# ON THE DE GIORGI-NASH-MOSER REGULARITY THEORY FOR KINETIC EQUATIONS

# SULLA TEORIA DELLA REGOLARITÀ ALLA DE GIORGI-NASH-MOSER PER EQUAZIONI CINETICHE

FRANCESCA ANCESCHI

Abstract. In this note we review some recent results regarding the De Giorgi-Nash-Moser weak regularity theory for Kolmogorov operators obtained in [10] in collaboration with A. Rebucci. To simplify the treatment, we focus on the model case of the Fokker-Planck equation with rough coefficients and we highlight the main steps of the proof of a Harnack inequality for weak solutions.

Sunto. In questa nota si presentano alcuni recenti risultati relativi alla teoria della regolarità debole alla De Giorgi-Nash-Moser per operatori di Kolmogorov ottenuti in [10] in collaborazione con A. Rebucci. Per semplificare la trattazione, la nostra analisi si incentra sul caso modello dell'equazione di Fokker-Planck a coefficienti misurabili e si propone di presentare i passi fondamentali della dimostrazione di una disuguaglianza di Harnack per soluzioni deboli.

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## 1. Introduction

The Hilbert's $19^{th}$ problem regarding the analytical regularity of minimizers for an energy functional associated to an elliptic Euler-Lagrange equation gave rise to the study of the regularity theory for elliptic PDEs with rough coefficients in divergence form. In this setting, the Hölder continuity of weak solutions was independently proved by De Giorgi [15, 16], via an iterative argument to gain integrability combined with an isoperimetric inequality to control the decay of oscillation of the solution, and Nash [35], who considered

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Dipartimento di Matematica, Università di Bologna

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sharp estimates for the fundamental solution together with a $L\log L$ energy estimate. Later on, Moser proposed an alternative proof [32] of the already established results and further extended them to the case of parabolic equations in divergence form [33]. Given the outstanding contributions of these authors, this branch of mathematical analysis is named after them as De Giorgi-Nash-Moser regularity theory. We remark that these techniques are only feasible for the study of the regularity for solutions to elliptic and parabolic equations in divergence form, and for this reason Krylov and Safonov later on developed in [28] new methods to tackle with non-divergence form equations, both in the elliptic and parabolic settings.

After these fundamental breakthroughs, the study of the weak regularity theory had been confined to the uniformly elliptic and parabolic setting for many years, and it has only been recently extended to some classes of hypoelliptic operators where the diffusion does not act in every direction, a family to which the Kolmogorov operator and more specifically the Fokker-Planck operator subject of our analysis belong to.

An operator $\mathscr{L}$ is hypoelliptic if, for every distributional solution $u \in L^1_{loc}(\Omega)$ to $\mathscr{L}u = f$ on $\Omega$, it is true that $f \in C^\infty(\Omega)$ implies $u \in C^\infty(\Omega)$. Hence, operators of this type possess strong regularizing properties, may be characterized through a rank condition proved by Hörmander in [22] and may be rewritten in the very specific form $\mathscr{L} := \sum\limits_{i=1}^{k} X_i^2 + Y$, where $Y$ and $X_i$, for $i = 1, \dots, k$, are smooth vector fields. This class of operators is wide and it can be subdivided into two subclasses depending on the geometrical structure of the underlying Lie group, for which the two non degenerate prototypes one should have in mind are the Laplace operator $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ on $\mathbb{R}^n$ and the heat operator $\mathcal{H} := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + \frac{\partial}{\partial t}$ on $\mathbb{R}^{n+1}$, see [12]. In this work, we focus on the latter class, for which the constant coefficients Kolmogorov operator $\mathscr{K}_0$ associated to the second order partial differential equation

$$\mathscr{K}_0 u(v,x,t) = \Delta_v u(v,x,t) + v \cdot \nabla_x u(v,x,t) - \partial_t u(v,x,t) = 0, \qquad (v,x,t) \in \mathbb{R}^{2n+1}, \tag{1}$$

is the degenerate prototype. Equation (1) was firstly introduced in 1934 by Kolmogorov in [26] to describe the dynamics of the distribution of Brownian test particles immersed in a fluid in thermodynamical equilibrium provided that the test particle is much heavier than the molecules of the fluid. In this setting, a solution $u$ to (1) is the distribution function of particles with velocity $v$ at position $x$ at time $t$ and equation (1) is the backward Kolmogorov equation of the stochastic process

$$\begin{cases} dV_t = \sqrt{2} dW_t, \\ dX_t = V_t dt, \end{cases}$$

where $(W_t)_{t \geq 0}$ is a $n-$dimensional Wiener process. We refer the reader to [7, 34] for a survey on Fokker-Planck equations and their applications.

The operator $\mathscr{K}_0$ can be written in Hörmander form $\sum_{i=1}^{n} X_i^2 + Y$ provided that

$$X_i = \partial_{v_i} \quad \text{with } i = 1, \ldots, n, \qquad Y = v \cdot \nabla_x - \partial_t,$$

and it is hypoelliptic because it satisfies the Hörmander's rank condition, i.e.

$$\text{rank Lie}\{X_i, \ i = 1, \ldots, n, Y\}(v, x, t) = 2n + 1 \quad \text{for every } \ (v, x, t) \in \mathbb{R}^{2n+1},$$

see [7]. Hence, as firstly pointed out in [14], the correct geometrical framework to study the operator $\mathscr{K}_0$ are Lie groups. When considering (1), the associated Lie group has a quite natural interpretation. Indeed, the composition law

$$(v, x, t) \circ (v_0, x_0, t_0) = (v_0 + v, x_0 + x + t v_0, t_0 + t), \tag{2}$$

where $(v_0, x_0, t_0), (v, x, t) \in \mathbb{R}^{2n+1}$, with respect to which $\mathscr{K}_0$ is invariant, agrees with the Galilean change of variables. By this we mean: if $w(v, x, t) = u(v_0 + v, x_0 + x + t v_0, t_0 + t)$ and $g(v, x, t) = f(v_0 + v, x_0 + x + t v_0, t_0 + t)$, then $\mathscr{K}_0 u = f$ if and only if $\mathscr{K}_0 w = g$ for every $(v_0, x_0, t_0) \in \mathbb{R}^{2n+1}$.

