisfied by v

just involves the volume density f (and not the perimeter density h). However, if $\gamma \geq \alpha$, we can replace all exponents γ by α in (14) and this simplify a bit the whole argument.

Indeed, in such a case, being $\alpha \geq \frac{\alpha}{2-\alpha}$, in order to conlude the proof, it is enough to show that the error that we make passing from the decay estimates for v to decay estimates for u is of order $r^{n-1+\frac{2\alpha}{2-\alpha}}$. This is the core of Step 2.

Step 2. The error estimate reads as follow:

Lemma 3.4 (Error estimate, [3])**.** *Let* f, h *satisfy* (5).
Then,

$$\int_{B_R} |Du - Dv|^2 \, dx' \lesssim R^{n-1+\frac{2\alpha}{2-\alpha}} + R^{n-1+\frac{2\gamma}{1-\gamma}}. \tag{15}$$

In particular, if $\gamma \geq \alpha$, *we have*

$$\int_{B_R} |Du - Dv|^2 \, dx' \lesssim R^{n-1+\frac{2\alpha}{2-\alpha}}. \tag{16}$$

This statement corresponds to the case $\delta = 0$ in [3, Lemma 3.5]. We give here just a sketch of the proof and refer to [3] for the details. Also in this step the estimate (13) on λ plays a crucial role: it ensures that the assumption of [3, Lemma 3.5] is satisfied for $\delta = 0$. Observe again that, if $\gamma \geq \alpha$, then the leading term in (15) is $R^{n-1+\frac{2\alpha}{2-\alpha}}$, which is already the good one!

Sketch of the proof of Lemma . We test the Euler–Lagrange equation (12) satisfied by v with $\varphi = u - v$, and we get

$$\frac{\mu}{2}\int_{B_R} |Du - Dv|^2 \, dx' \leq \int_{B_R} \big(a_K(Du) - a_K(Dv)\big) \, dx' + \lambda \int_{B_R} f(x', v(x'))(u(x') - v(x')) \, dx'. \tag{17}$$

Now we estimate the two integrals on the right-hand side separately.

Let us start with the first term, which is the one responsable for the term $R^{n-1+\frac{2\alpha}{2-\alpha}}$ in (15). Using the Lipschitz bound on u, the definition of a_K and the lower bound on h, we have that

$$\begin{aligned}
\frac{1}{M}\int_{B_R} \big(a_K(Du) - a_K(Dv)\big) \, dx' &\leq h(0', u(0')) \int_{B_R} (a(Du) - a(Dv)) \, dx' \\
&= \int_{B_R} \big(h(0', u(0')) - h(x', u(x'))\big)\big(a(Du) - a(Dv)\big) \, dx' \\
&\quad + \int_{B_R} \big(h(x', u(x'))a(Du) - h(x', v(x'))a(Dv)\big) \, dx' \\
&\quad + \int_{B_R} \big(h(x', v(x')) - h(x', u(x'))\big) a(Dv) \, dx'.
\end{aligned}$$

The second term on the right-hand side is non-positive since u is a minimizer for the initial weighted problem and v is an admissible competitor. Using now that h is Hölder of order α, u is Lipschitz with constant K, a is Lipschitz with constant 1, Young's inequality, one can get

$$\begin{aligned}&\frac{1}{M}\int_{B_R}\big(a_K(Du)-a_K(Dv)\big)\,dx'\\&\leq \varepsilon\int_{B_R}\big(|Du-Dv|^2+R^{-2}|u-v|^2\big)\,dx'+C\left(\int_{B_R}|x'|^{2\alpha}\,dx'+R^{\frac{2\alpha}{2-\alpha}}\int_{B_R}a(Dv)^{\frac{2}{2-\alpha}}\,dx'\right),\end{aligned}$$

where ε is some small constant. Clearly, the second term on the right-hand side is of order $R^{n-1+2\alpha}$. Moreover, using the minimality of v and the Lipschitz bound on u, we have that

$$\int_{B_R}a(Dv)^{\frac{2}{2-\alpha}}\,dx'\lesssim\int_{B_R}a_K(Dv)\,dx'\leq\int_{B_R}a_K(Du)\,dx'\leq CR^{n-1}.$$

For $R\leq 1$, we thus obtain

$$\frac{1}{M}\int_{B_R}\big(a_K(Du)-a_K(Dv)\big)\,dx'\leq\varepsilon\int_{B_R}\big(|Du-Dv|^2+R^{-2}|u-v|^2\big)\,dx'+CR^{n-1+\frac{2\alpha}{2-\alpha}}.$$

It remains to estimate the term coming from the volume constraint. With similar computations, using that $f\in C^{0,\gamma}$, Young's inequality, and again the crucial estimate (3.2) for λ, one can show that

$$\begin{aligned}|\lambda|\int_{B_R}f(x',v(x'))(u(x')-v(x'))\,dx'&\leq CR^{-1}\int_{B_R}|u(x')-v(x')|^{1+\gamma}\,dx'\\&\leq\varepsilon\int_{B_R}R^{-2}|u-v|^2\,dx'+CR^{n-1+\frac{2\gamma}{1-\gamma}}.\end{aligned}$$

Finally, combining the two terms together, using Poincaré's inequality and choosing ε sufficiently small, we conlude the proof.

