NONLOCAL NEUMANN BOUNDARY CONDITIONS
CONDIZIONI AL BORDO DI NEUMANN NON LOCALI

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ABSTRACT. We present some properties of a nonlocal version of the Neumann boundary conditions associated to problems involving the fractional p-Laplacian. For this problems, we show some regularity results for the general case and some existence results for particular types of problems. When $p = 2$, we give a generalization of the boundary conditions in which both the nonlocal and the classic Neumann conditions are present, and we consider problems involving both nonlocal and local interactions.

SUNTO. Presentiamo alcune proprietà di una versione non locale delle condizioni al bordo di Neumann associate a problemi in presenza del p-Laplaciano frazionario. Per questa tipologia di problemi, mostriamo alcuni risultati di regolarità per il caso generale e qualche risultato di esistenza per alcuni problemi in particolare. Quando $p = 2$, diamo una generalizzazione delle condizioni al bordo dove sono presenti sia le condizioni di Neumann non locali che quelle classiche, considerando problemi con interazioni sia non locali che locali.

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1. Introduction

Consider a bounded domain Ω of $\mathbb{R}^N$, $N \geq 1$, with Lipschitz boundary. We want to investigate problems of the form

$$\begin{cases} (-\Delta)_p^s u = f(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = g(x) & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{1}$$

where

$$(-\Delta)_p^s u(x) = C_{N,s,p} PV \int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+ps}}\, dy$$

is the fractional p-Laplacian and

$$\mathscr{N}_{s,p} u(x) := C_{N,s,p} \int_\Omega |u(x)-u(y)|^{p-2} \frac{u(x)-u(y)}{|x-y|^{N+ps}}\, dy, \quad x \in \mathbb{R}^N \setminus \overline{\Omega}, \tag{2}$$

is the *nonlocal normal $p-$derivative*, or *$p-$Neumann boundary condition* and describes the natural Neumann boundary condition in presence of the fractional $p-$Laplacian. It extends the notion of nonlocal normal derivative introduced in [5] for the fractional Laplacian, i.e. for $p = 2$. The definition in (2) was introduced in [1], where basic integration by parts were given. In our situation, $p > 1$, $s \in (0,1)$ and $C_{N,s,p}$ is the constant appearing in the definition of the fractional $p-$Laplacian; however, for the sake of simplicity, from now on, we will set $C_{N,s,p} = 1$.

We recall here the analogous of the divergence theorem and of the integration by parts formula for the nonlocal case first introduced in [1]:

Proposition 1.1. *Let u be any bounded C^2 function in $\mathbb{R}^N$. Then,*

$$\int_\Omega (-\Delta)_p^s u\, dx = -\int_{\mathbb{R}^N \setminus \Omega} \mathscr{N}_{s,p} u\, dx.$$

Proposition 1.2. *Let u and v be bounded C^2 functions in $\mathbb{R}^N$. Then,*

$$\frac{1}{2} \int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} |u(x)-u(y)|^{p-2} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\, dxdy$$
$$= \int_\Omega v(-\Delta)_p^s u\, dx + \int_{\mathbb{R}^N \setminus \Omega} v \mathscr{N}_{s,p} u\, dx.$$

Throughout the paper we give properties and results regarding the topics discussed above.

We start in Section 2 giving the definition of the functional space for the purely nonlocal case as introduced in [1] and then extended in [10], as well as some useful embedding results. Moreover, we give a characterization for functions satisfying the homogeneous Neumann condition when $p = 2$, that is $\mathscr{N}_{s,2}u = 0$, and we give the definition of weak solution in the general framework. Finally we give some regularity results. The main one is an L^∞ result for weak solutions and relies on an iteration method, see also [7] for details on the Dirichlet case and more references.

In Section 3 we deal with different kinds of nonlocal problems, starting with the eigenvalue problem. In this case, we prove the existence of an unbounded sequence of eigenvalues and that all eigenfunctions not corresponding to the first eigenvalue changes sing. The sequence of eigenvalues is constructed in a variational way using the cohomological index of Fadell and Rabinowitz, see respectively [8] for the Dirichlet case and [6] for more details on the index. Then we deal with problems in presence of the Ambrosetti-Rabinowitz condition and, depending on the hypotheses on the nonlinearity, we have two different cases. In the first one we prove existence of solution using a linking over cones argument, see also [2] for the local case. In the second case, to prove the existence result we rely on critical group theory and in particular on the notion of cohomological local splitting, see [7] for more details and the Dirichlet case. We also deal with problems without Ambrosetti-Rabinowitz condition. In this case we have assume a different condition, which we will give in detail later, introduced in [9] and generalized in [10]. The proof of the existence result relies on the mountain pass Theorem. Finally, we deal with a nonlocal problem with a source term which is asimptotically p-linear at infinity. Here we prove existence results both in the resonant and non-resonant case. The proofs of both this results relies on critical group theory, see also [7] for the Dirichlet case.