Additionally, $\mathscr{K}_0$ is invariant with respect to the family of dilations defined as

$$\delta_r(v, x, t) := (rv, r^3 x, r^2 t), \qquad \text{with } (v, x, t) \in \mathbb{R}^{2n+1} \text{ and } r > 0. \tag{3}$$

We remark these dilations act as the usual parabolic scaling with respect to variables $v$ and $t$; whereas, the term $r^3$ in front of $x$ is due to the fact that the velocity $v$ is the derivative of the position $x$ with respect to time $t$.

Hence, starting from (2), (3) and the unit past cylinders

$$\mathcal{Q}_1 := B_1 \times B_1 \times (-1, 0), \qquad \widetilde{\mathcal{Q}}_1 := B_1 \times B_1 \times (-1, 0]$$

defined through Euclidean open balls $B_1 = \{y \in \mathbb{R}^n : |y| \leq 1\}$, for every $z_0 \in \mathbb{R}^{2n+1}$ and $r > 0$, we set the cylinder centered at $z_0$ of radius $r$ as

$$\mathcal{Q}_r(z_0) := z_0 \circ (\delta_r(\mathcal{Q}_1)) = \{z \in \mathbb{R}^{2n+1} \, : \, z = z_0 \circ \delta_r(\zeta), \zeta \in \mathcal{Q}_1\}. \tag{4}$$

This definition of slanted cylinders admits an equivalent ball representation, see [41, equation (21)], that it is sometimes preferred. Specifically, if we consider the cylinder $\mathcal{Q}_r$ centered at the origin and of radius $r$, then there exists a constant $\overline{c} > 0$ such that

$$B_{r_1} \times B_{r_1^3} \times (-r_1^2, 0] \subset \mathcal{Q}_r \subset B_{r_2} \times B_{r_2^3} \times (-r_2^2, 0], \tag{5}$$

where $r_1 = r/\overline{c}$ and $r_2 = \overline{c}r$. Note that if we consider a cylinder centered in a generic point $z_0 = (v_0, x_0, t_0) \in \mathbb{R}^{2n+1}$ instead of the origin, then the constant $\overline{c}$ depends both on $n$ and $|v_0|$. We refer to [7, 12] and the references therein for additional information on this subject.

## 2. The problem at hand

Our analysis specifically considers the kinetic Kolmogorov-Fokker-Planck equation

$$\nabla_v \cdot (A_0(v,x,t)\nabla_v u(v,x,t)) + \nabla_v u(v,x,t) + (v \cdot \nabla_x - \partial_t)\, u(v,x,t) = f(v,x,t), \tag{6}$$

where $(v, x, t) \in \mathbb{R}^{2n+1}$ and $A_0$ is a $2n \times 2n$ symmetric matrix made of real measurable entries satisfying the following ellipticity condition.

**(E)** There exist two real positive constants $\lambda, \Lambda > 0$ such that

$$\lambda|\xi|^2 \leq \langle A_0(v,x,t)\xi, \xi\rangle \leq \Lambda|\xi|^2 \qquad \text{for every } \xi \in \mathbb{R}^n.$$

Equation (6) is a more elaborate version of (1) and it has various applications to the modeling of real life problems arising in physics and economics, see for instance [34, 39] and [4, 36], respectively. To carry out our analysis, throughout this work we will consider some additional assumptions.

**(D)** $\Omega = \Omega_v \times \Omega_{xt}$ of $\mathbb{R}^{2n+1}$, where $\Omega_v$ is a bounded Lipschitz domain of $\mathbb{R}^n$ and $\Omega_{xt}$ is a bounded Lipschitz domain of $\mathbb{R}^{n+1}$.

**(F)** $f \in L^q_{loc}(\Omega)$, with $q > \frac{(Q+2)}{2}$.

Assumption **(D)** is not restrictive since the cylinders $\mathcal{Q}$ that we consider in our local analysis (see (4)) naturally satisfy it. On the other hand, we have at our disposal various results considering a more general version of **(F)**, see for instance [10, 42], but in order to simplify the present treatment we restrict ourselves to the case of bounded lower order coefficients and locally bounded right-hand side.

From now on, we denote by $\mathcal{D}(\Omega)$ the set of $C^\infty$ functions compactly supported in $\Omega$ and by $H^1_v$ the Sobolev space of functions $u \in L^2(\Omega_v)$ with distribution gradient $\nabla_v u \in (L^2(\Omega_v))^n$, i.e. $H^1_v(\Omega_v) := \{u \in L^2(\Omega_v) : \nabla_v u \in (L^2(\Omega_v))^n\}$, and we set

$$\|u\|^2_{H^1_v(\Omega_v)} := \|u\|^2_{L^2(\Omega_v)} + \|\nabla_v u\|^2_{L^2(\Omega_v)}.$$

In a standard manner, see also [2, 30], $\mathcal{W}$ denotes the closure of $C^\infty(\overline{\Omega})$ in the norm

$$\|u\|^2_{\mathcal{W}} = \|u\|^2_{L^2(\Omega_{xt};H^1_v(\Omega_v))} + \|Yu\|^2_{L^2\left(\Omega_{xt};H^{-1}_v(\Omega_v)\right)}, \tag{7}$$

where the previous norm can be explicitly computed as follows:

$$\|u\|^2_{\mathcal{W}} = \int_{\Omega_{xt}} \|u(\cdot,x,t)\|^2_{H^1_v(\Omega_v)} \,\mathrm{d}x\,\mathrm{d}t + \int_{\Omega_{xt}} \|Yu(\cdot,x,t)\|^2_{H^{-1}_v(\Omega_v)} \,\mathrm{d}x\,\mathrm{d}t.$$

The space of functions $\mathcal{W}$ is the most natural framework for the study of the weak regularity theory for (6). It was firstly formalized in a preliminary version of [2] in 2021 for the study of the kinetic Kolmogorov-Fokker-Planck equation and later on considered by various authors in different contexts, see for instance [10, 30]. Now, we are in a position to introduce the definition of weak solution we consider in our work.