□

Combing the two Steps together, we can finally prove our main result, in the case $\gamma\geq\alpha$.

Proof of Theorem 1.4. Assume $\gamma\geq\alpha$. By Proposition 3.3 and Lemma 3.4, we have

$$\begin{aligned}\int_{B_r}|\partial_i u-(\partial_i u)_r|^2\,dx'&\leq\int_{B_r}|\partial_i u-(\partial_i v)_r|^2\,dx'\\&\lesssim\int_{B_r}|\partial_i v-(\partial_i v)_r|^2\,dx'+\int_{B_R}|Du-Dv|^2\,dx'\\&\lesssim\left(\frac{r}{\rho}\right)^{n-1+2\gamma}\int_{B_\rho}|\partial_i v-(\partial_i v)_\rho|^2\,dx'+r^{n-1+2\alpha}+R^{n-1+\frac{2\alpha}{2-\alpha}},\end{aligned}\tag{18}$$

for any $0 < r < \rho \leq \rho_0 = \varepsilon_0 R$.

Using again (16), we can pass on the right-hand side to $\partial_i u$ instead of $\partial_i v$, and obtain:

$$\int_{B_r} |\partial_i u - (\partial_i u)_r|^2 \, dx' \lesssim \left(\frac{r}{\rho}\right)^{n-1+2\alpha} \int_{B_\rho} |\partial_i u - (\partial_i u)_\rho|^2 \, dx' + R^{n-1+\frac{2\alpha}{2-\alpha}}, \tag{19}$$

for any $0 < r < \rho \leq \rho_0$. We choose now $\rho = \rho_0 = \varepsilon_0 R$ to deduce that:

$$\begin{aligned} \int_{B_r} |\partial_i u - (\partial_i u)_r|^2 \, dx' &\lesssim \left(\frac{r}{R}\right)^{n-1+2\alpha} \int_{B_{\varepsilon_0 R}} |\partial_i u - (\partial_i u)_{\varepsilon_0 R}|^2 \, dx' + R^{n-1+\frac{2\alpha}{2-\alpha}} \\ &\lesssim \left(\frac{r}{R}\right)^{n-1+2\alpha} \int_{B_R} |\partial_i u - (\partial_i u)_R|^2 \, dx' + R^{n-1+\frac{2\alpha}{2-\alpha}}, \end{aligned} \tag{20}$$

for any $0 < r \leq \varepsilon_0 R$. The same estimate trivially extends to $r \in (\varepsilon_0 R, R)$ and thus holds for any $0 < r < R$.

We can now apply the iteration Lemma 1.6, to get

$$\int_{B_r(x_0')} |\partial_i u - (\partial_i u)_r|^2 \, dx' \lesssim \left(\frac{r}{R}\right)^{n-1+\frac{2\alpha}{2-\alpha}} \int_{B_R} |\partial_i u - (\partial_i u)_R|^2 \, dx' + r^{n-1+\frac{2\alpha}{2-\alpha}}.$$

Using the Campanato characterization of Hölder functions, this concludes the proof of the Theorem.

□

4. Optimal regularity for the case of a double density

In this last Section, we describe how to get the optimal regularity result in the case $\gamma < \alpha$. Since our statement is independent on γ, we may assume γ small. Hence, in the following we assume that

$$\gamma < \min\left\{\frac{\alpha}{2}, \frac{2(1-\alpha)}{2-\alpha}\right\}. \tag{21}$$

We start with the obvious observation that, when $\gamma < \alpha$ estimates (14) and (15) are not good enough. Indeed, combining them together and arguing as before, we would get $u \in C^{1,\min\{\frac{\alpha}{2-\alpha},\gamma\}}$, which is far from being optimal when γ is small. The main issue is that the bound (13) on λ is too rough. The idea is then to improve such bound and this improvement would be reflected in both Proposition 3.3 and Lemma 3.4. As we will see, the needed improvement in the final regularity of u is achieved by a double iteration. Let us discuss this strategy more in detail.

We start with a Lemma (see [3, Lemma 3.4]) which basically states that is possible to give a better bound on λ if we already knew that Du and Dv are sufficiently close in $L^2(B_R)$. In such result, also the initial $C^{1,\sigma}$ regularity of our minimizer u plays an important role.