In Section 4 we deal with a different kind of operator in presence of both local and nonlocal interactions. In order to do so, we first introduce a new functional space, give the definition of weak solution and describe the Neumann boundary condition in this framework. Then, we deal with a weighted eigenvalue problem and prove the existence of two unbounded sequences of eigenvalues, one being positive and the other negative. Moreover, we show some properties of the first positive eigenvalue. To conclude, we deal with a problem involving a logistic equation in this setting, and we first prove the existence of a nonnegative solution. Then, depending on the term m of the equation, which will be better described later, we give a threshold to determine if the solution is the trivial one or not. Since this threshold depends on the first positive eigenvalue of the associated eigenvalue problem, we naturally consider an optimization problem for such an eigenvalue depending on m, and show when it is possible to have a nontrivial solution or when the only solution is the trivial one. For more information on this type of optimization problem see also [13].

2. Functional space, properties and regularity

In this section we introduce the functional setting and some basic properties, together with some regularity results.

Fix a bounded domain with Lipschitz boundary $\Omega \subset \mathbb{R}^N$, $N \geq 1$, and for $u : \mathbb{R}^N \to \mathbb{R}$ measurable, set

$$\|u\|_X := \left(\|u\|^p_{L^p(\Omega)} + \||g|^{\frac{1}{p}} u\|^p_{L^p(\mathbb{R}^N \setminus \Omega)} + \int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} \frac{|u(x) - u(y)|^p}{|x - y|^{N+ps}} \, dxdy \right)^{\frac{1}{p}},$$

where $C\Omega = \mathbb{R}^N \setminus \Omega$, and

$$X := \{u : \mathbb{R}^N \to \mathbb{R} \quad \text{measurable such that } \|u\|_X < \infty\}.$$

Remark 2.1. *It is clear that, Ω being "nice enough", in the previous setting we can equally write $\mathbb{R}^N \setminus \Omega$ in place of $\mathbb{R}^N \setminus \overline{\Omega}$. The abstract setting can be faced also for Ω less regular, replacing $\||g|^{\frac{1}{p}} u\|_{L^p(\mathbb{R}^N \setminus \Omega)}$ with $\||g|^{\frac{1}{p}} u\|_{L^p(\mathbb{R}^N \setminus \overline{\Omega})}$, which is the natural norm in the general framework.*

As a consequence of this definition, the space X has the following.

Proposition 2.1. *X is a reflexive Banach space with norm $\| \cdot \|_X$.*

Remark 2.2. *X is embedded in $L^p(B(0, R))$ for every $R > 0$.*

Remark 2.3. *X is embedded continuously in $W^{s,p}(\Omega)$. As a consequence, recalling the standard compact embeddings of $W^{s,p}(\Omega)$ in $L^q(\Omega)$ for a suitable q, we have that the space X is embedded compactly in $L^q(\Omega)$ for the same values of q.*

For the proof of Proposition 2.1 and further details on Remarks 2.2 and 2.3 see [10].

When $p = 2$, we can give an interesting property concerning homogeneous boundary conditions, that is, all functions satisfying $\mathscr{N}_s u = \mathscr{N}_{s,2} u = 0$ minimize the Gagliardo seminorm. Setting

$$E_u(x) := \int_\Omega \frac{u(z)}{|x - z|^{N+2s}}\, dz.$$

for all $x \in \mathbb{R}^N \setminus \overline{\Omega}$, it is easy to see that if

$$u(x) = \frac{E_u(x)}{E_1(x)},$$

for all $x \in \mathbb{R}^N \setminus \overline{\Omega}$, then $\mathscr{N}_s u = 0$. Then we have the following result:

Theorem 2.1. *Let $u : \mathbb{R}^N \to \mathbb{R}$ with $u \in L^1(\Omega)$, if we define*

$$\tilde{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ \\ \dfrac{E_u(x)}{E_1(x)} & \text{if } x \in \mathbb{R}^N \setminus \overline{\Omega}, \end{cases}$$

we have

$$\int_{\mathcal{Q}} \frac{|\tilde{u}(x) - \tilde{u}(y)|^2}{|x - y|^{N+2s}}\, dx\, dy \le \int_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}}\, dx\, dy.$$

In addition, the equality holds if and only if $\mathscr{N}_s u = 0$.