**Definition 2.1.** *A function $u \in \mathcal{W}$ is a weak solution to* (6) *under the assumptions* **(D)**, **(E)** *and* **(F)**, *if for every $\varphi \in \mathcal{D}(\Omega)$, we have*

$$\int_\Omega -\langle A_0 \nabla_v u, \nabla_v \varphi\rangle - uY\varphi + \nabla_v u\,\varphi = \int_\Omega f\,\varphi.$$

*A function $u \in \mathcal{W}$ is a weak sub-solution to* (6) *if for every $\varphi \in \mathcal{D}(\Omega)$, $\varphi \geq 0$, it holds*

$$\int_\Omega -\langle A_0 \nabla_v u, \nabla_v \varphi\rangle - uY\varphi + \nabla_v u\,\varphi \geq \int_\Omega f\,\varphi. \tag{8}$$

*A function $u$ is a super-solution to* (6) *if it satisfies* (8) *with $\leq$.*

The major issue one has to tackle with considering solutions belonging to the space $\mathcal{W}$ is the handling of the duality pairing between $L^2H^1$ and $L^2H^{-1}$. To this end, in a local setting the following useful remark is extensively employed, see [25, Chapter 4]. *For every open subset $A \subset \mathbb{R}^n$ and for every function $g \in H^{-1}(A)$ there exist two functions $H_0 \in L^2(A)$, $H_1 \in (L^2(A))^n$ such that $g = \mathrm{div} H_1 + H_0$ and $\|H_0\|_{L^2(A)} + \|H_1\|_{L^2(A)} \leq 2\|g\|_{H^{-1}(A)}$.*

2.1. **The Harnack inequality.** The aim of this work is to review the results presented in [10] and to analyze the fundamental ingredients required to prove the following Harnack inequality for weak solutions to (6).

**Theorem 2.1.** *Let $u$ be a non-negative weak solution to $\mathscr{L}u = f$ in $\Omega \supset \widetilde{\mathcal{Q}}_1$ under the assumptions* **(D)**, **(E)** *and* **(F)**. *Then there exist three positive constants $C, \omega$ and $\rho$ only depending on $n, \lambda, \Lambda$ such that*

$$\sup_{\widetilde{\mathcal{Q}}_-} u \leq C \left( \inf_{\mathcal{Q}_+} u + \|f\|_{L^q(\mathcal{Q}^0)} \right),$$

*where $0 < \omega < 1$ and $0 < \rho < \frac{\omega}{\sqrt{2}}$, and the cylinders $\widetilde{\mathcal{Q}}_-$ and $\mathcal{Q}_+$ are defined as $\mathcal{Q}_+ = B_\omega \times B_{\omega^3} \times (-\omega^2, 0]$ and $\widetilde{Q}_- = B_\rho \times B_{\rho^3} \times (-1+\rho^2, -1+2\rho^2)$.*

The proof of a Harnack inequality for (6) had been an open problem for decades. A first step in this direction was the proof of a $L^2 - L^\infty$ estimate by Pascucci and Polidoro in [38], where the authors considered a stronger notion of weak solution than the one of the present work, i.e. $u \in L^2(\Omega)$ and $\nabla_v u, Yu \in L^2(\Omega)$. Then, starting from this result, various authors contributed to complete the theory. On one hand, considering the same "strong" notion of weak solution the Hölder regularity was proved in [40, 41, 42]. On the other hand, another notable breakthrough was obtained in [18], where a by contradiction proof of a Harnack inequality for weak solutions to (6) in the sense of Definition 2.1 was presented, see also [3]. This result was later on refined in [20, 21] in the case of the Fokker-Planck equation, where the authors provide two different quantitative approaches to obtain analogous results for weak solutions in the sense of the present work.

To our knowledge, the only Harnack inequality for weak solutions to ultraparabolic Kolmogorov equations in $\mathbb{R}^{N+1}$ is proved in the work [10] by the author in collaboration with A. Rebucci, and the statement is the one reported in Theorem 2.1. Our proof is based on three fundamental ingredients - boundedness of weak solutions, weak Poincaré inequality and Log-transform -, that are combined to firstly obtain a weak Harnack inequality that combined with the $L^2 - L^\infty$ estimate provides us with the desired result. In the present work, we aim at reviewing the most important steps and the major difficulties one has to take into consideration in this route towards the Harnack inequality for (6). We refer the reader to [10] if interested in the more general case.

## 3. Useful estimates: Sobolev and Poincaré type inequalities

3.1. **Sobolev type inequality.** The space $\mathcal{W}$ to which weak solutions belong to is not an usual Sobolev space, because only the $L^2$ norm of the partial gradient in the $v$ direction is controlled. Hence, for many years a proper Sobolev embedding was not available for this space and many authors ran around this issue proving a Sobolev embedding for weak sub-solutions to (6). This technique was firstly proposed by Pascucci and Polidoro in [38] and it is based on potential estimates for the fundamental solution of the principal part operator associated to $\mathscr{K}$, which in in this case is the operator $\mathscr{K}_0$ as defined in (1). Locally, the geometry of $\mathscr{K}$ coincides with the one of $\mathscr{K}_0$ (see [29]). Furthermore, it is always true that if $u$ is a solution to (6), then

$$\mathscr{K}_0 u = (\mathscr{K}_0 - \mathscr{K})u + f = \mathrm{div}_v\left((\mathbb{I}_n - A_0)\nabla_v u\right) - \nabla_v u + f. \tag{9}$$

Hence, *"[...] since the classical Sobolev inequality can be proved by representing any function $u \in H^1$ as a convolution with the fundamental solution of the Laplace operator"*, see [38, p. 396], it is natural to consider a representation formula of $u$ in terms of the fundamental solution of $\mathscr{K}_0$ defined as (see [26])

$$\Gamma((v,x,t),(0,0,0)) = \left(\frac{3}{4\pi^2 t^4}\right)^{\frac{n}{2}} \exp\left(-\frac{3|x-(t/2)v|^2}{t^3} - \frac{|v|^2}{4t}\right), \qquad \text{if } t>0,$$

and equal to zero for $t \le 0$. $\Gamma$ is well-defined and, since $\mathcal{K}_0$ is dilation invariant with respect to $(\delta_r)_{r>0}$, it is also a homogeneous function of degree $-4n$. This property implies a $L^p$ estimate for Newtonian potentials, see [4]. Thus, by directly applying [13, Proposition 3], with $\alpha = 1$ and $\alpha = 2$ when considering the $\Gamma$-potential for $f$ and $\nabla_v f$, respectively, it is possible to derive explicit potential estimates, that combined with the representation formula (9) and the use of the fundamental solution $\Gamma$ lead to the following Sobolev embedding for weak sub-solutions, see [10, Theorem 3.3].