Lemma 4.1 (Improved bound on λ [3])**.** *Let f, h satisfy* (5)*. Suppose that u is $C^{1,\sigma}(B_R)$ and that*

$$\int_{B_R} |Du - Dv|^2 \, dx \lesssim R^{n-1+2\theta} \tag{22}$$

for some σ and $\theta \in [0,1)$, then

$$|\lambda| \lesssim R^{\theta-1} + R^{\sigma-1}. \tag{23}$$

Proof. We give the detailed proof for the convenience of the reader. Consider a smooth cut-off function $\eta \in C_0^\infty(B_R, [0,1])$ satisfying $\eta \equiv 1$ in $B_{R/2}$ and $\|D\eta\|_{L^\infty} \lesssim R^{-1}$ and use it as a test function in the Euler–Lagrange equation (12). We have that:

$$|\lambda| R^{n-1} \lesssim \left| \int_{B_R} D_z a_K(Dv) \cdot D\eta \, dx' \right|.$$

We now use the fact that η is compactly supported in B_R to observe that $\int_{B_R} \xi \cdot D\eta \, dx = 0$ for any $\xi \in \mathbb{R}^{n-1}$. In particular, for $\xi = D_z a_K((Du)_R)$, we find that

$$\begin{aligned} |\lambda| R^{n-1} &\lesssim \left| \int_{B_R} (D_z a_K(Dv) - D_z a_K((Du)_R)) \cdot D\eta \, dx' \right| \\ &\lesssim R^{-1} \int_{B_R} |D_z a_K(Dv) - D_z a_K((Du)_R)| \, dx'. \end{aligned}$$

Using that $D_z a_K$ is Lipschitz with a Lipschitz constant depending just on K, and the triangle inequality, we thus obtain

$$|\lambda| R^n \lesssim \int_{B_R} |Dv - Du| \, dx' + \int_{B_R} |Du - (Du)_R| \, dx'.$$

For the first term, we use Jensen's inequality and the hypothesis (22) to bound

$$\int_{B_R} |Dv - Du| \, dx \lesssim R^{n-1+\theta}.$$

For the second term, we use that Du is σ-Hölder continuous, and thus

$$\int_{B_R} |Du - (Du)_R| \, dx \lesssim R^{n-1+\sigma}.$$

A combination of the previous bounds yields the statement of the lemma. □

The following Lemma (see [3, Lemma 3.5]) is just a straightforward modification of Lemma 3.4.

Lemma 4.2 (First error estimate)**.** *Suppose that there exists $\delta \in [0,1)$ such that*

$$|\lambda| \lesssim R^{\delta-1}.$$

Then,

$$\int_{B_R} |Du - Dv|^2 \, dx' \lesssim R^{n-1+\frac{2\alpha}{2-\alpha}} + R^{n-1+\frac{2}{1-\gamma}(\gamma+\delta)}. \tag{24}$$

The proof is exatly the same as the one of Lemma 4.2, using now the better bound on λ, which is taken here as an assumption.

We are now ready to improve the bound on the Lagrange multiplier. Indeed, iterating Lemma 4.2 and 4.1, we get (see [3, Corollary 3.6]):

Corollary 4.3 (Improved error estimate [3])**.** *Let $u \in C^{1,\sigma}(B_R)$ with $\sigma \leq \frac{\alpha}{2-\alpha}$, then we have:*

$$|\lambda| \lesssim R^{\sigma-1} \tag{25}$$

and

$$\int_{B_R} |Du - Dv|^2 \, dx' \lesssim R^{n-1+\frac{2\alpha}{2-\alpha}} + R^{n-1+\frac{2}{1-\gamma}(\gamma+\sigma)}. \tag{26}$$

Proof. By Lemma 3.2 we know that $|\lambda| \lesssim R^{-1}$, hence we can apply Lemma 4.2 with $\delta = 0$ and deduce that

$$\int_{B_R} |Du - Dv|^2 \lesssim R^{n-1+\frac{2\alpha}{2-\alpha}} + R^{n-1+\frac{2\gamma}{1-\gamma}}.$$

Under the assumption (21), the second term on the right-hand side is the leading order term for $R \leq 1$. Hence, applying Lemma 4.1 (with $\theta = \frac{\gamma}{1-\gamma}$), we deduce that

$$|\lambda| \lesssim R^{\sigma-1} + R^{\frac{\gamma}{1-\gamma}-1}.$$

If $\frac{\gamma}{1-\gamma} \geq \sigma$, the first bound (25) is proved and (26) follows directly from Lemma 4.2. Otherwise, we can iterate the above procedure and after a finite number of steps we will reach the estimate

$$|\lambda| \lesssim R^{\sigma-1},$$

which, again, will imply, by using Lemma 4.2, the desired estimate (26). □

With the above Corollary, we have thus improved the error estimate from (15) to (26) by an exponent σ in the second term on the right-hand side. We also need an improvement on the decay estimates for the solution v of the comparison problem and this is possible, again, using (25). The result is the following (see [3, Proposition 4.2]):