The proof of Theorem 2.1 can be found in [4].

The integration by parts formula in Proposition 1.2 leads to this natural definition:

Definition 2.1. *Let $f \in L^{p'}(\Omega)$ and $g \in L^1(\mathbb{R}^N \setminus \overline{\Omega})$. We say that $u \in X$ is a weak solution of problem (1) whenever*

$$\frac{1}{2} \int_{\mathbb{R}^{2N} \setminus (C\Omega)^2} \frac{J_p(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+ps}}\, dxdy = \int_\Omega f v\, dx + \int_{\mathbb{R}^N \setminus \overline{\Omega}} g v\, dx$$

for every $v \in X$, where

$$J_p(u(x) - u(y)) := |u(x) - u(y)|^{p-2}(u(x) - u(y)).$$

As usual, to find solutions for a problem we will seek critical points of suitable functionals.

Now we show some regularity results for this type of problems. The first one is a consequence of Definition 2.1, and is the following

Theorem 2.2. *Let u be a weak solution of* (1)*. Then, $\mathcal{N}_{s,p}u = g$ a.e. in $\mathbb{R}^N \setminus \overline{\Omega}$.*

This result gives information only outside of Ω, but it is useful to prove an L^∞ regularity result, which is the following

Proposition 2.2. *If there exists $a > 0$ such that*

$$|f(x,t)| \leq a(|t|^{p-1} + |t|^{r-1})$$

for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$, with $1 \leq p \leq r \leq \frac{pN}{N-ps}$ if $N > ps$, then for every weak solution $u \in X$ of

$$\begin{cases} (-\Delta)^s_p u = f & \text{in } \Omega, \\ \mathcal{N}_{s,p}u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega} \end{cases}$$

we have $u \in L^\infty(\mathbb{R}^N)$.

In particular, when $p = 2$, we have continuity of solutions outside of Ω.

Proposition 2.3. *Let $u \in X$ be a weak solution of*

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ \mathcal{N}_s u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}. \end{cases}$$

Then, $u \in C(\mathbb{R}^N \setminus \Omega)$.

A proof of Theorem 2.2 can be found in [10], while for the proofs of Propositions 2.2 and 2.3 see [12].

3. Nonlocal problems

In this section we present different kinds of problems in presence of an operator with only nonlocal interactions.

3.1. **The eigenvalue problem.** In this subsection we consider the nonlinear eigenvalue problem

$$\begin{cases} (-\Delta)^s_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{3}$$

depending on parameter $\lambda \in \mathbb{R}$.

Definition 3.1. *We say that λ is an eigenvalue of $(-\Delta)^s_p$ with $p-$Neumann boundary conditions and associated λ-eigenfunction u if* (3) *admits a weak solution $u \in X$ (notice that now $g \equiv 0$), that is*

$$\frac{1}{2}\int_{\mathbb{R}^{2N}\setminus(C\Omega)^2} \frac{J_p(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+ps}}\,dxdy = \lambda \int_\Omega |u|^{p-2}uv\,dx$$

for all $v \in X$.

As in the classical case, we call the set of all the eigenvalues the point spectrum of $(-\Delta)^s_p$ in X and we denote it by $\sigma(s,p)$. It is easy to see that for $\lambda = 0$ constant functions are all 0-eigenfunctions. Then we give the following results

Proposition 3.1. *There exists a sequence of eigenvalues $(\lambda_k)_k$ with $k \in \mathbb{N}$ such that $\lambda_k \to \infty$.*

Proposition 3.2. *Let $v \in X$ be a solution to* (3) *such that $v > 0$ in Ω. Then $\lambda = 0$, hence v is constant.*

For more details and references on the eigenvalue problem, see [10].