**Theorem 3.1.** *Let **(D)**, **(E)** and **(F)** hold. Let $v$ be a non-negative weak sub-solution of* (6) *in $\mathcal{Q}_1$. Then there exists a constant $C = C(n, \lambda, \Lambda) > 0$ such that the following inequality holds*

$$\parallel v \parallel_{L^{2\alpha}(\mathcal{Q}_\rho(z_0))} \leq C \left( \frac{r-\rho+1}{r-\rho} \right) \parallel \nabla_v v \parallel_{L^2(\mathcal{Q}_r(z_0))} + \frac{C\,(\rho+1)}{\rho(r-\rho)} \parallel v \parallel_{L^2(\mathcal{Q}_r(z_0))} + C \parallel f \parallel_{L^2(\mathcal{Q}_r(z_0))}$$

*for every $\rho, r$ with $\frac{1}{2} \leq \rho < r \leq 1$ and for every $z_0 \in \Omega$, where $\alpha = 1 + \frac{1}{2n}$. The same statement holds for non-negative super-solutions.*

Recently, many authors investigated the possibility to prove a proper Sobolev embedding for $\mathcal{W}$ and various results are now available in literature, see for instance [6, 17, 37], but it is still unclear if it is possible to improve the results of [10] considering these new embeddings, in particular with respect to the weak Harnack inequality when the lower order coefficients are sign changing (see [10, Remark 5.3]).

3.2. **Weak Poincaré inequality.** The importance of the Poincaré inequality for the study of the regularity of weak solutions firstly became clear in [27], where it was observed that a Poincaré inequality can be employed instead of a John-Niremberg type bounded mean oscillation lemma to prove a weak Harnack inequality for weak solutions to parabolic equations in divergence form.

In a hypoelliptic setting such as the one considered in this work it is not possible to replicate classical proofs of the Poincaré inequality due to the non-Euclidean geometry underlying the space of functions $\mathcal{W}$. Hence, other techniques need to be taken into consideration. A first Poincaré type inequality for "strong" weak sub-solutions in the Fokker-Planck and ultraparabolic settings (see Section 1) was introduced in [41] and [42], respectively. Later on, in [2] a purely functional proof of a proper Poincaré inequality for functions belonging to $\mathcal{W}$ in the Fokker-Planck case was provided. It is still an open problem wether it is possible to extend this result to the ultraparabolic setting, since the argument of the proof highly relies on the separation of variables that is compatible with the Fokker-Planck case, but not possible in the ultraparabolic one. Because of this, in [10] we were forced to consider a different path to prove a Poincaré type inequality for functions belonging to $\mathcal{W}$ in the ultraparabolic setting. So, we considered the work [20] and extended the weak Poincaré inequality presented there to our case. In order to state our result, see [10, Theorem 4.1], we first need to introduce the following sets

$$\mathcal{Q}_{zero} = \{(v,x,t) : |v| \le \eta,\ |x| \le \eta^3,\ -1-\eta^2 < t \le -1\},$$
$$\mathcal{Q}_{ext} = \{(v,x,t) : |v| \le 2R,\ |x| \le 8R,\ -1-\eta^2 < t \le 0\}, \tag{10}$$

where $R > 1$, $\eta \in (0,1)$.

**Theorem 3.2.** *Let $\eta \in (0,1)$ and let $\mathcal{Q}_{zero}$, $\mathcal{Q}_{ext}$ be defined as in (10). Let $\mathcal{Q}_1 = B_1 \times B_1 \times (-1,0)$. Then there exist $R > 1$ and $\vartheta_0 \in (0,1)$ such that for any non-negative function $u \in \mathcal{W}$ such that $u \le M$ in $\mathcal{Q}_1$, for a positive constant $M$, and*

$$|\{u = 0\} \cap \mathcal{Q}_{zero}| \ge \frac{1}{4} |\mathcal{Q}_{zero}|, \tag{11}$$

*then there exists a constant $C_P = C_P(n) > 0$ such that* [1]

$$\|(u - \vartheta_0 M)_+\|_{L^2(\mathcal{Q}_1)} \le C_P \left( \|\nabla_v u\|_{L^2(\mathcal{Q}_{ext})} + \|Yu\|_{L^2H^{-1}(\mathcal{Q}_{ext})} \right). \tag{12}$$

Inequality (12) strongly differs from a classical Poincaré inequality because it only holds for bounded functions in $\mathcal{W}$ that are equal to zero in a big enough portion of the cylinder $\mathcal{Q}_{zero}$. Moreover, the left-hand side does not involve the mean of the function $u$, but considers the difference between $u$ and a finite constant $\theta_0 M$. Lastly, the set on which is computed the $L^2$ norm of the left-hand side is smaller than the set on which the norms of the right-hand side are computed. Nevertheless, as it will be clear in forthcoming Section 4, Theorem 3.2 is strong enough to prove a weak Harnack inequality for solutions to (6).

In order to prove Theorem 3.2, the idea is to firstly derive a local Poincaré inequality in terms of an error function $h$ defined as the solution to the Cauchy problem

$$\begin{cases} -\mathscr{K}_0 h = u(-\mathscr{K}_0 \psi), & \text{in } \mathbb{R}^{2n} \times (-\rho^2, 0) \\ h = 0, & \text{in } \mathbb{R}^{2n} \times \{-\rho^2\} \end{cases} \tag{13}$$

where $\mathscr{K}_0$ is defined in (1) and $\psi$ is a function that will be chosen later on, see (14) and subsequent lines. Note that we are allowed to work with the Kolmogorov operator $\mathscr{K}_0$ because the definition of the functional space $\mathcal{W}$ only involves the partial gradient $\nabla_v$

[1] The notation we consider here needs to be intended in the sense of (7). In particular, we have that $L^2H^{-1}(\mathcal{Q}_{ext}) = L^2(B_{8R} \times (-1-\eta^2, 0], H_v^{-1}(B_{2R}))$.

and the Lie derivative $Y$. This allows us to forget the equation under study and to obtain a purely functional result. Then the local Poincaré inequality reads as follows, see [10, Lemma 4.2].

*Let $\mathcal{Q}_{ext}$ be as in* (10) *and let $\psi : \mathbb{R}^{2n+1} \to [0,1]$ be a $C^\infty$ function, with support in $\mathcal{Q}_{ext}$ and such that $\psi = 1$ in $\mathcal{Q}_1$. Then for any $u \in \mathcal{W}$, the following holds*

$$\|(u-h)_+\|_{L^2(\mathcal{Q}_1)} \leq C \left( \|\nabla_v u\|_{L^2(\mathcal{Q}_{ext})} + \|Yu\|_{L^2 H^{-1}(\mathcal{Q}_{ext})} \right)$$

*where $h$ is the solution to* (13)*, $C$ is a constant only depending on $|\rho^2|$ and $\|\nabla_v \psi\|_{L^\infty(\mathcal{Q}_{ext})}$.*

The proof of this local result is mainly based on the properties of $\mathscr{K}_0$ and $\mathcal{W}$, and for this reason it is suitable for an extension to the more general setting of ultraparabolic equations in $\mathbb{R}^{N+1}$.