Proposition 4.4 ([3])**.** *Let $u \in C^{1,\sigma}(B_R)$ with $\sigma \leq \frac{\alpha}{2-\alpha}$ and let $v \in u + W_0^{1,2}(B_R)$ be a solution of* (12)*, under the volume constraint* (11)*. Then there exists $\rho_0 > 0$ of the* *form* $\rho_0 = \varepsilon_0 R$ *(with* ε_0 *depending only on* n, K, M, α, γ, $[f]_{C^{0,\gamma}}$ *and* $[h]_{C^{0,\alpha}}$*) such that* $B_{\rho_0}(x_0') \subset B_{R/4}$*, and for all* $0 < r < \rho \leq \rho_0$*, we have*

$$\int_{B_r(x_0')} |\partial_i v - (\partial_i v)_r|^2 \, dx' \lesssim \left(\frac{r}{\rho}\right)^{n-1+2(\gamma+\sigma)} \int_{B_\rho(x_0')} |\partial_i v - (\partial_i v)_\rho|^2 \, dx' + r^{n-1+2(\gamma+\sigma)}, \tag{27}$$

for any $i = 1, \dots, n-1$.

As before, we omit the proof and we refer to [3, Section 4]. We just emphisize that we have improved the decay estimate for the oscillation of Dv from $r^{n-1+2\gamma}$ to $r^{n-1+2(\gamma+\sigma)}$.

We can now conclude with the

Proof of Theorem 1.4. Combining together Corollary 4.3 and Proposition 4.4, as we did at the end of the previous Section, we deduce that

$$u \in C^{1,\min\{\gamma+\sigma, \frac{\alpha}{2-\alpha}\}}.$$

In order to conclude, we observe that if $\gamma + \sigma \geq \frac{\alpha}{2-\alpha}$ the proof is completed. Otherwise we can iterate the above reasoning: setting $\sigma_j := \sigma + \frac{j\gamma}{2}$, we can iteratively apply Proposition 4.4 and Corollary 4.3, with $u \in C^{1,\sigma_j}$ and plug the new improved estimate (26) (with $\sigma_j > \sigma$ in place of σ) into (18). After a finite number N of steps (in particular when $N\gamma/2 + \sigma \geq \frac{\alpha}{2-\alpha}$) we reach the exponent $\frac{\alpha}{2-\alpha}$. This concludes the proof. □

We conclude this note, giving an explicit example which shows optimality of our result (see [3, Example 1.2]).

Example 4.5. *For simplicity, we consider the two-dimensional problem and an isoperimetric set which locally can be written as the graph of a function* w *over the interval* $(0, \ell)$*, with* $w > 0$ *in* $(0, \ell)$ *and* $w(0) = 0$*. We suppose that the volume density is locally constant, with* $f \equiv 1$*, and that the perimeter density is locally of the form* $h(x_1, x_2) = H(x_1, |x_2|^\alpha)$ *for some* $\alpha \in (0, 1)$ *and some function* H *such that both* H *and its derivative* $\partial_2 H$ *with respect to the second variable are bounded from below and from above by positive constants. Then, according to* (6) *and* (7)*,* w *minimizes the functional*

$$w \mapsto \int_0^\ell H(z, |w(z)|^\alpha) \sqrt{1 + (w'(z))^2} \, dz,$$

among all w *satisfying the constraint*

$$\int_0^\ell w(z) \, dz = m.$$

In this situation, we can write the Euler–Lagrange equation in $(0, \ell)$*:*

$$\left(H(z, |w(z)|^\alpha) \frac{w'}{\sqrt{1 + (w')^2}} \right)' + \partial_2 H(z, |w(z)|^\alpha) \alpha w^{\alpha-1} \sqrt{1 + (w')^2} = \lambda,$$

where $\lambda \in \mathbb{R}$ is the Lagrange multiplier. Since w is $C^{1,\sigma}$ and $w(0) = 0$, we have that $w(z) \approx z^{1+\sigma}$ near $z = 0$. Plugging this Ansatz into the Euler–Lagrange equation, we see that the first term is of the order $O(z^{\sigma-1})$, while the second one is $O(z^{(\alpha-1)(1+\sigma)})$. The singularity of both terms enforces the two exponents to coincide, $\sigma - 1 = (\alpha - 1)(1 + \sigma)$, which yields $\sigma = \frac{\alpha}{2-\alpha}$.

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