Now, to complete the proof of Theorem 3.2 we are simply left with the estimate of the error function $h$ defined in (13). More specifically, we aim to show that there exists a constant $\vartheta_0 \in (0,1)$ only depending on $n$, $\lambda$ and $\Lambda$ such that the error function $h$ is bounded from above by $\vartheta_0 \|u\|_{L^\infty(\mathcal{Q}_{ext})}$. To do this, the fundamental ingredient is a specific cut-off function defined as

$$\psi_1(v,x,t) = \chi_1(\|v\|)\chi_2(x,t)\Phi_t(t), \tag{14}$$

where $\chi_1, \chi_2$ and $\Phi_t$ are defined as follows. The cut-off function $\chi_1 \in C^\infty([0,+\infty))$ is

$$\chi_1(s) = \begin{cases} 0, & \text{if } s > 2, \\ 1, & \text{if } s \leq \sqrt{2}, \end{cases} \qquad \text{and } \chi_1' \leq 0.$$

On the other hand, $\chi_2 \in C^\infty(\mathbb{R}^{2n+1})$ is a cut-off function defined as

$$\chi_2(x,t) = \chi_1 \left( \sum_{j=1}^n \frac{x_j^2}{2^5\sqrt{2}} - C\,t \right),$$

which is supported in $\mathcal{Q}_{ext}$ and equal to 1 in $\mathcal{Q}_1$, and where $C$ is a positive constant suitably chosen, see [20, Lemma 3.2] and [10, Lemma 4.3]. Finally, $\Phi_t : [-1-\eta^2, 0] \to [0,1]$ is a smooth function equal to 1 in $[-1,0]$, with $\Phi_t(0) = 1$, $\Phi_t' \geq 0$ in $[-1-\eta^2, 0]$ and $\Phi_t' = 1$ in $[-1-\eta^2, -1-T]$, where $\eta \in (0,1]$ and $T \in (0, \eta^2)$.

Then $\psi_1 : \mathbb{R}^{2n} \times [-1-\eta^2, 0]$ is supported in $\{(v,x,t) : \ |v| \le 2, \, |x| \le 8, \, t \in [-1-\eta^2, 0]\}$, is equal to 1 in $\mathcal{Q}_1$, and is such that the following conditions hold

$$(-v \cdot \nabla_x + \partial_t)\, \psi_1 \ge 0 \quad \text{everywhere}, \quad (-v \cdot \nabla_x + \partial_t)\, \psi_1 \ge 1 \quad \text{if } t \in (-1-\eta^2, -1-T].$$

Thus, considering $\psi(v,x,t) = \psi_1(v/R, x/R, t)$ as the right-hand side in (13), where $R$ is a large enough constant, one can prove a control for the localization term $h$, see [10, Lemma 4.4].

*Let $\eta \in (0,1]$ and let $\mathcal{Q}_{ext}$ be as defined in* (10)*. Then there exist $R = R(Q,\eta) > 1$, $\vartheta_0 = \vartheta_0(Q,\eta) \in (0,1)$ and a $C^\infty$ cut-off function $\psi : \mathbb{R}^{2n+1} \to [0,1]$, with support in $\mathcal{Q}_{ext}$ and equal to 1 in $\mathcal{Q}_1$, such that for all $u \in \mathcal{W}$ non-negative bounded functions defined on $\mathcal{Q}_{ext}$ and satisfying* (11)*, then the function $h$ solution to the Cauchy problem* (13) *with $\rho^2 = 1 + \eta^2$ satisfies $h \le \vartheta_0 \|u\|_{L^\infty(\mathcal{Q}_{ext})}$ in $\mathcal{Q}_1$.*

## 4. Harnack inequality

The proof of Theorem 2.1 is based on three main ingredients: a $L^2 - L^\infty$ estimates for weak sub-solutions, the Log-transform and Theorem 3.2, which was already discussed in the previous section. Hence, we now focus on the first two tools of our list, with a particular attention to the second one since the boundedness of weak subsolutions is a well-established result counting many contributions in the field in recent years, see [8, 10, 13, 38, 40, 41, 42]. Then we briefly sketch the route towards a weak Harnack inequality for non-negative super-solutions to (6), and we refer the reader to [10] for complete computations.

4.1. **Boundedness of weak sub-solutions.** To prove the boundedness of weak sub-solutions to (6) one has to combine a Caccioppoli inequality with the Sobolev type inequality introduced in Theorem 3.1.

The Caccioppoli inequality is an useful apriori estimate, also known as energy estimate, which classically provides us with quantitative information on the full gradient of the solution. Given the degenerate structure of the diffusion in (6), in the hypoelliptic (and ultraparabolic) framework it only provides us with information on the partial gradient $\nabla_v u$ of the solution. This lack of information is counterbalanced by the structure of the

Sobolev inequality we consider in this setting. The proof of this energy estimate is carried out in a standard manner, see [10, Theorem 3.4], by testing the definition of weak solution (8) against the test function

$$\varphi = p\psi^2 u_l^{2p-1}, \quad \text{with } p \geq \frac{1}{2}, \quad \text{and } u_l := u + l,$$

for a suitable constant $l > 0$, that we will chose later on, and where $\psi$ is a suitable cut-off function. By the definition of the space $\mathcal{W}$ it is possible to carry out the computations assuming $u \in C_c^\infty(\mathcal{Q}_r)$ to be a bounded weak sub-solution to (6). Indeed, if $u$ were not bounded, then we would consider the test function

$$\varphi = p\psi^2 u_{l,M}^{2p-1}, \quad \text{where } u_{l,M} = \min\{u_l, M\} \quad \text{and let } M \text{ go to infinity.}$$

Then the computations are rather standard and mainly based on a combination of Young's inequality, Hölder's inequality and assumption **(E)**. The only interesting term is the one involving the right-hand side $f$ and which is responsible for the choice of $l$. Indeed, it morally needs to be estimated in the following way

$$-\int_{\mathcal{Q}_r} f\varphi \leq p\int_{\mathcal{Q}_r} |f| u_l^{2p-1}\psi^2 = p\int_{\mathcal{Q}_r} \mathcal{C}_f u_l^{2p}\psi^2 \leq p\|\mathcal{C}_f\|_{L^\infty(\mathcal{Q}_r)} \|u_l^p\|^2_{L^2(\mathcal{Q}_r)} \leq p\|u_l^p\|^2_{L^2(\mathcal{Q}_r)},$$

where by choosing $\mathcal{C}_f$ such that $\frac{|f|}{u_l} \leq \mathcal{C}_f$ and $l := \|f\|_{L^\infty(\mathcal{Q}_r)}$ we have $\|\mathcal{C}_f\|_{L^q(\mathcal{Q}_r)} \leq 1$. Then the a priori estimate we are interested into reads as follows.

*Let* **(D)**, **(E)**, **(F)** *hold and $u$ be a non-negative weak sub-solution to* (6) *in $\mathcal{Q}_r$, with* $0 < \rho < r \leq 1$. *For any $p \in ]\frac{1}{2}, +\infty[$ such that $u^p \in L^2(\mathcal{Q}_r)$ it holds*

$$\frac{2p-1}{p}\lambda\|\nabla_v u_l^p\|^2_{L^2(\mathcal{Q}_\rho)} \leq \left(\frac{c_1}{(r-\rho)^2}\frac{p}{2p-1}\frac{\Lambda}{\lambda} + \frac{c_0}{\rho(r-\rho)}\right)\|u_l^p\|^2_{L^2(\mathcal{Q}_r)} + \left(\frac{c_0 + p(r-\rho)}{r-\rho}\right)\|u_l^p\|^2_{L^{2\beta}(\mathcal{Q}_r)} \tag{15}$$

*where $u_l = u + \|f\|_{L^\infty(\mathcal{Q}_r)}$ and $c_0, c_1$ are positive constants.*

Now, combining Theorem 3.1 with (15) for a non-negative sub-solution $u$, we obtain the following estimate. If $s > 1$, $\delta > 0$ verify the condition

$$\left|s - \frac{1}{2}\right| \geq \delta, \qquad \text{then we have} \qquad \| u_l^s \|_{L^{2\alpha}(\mathcal{Q}_\rho)} \leq \widetilde{C}\,(s, \lambda, \Lambda, n, )\, \| u_l^s \|_{L^2(\mathcal{Q}_r)}, \tag{16}$$

where $\widetilde{C}$ is a positive constant such that $\widetilde{C}(s,\lambda,\Lambda,n) \le \frac{K(\lambda,\Lambda,n)\sqrt{s}}{(r-\rho)^2}$. Then we set $v_n = u_l^{\frac{p_n}{2}}$ and we iterate inequality (16) by choosing

$$\rho_n = \frac{1}{2}\left(1+\frac{1}{2^n}\right), \qquad p_n = \left(\frac{\alpha}{2}\right)^n \frac{p}{2}, \qquad n \in \mathbb{N}\cup\{0\}.$$

Thus, by combining (16) with useful estimates on $\rho_n - \rho_{n+1}$, for every $n \in \mathbb{N}\cup\{0\}$ the following holds

$$\| v \|_{L^{\alpha^{n+1}}(\mathcal{Q}_{\rho_{n+1}})} \le \left(\frac{K\sqrt{p}}{(\rho_n-\rho_{n+1})^2}\right)^{\left(\frac{2}{\alpha}\right)^n} \| v \|_{L^{\alpha^n}(\mathcal{Q}_{\rho_n})}$$

Iterating this inequality and letting $n$ go to infinity, we get

$$\sup_{\mathcal{Q}_\rho(z_0)} u \le \frac{C}{(r-\rho)^{2n+1}} \left(\| u \|_{L^2(\mathcal{Q}_r(z_0))} + \|f\|_{L^q(\mathcal{Q}_r(z_0))}\right), \tag{17}$$

These are the main steps for this proof, and for more specific details we refer the reader to [10, Theorem 3.1].

4.2. **Log-transform.** The method of proof we follow needs to combine the information retrieved by the application of a Poincaré inequality with an expansion of positivity argument. The latter is proved via the application of a Log-transform to the equation defined through a convex function whose properties parallel the ones of the less regular function $\max(0,-\ln)$. Such a function was first constructed by Kruzhkov in [27], later on employed in [20, 42], and it is a $C^2$ convex non-increasing function $G : ]0,+\infty[ \to ]0,+\infty[$ such that

- $G'' \ge (G')^2$ and $G' \le 0$ in $]0,+\infty[$,
- $G$ is supported in $]0,1]$, with $G(t)=0$ for $t \ge 1$,
- $G(t) \sim -\ln t$ as $t \to 0^+$, $-G'(t) \le \frac{1}{t}$ for $t \in ]0,\frac{1}{4}]$.

Additionally, given a non-negative weak super-solution $u$ to (6), the function $G$ transforms it into a non-negative weak sub-solution to a suitably defined equation.

*If $\varepsilon \in \left]0,\frac{1}{4}\right]$ and $u$ is a non-negative weak super-solution to* (6) *under the assumptions* ***(D)-(E)-(F)*** *in $\mathcal{Q}_{ext} = B_{R_0} \times B_{R_1} \times ]t_0,T] \subset \Omega$, where $R_i > 0$ for $i = 0,1$. Then $g := G(u+\varepsilon^\gamma)$, for $\gamma > 0$, is a non-negative weak-sub-solution to the following equation:*

$$\mathrm{div}(A_0 \nabla_v g) + v\cdot\nabla_x g - \partial_t g + \nabla_v g + \lambda |D_{m_0} g|^2 = \varepsilon^{-\gamma}|f|. \tag{18}$$

This directly follows form the definition of $G$ and the weak formulation of (6), see [10, Lemma 5.5]. Hence, by proceeding analogously as in the previous sub-section, we can additionally derive the following Caccioppoli type inequality for $g$:

$$\frac{\lambda}{2} \int\limits_{\mathcal{Q}_{int}} |D_{m_0} g|^2 \leq C_G \int\limits_{\mathcal{Q}_{ext}} g + \varepsilon^{-\gamma} \left( \|c\|_{L^2(\mathcal{Q}_{ext})} \|u\|_{L^2(\mathcal{Q}_{ext})} + \|f\|_{L^1(\mathcal{Q}_{ext})} \right), \tag{19}$$

where $\mathcal{Q}_{int} = B_{r_0} \times B_{r_1} \times ]t_1, T_1]$, with $t_0 < t_1 < T_1 < T_0$, and $C_G = C_G(n, \Lambda, Q)$.

Then, the main application of this Log-transform is in the proof of the weak expansion of positivity, an useful result quantifying the ability of (6) to spread positivity information for super-solutions. More precisely, let us introduce the cylinders

$$\mathcal{Q}_{pos} = B_\theta \times B_{\theta^3} \times (-1 - \theta^2, -1], \qquad \widetilde{\mathcal{Q}}_{ext} = B_{3R} \times B_{3^3 R} \times (-1 - \theta^2, 0],$$

where $R = R(\theta, n, \lambda, \Lambda)$ is the constant introduced in the proof of Theorem 3.2 and $\theta \in (0, 1]$ is a parameter we will choose later on. Then our expansion of positivity lemma reads as follows, see [10, Lemma 5.8].

**Lemma 4.1.** *Let $\theta \in (0, 1]$. Then there exist a small positive constant $\eta_0 = \eta_0(\theta, n, \lambda, \Lambda) \in (0, 1)$ such that for any non-negative weak super-solution $u$ of (6) under assumptions **(D)**-**(E)**-**(F)** in some cylindrical open set $\Omega \supset \widetilde{\mathcal{Q}}_{ext}$ such that*

$$|\{u \geq 1\} \cap \mathcal{Q}_{pos}| \geq \frac{1}{2} |\mathcal{Q}_{pos}|,$$

*we have $u \geq \eta_0$ in $\mathcal{Q}_1$.*

The proof of this lemma considers $g = G(u + \varepsilon^\gamma)$ for $\varepsilon \in \left]0, \frac{1}{4}\right]$ and $\gamma = \frac{1}{8}$. Then by definition of $G$ we have that $g$ is non-negative and a sub-solution to (18). Since $G$ is non-increasing and $\varepsilon \in \left]0, \frac{1}{4}\right]$ we also have $g \leq G(\varepsilon^{\frac{1}{8}}) \leq G(\varepsilon)$. Then the idea of the proof is to apply (17) to the function $g$ (with $\mathcal{Q}_1$ as the small cylinder and $\mathcal{Q}_{1+\iota}$ as the big cylinder with an accurate choice of $\iota$ such that $\mathcal{Q}_{ext} \subset \mathcal{Q}_{1+\iota} \subset \delta_{(1+\iota)^2} \mathcal{Q}_{ext} \subset \widetilde{\mathcal{Q}}_{ext}$,) combined with Theorem 3.2 scaled on the cylinder $\mathcal{Q}_{1+\iota} = \delta_{1+\iota}(\mathcal{Q}_{ext})$. Finally, the $L^2$-norm of $\nabla_v g$ is estimated via the square root of the square of its mass on a larger cylinder.

We point out that the key ingredient is the application of Theorem 3.2 to $g$, which is possible because the function $g = G(u + \varepsilon^{\frac{1}{8}})$, by definition, is equal to zero if $u + \varepsilon^{\frac{1}{8}} > 1$, and this happens if and only if $u > 1 - \varepsilon^{\frac{1}{8}}$. Then we choose $\iota$ in such a way that

$$| \{g = 0\} \cap \delta_{(1+\iota)}(\mathcal{Q}_{zero})| \geq | \{u \geq 1\} \cap \delta_{(1+\iota)}(\mathcal{Q}_{zero})| \geq \frac{1}{4} |\delta_{(1+\iota)}(\mathcal{Q}_{zero})|.$$

Hence, the function $g$ satisfies inequality (12) with $\delta_{(1+\iota)}(\mathcal{Q}_{zero})$ taking the role of $\mathcal{Q}_{zero}$ and we are therefore allowed to apply Theorem 3.2. Thus, we have a chain of inequalities

$$\begin{aligned}
g - \theta_0 G(\varepsilon) &\leq (g - \theta_0 G(\varepsilon))_+ \leq \sup_{\mathcal{Q}_1} (g - \theta_0 G(\varepsilon))_+ \\
&\leq C_\iota C_M \parallel (g - \theta_0 G(\varepsilon))_+ \parallel_{L^2(\mathcal{Q}_{1+\iota})} + C_\iota C_M \varepsilon^{-\frac{1}{8}} \parallel f \|_{L^q(\mathcal{Q}_{1+\iota})} && \text{by (17)} \\
&\leq C_P C_\iota C_M \|\nabla_v (g - \theta_0 G(\varepsilon))_+\|_{L^2(\mathcal{Q}_{ext})} && \text{Theorem 3.2} \\
&\qquad + C_P C_\iota C_M \|Y(g - \theta_0 G(\varepsilon))_+\|_{L^2 H^{-1}(\mathcal{Q}_{ext})} + C_\iota C_M \varepsilon^{-\frac{1}{8}} \parallel f \|_{L^q(\mathcal{Q}_{1+\iota})} \\
&\leq C \left( C_P C_\iota C_M C_G \right)^{\frac{1}{2}} \left( \int_{\mathcal{Q}_{ext}} g \right)^{\frac{1}{2}} && \text{by (19)} \\
&\qquad + C \left( C_P C_\iota C_M C_G \right)^{\frac{1}{2}} \varepsilon^{-\frac{1}{16}} \left( \|f\|_{L^1(\mathcal{Q}_{ext})} \right)^{\frac{1}{2}} + C_\iota C_M \varepsilon^{-\frac{1}{8}} \parallel f \|_{L^q(\mathcal{Q}_{1+\iota})} \\
&\leq C_1 \sqrt{G(\varepsilon)} + C_2 \varepsilon^{-\frac{1}{8}} && G \text{ is non-increasing}
\end{aligned}$$

where $C_\iota$ is a constant due to the scaling on the cylinder $\mathcal{Q}_{(1+\iota)}$ and $C_1$, $C_2$ are two positive constants depending on $n$, $\Lambda$, $\lambda$, $\|f\|_{L^q(\mathcal{Q}_{ext})}$ and $\iota$ and we used inequality $\varepsilon^{-\frac{1}{16}} \leq \varepsilon^{-\frac{1}{8}}$ for $\varepsilon$ sufficiently small. All in all, we obtain the following estimate

$$G(u + \varepsilon^{\frac{1}{8}}) - \theta_0 G(\varepsilon) \leq C_1 \sqrt{G(\varepsilon)} + C_2 \varepsilon^{-\frac{1}{8}} \qquad \text{in } \mathcal{Q}_1.$$

The rest of the proof follows by applying $g$ properties listed above. This is just a sketch of the proof, for a complete result we refer to [10].

4.3. **Harnack inequality.** Finally, the proof of Theorem 2.1 directly follows combining a weak Harnack inequality with the boundedness estimate (17). Hence, we are now interested into the proof of the following weak Harnack inequality via a covering argument based on an adaptation of the Ink Spot theorem, or Caldérun-Zigmund decomposition, to the hypoelliptic setting, see [10, 20, 24].

**Theorem 4.1.** *Let* $\mathcal{Q}^0 = B_{R_0} \times B_{R_0} \times (-1, 0]$ *and let* $u$ *be a non-negative weak super-solution to* (6) *in* $\Omega \supset \mathcal{Q}^0$ *under assumptions* ***(D)-(E)-(F)****. Then*

$$\left( \int_{\mathcal{Q}_-} u^p \right)^{\frac{1}{p}} \leq C \left( \inf_{\mathcal{Q}_+} u + \| f \|_{L^q(\mathcal{Q}^0)} \right), \tag{20}$$

*where* $\mathcal{Q}_+ = B_\omega \times B_{\omega^3} \times (-\omega^2, 0]$ *and* $\mathcal{Q}_- = B_\omega \times B_{\omega^3} \times (-1, -1 + \omega^2]$*. Moreover, the constants* $C$*,* $p$*,* $\omega$ *and* $R_0$ *only depend on* $n$*,* $q$ *and* $\lambda$*,* $\Lambda$*.*

The proof of this result is obtained combining the fact that super-solutions to (6) expand positivity along times (Lemma 4.1) with the covering argument based on the family of stacked cylinders described in [10, Lemma Appendix B.1]. We mention that a similar argument was introduced for the Boltzmann equation in [24] and for the study of the Kolmogorov operator in trace form with Cordes-Landis assumptions in [1]. The result stated here, which is a specific version of [10, Theorem 5.1] is the only weak Harnack inequality available for Kolmogorov equations in the ultraparabolic setting. For the proof of this result we refer to [10], but it is worth recalling that $\omega$ is small enough so that, when expanding positivity from a given cylinder $\mathcal{Q}_r(z_0)$ in the past, the union of the stacked cylinders where the positivity is spread includes $\mathcal{Q}_+$. Additionally, $R_0$ is big enough so that Lemma 4.1 can be applied to every stacked cylinder.

## 5. Future developments

As already mentioned in the introduction of this work, our results already apply to the general case of ultraparabolic Kolmogorov equations with rough coefficients of the type

$$\begin{aligned} \mathscr{L} u(x,t) :=& \sum_{i,j=1}^{m_0} \partial_{x_i} \left( a_{ij}(x,t) \partial_{x_j} u(x,t) \right) + \sum_{i,j=1}^{N} b_{ij} x_j \partial_{x_i} u(x,t) - \partial_t u(x,t) \\ &+ \sum_{i=1}^{m_0} b_i(x,t) \partial_i u(x,t) + c(x,t) u(x,t) = f(x,t), \end{aligned}$$

where $z = (x,t) = (x_1, \ldots, x_N, t) \in \mathbb{R}^{N+1}$, $1 \leq m_0 \leq N$, the matrix $A_0 = (a_{ij}(x,t))_{i,j=1,\ldots,m_0}$ satisfies an ellipticity assumption, and $B = (b_{ij})_{i,j=1,\ldots,N}$ is a constant matrix such that the principal part operator of $\mathscr{L}$ is dilation invariant with respect to a suitable family of dilations in $\mathbb{R}^{N+1}$. What remains an open problem is the study of Kolmorov operators in non-divergence form, or trace form, where the diffusion is given by $\mathrm{Tr}(A_0 D^2_{m_0} u)$. Indeed, the only available result in this direction is [1], where the authors prove a Harnack inequality for $C^2$ solutions under Cordes-Landis assumptions for the matrix $A_0$.

Nowadays, interesting applications of the De Giorgi-Nash-Moser regularity theory for Kolmogorov equations are found in kinetic theory, see [34]. In short, kinetic theory can be thought as a modeling theory for physical interactions involving number of elements so huge that it becomes impossible to track the individual behavior of each component. In particular, it is useful to describe the evolution of a system through the study of its statistical distribution over the phase space, where the only requirements are that the objects of the system share identical individual properties and their physical state is described by a phase space. Among the most interesting models of kinetic theory we recall the Boltzmann equation and the Landau equation, see for instance [19, 24, 34], which share the same transport operator of the Fokker-Planck equation (6). The study of these models still presents many open problems, and usually the Fokker-Planck equation (6) is viewed as a local approximation of the behavior of these more complex equations.

As far as we are concerned with the Boltzmann equation, many authors had contributed over the years, but the study of the regularity theory in this field has experienced a major breakthrough with a series of works by Imbert and Silvestre and a weak Harnack inequality for positive solutions is now available, see [24]. Among other recent contributions, it is worth recalling the work [31], where the author improves the results of [24] via a quantitative argument, and the work [5], where the authors consider a sub-class of the integro-differential operators considered in [24, 31] and prove the boundedness of sign-changing weak solutions. For a list of interesting open problems in this field we refer the reader to [34]. On the other hand, considering the Landau equation we find many interesting contributions studying the space-homogeneous case, see for instance [19], or a space inhomogeneous toy model introduced in [23], see for instance [11].

Lastly, we mention a very recent series of works considering kinetic equations where the transport term is a variation of the one appearing in (6):

$$\nabla_v \cdot (A_0(v,x,t)\nabla_v u(v,x,t)) + \nabla_v u(v,x,t) + (\mathbf{b}(v)\cdot\nabla_x - \partial_t)\, u(v,x,t) = f(v,x,t),$$

where $\mathbf{b}$ is a vector of $\mathbb{R}^n$ satisfying suitable non-degeneracy assumptions. This family includes various interesting models, such as the relativistic Kolmogorov equaiton [9], and the regularity theory for it was recently proved in [43].

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Dipartimento di Ingegneria Industriale e Scienze Matematiche, Università Politecnica delle Marche, Via Brecce Bianche, 12, 60131 Ancona, Italy

*Email address*: f.anceschi@staff.univpm.it