3.2. **p-superlinear problems with A-R condition.** The next problem that we consider is

$$\begin{cases} (-\Delta)^s_p u = \lambda |u|^{p-2}u + g(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}. \end{cases} \tag{4}$$

We recall that $p \in (1,\infty)$, Ω is a bounded domain with Lipschitz boundary, $\lambda \geq 0$ and $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is the map $x \mapsto g(x,t)$ is measurable for every $t \in \mathbb{R}$ and the map $t \mapsto g(x,t)$ is continuous for a.e. $x \in \Omega$. We assume the following hypotheses on g:

(g_1) there exist constants $a_1, a_2 > 0$ and $q > p$ such that for every $t \in \mathbb{R}$ and for a.e. $x \in \Omega$

$$|g(x,t)| \leq a_1 + a_2|t|^{q-1},$$

where $q < \frac{pN}{N-ps}$ if $N > ps$;

(g_2) $g(x,t) = o(|t|^{p-1})$ as $t \to 0$ uniformly a.e. in Ω;

(g_3) denoting $G(x,t) = \int_0^t g(x,\tau)\,d\tau$, there exist $\mu > p$ and $R \geq 0$ such that for every t with $|t| > R$ and for a.e. $x \in \Omega$

$$0 < \mu G(x,t) \leq g(x,t)t,$$

and there exist $\tilde{\mu} > p$, $a_3 > 0$ and $a_4 \in L^1(\Omega)$ such that for every $t \in \mathbb{R}$ and a.e. $x \in \Omega$,

$$G(x,t) \geq a_3|t|^{\tilde{\mu}} - a_4(x);$$

(g_4) if $R > 0$, then $G(x,t) \geq 0$ for every $t \in \mathbb{R}$ and a.e. $x \in \Omega$.

Condition (g_3) is commonly known as the Ambrosetti-Rabinowitz, A-R condition for short.

A first existence result is given in [11].

Theorem 3.1. *If hypotheses* (g_1) $-$ (g_4) *hold, then problem* (4) *admits a nontrivial weak solution.*

This result can actually be improved dropping condition (g_4), with some assumptions on the relation between λ and eigenvalues of problem (3). This is shown in the following result, and its proof can be found in [12].

Theorem 3.2. *If hypotheses* (g_1)-(g_3) *and one of the following hold:*

(*i*) $2\lambda \neq \lambda_k$ *for all* $k \in \mathbb{N}$,

(*ii*) $2\lambda = \lambda_k$, $k \in \mathbb{N}$, *and* $G(x,t) \geq 0$ *a.e. in* Ω *and for all* $|t| \leq \delta$ *for some* $\delta > 0$,

(*iii*) $2\lambda = \lambda_k$, $k \in \mathbb{N}$, *and* $G(x,t) \leq 0$ *a.e. in* Ω *and for all* $|t| \leq \delta$ *for some* $\delta > 0$,

then problem (4) *admits a nontrivial solution.*

For more information and references on problems in presence of the A-R condition, see [11] and [12].

3.3. **p-superlinear problems without A-R condition.** In this subsection we consider problems of the type

$$\begin{cases} (-\Delta)_p^s u + |u|^{p-2}u = f(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p}u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{5}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ a Carathéodory function such that $f(x,0) = 0$ for almost every $x \in \Omega$. In addition, we assume the following hypotheses:

(f_1) there exists $a \in L^q(\Omega)$, $a \geq 0$, with $q \in ((p_s^*)', p)$, $c > 0$ and $r \in (p, p_s^*)$ such that

$$|f(x,t)| \leq a(x) + c|t|^{r-1}$$

for a.e. $x \in \Omega$ and for all $t \in \mathbb{R}$;

(f_2) denoting $F(x,t) = \int_0^t f(x,\tau)d\tau$, we have

$$\lim_{t\to\pm\infty} \frac{F(x,t)}{|t|^p} = +\infty$$

uniformly for a.e. $x \in \Omega$;

(f_3) if $\sigma(x,t) = f(x,t)t - pF(x,t)$, then there exist $\vartheta \geq 1$ and $\beta^* \in L^1(\Omega)$, $\beta^* \geq 0$, such that

$$\sigma(x,t_1) \leq \vartheta\sigma(x,t_2) + \beta^*(x)$$

for a.e. $x \in \Omega$ and all $0 \leq t_1 \leq t_2$ or $t_2 \leq t_1 \leq 0$;

(f_4)

$$\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = 0$$

uniformly for a.e. $x \in \Omega$.

As usual, in (f_1) we have denoted by p_s^* the fractional Sobolev exponent of order s, that is

$$p_s^* = \begin{cases} \dfrac{pN}{N-ps} & \text{if } ps < N, \\ \infty & \text{if } ps \geq N, \end{cases}$$

so that the embedding in $L^q(\Omega)$ of $W^{s,p}(\Omega)$ (and thus of X) is compact for every $q < p_s^*$. The existence result is the following and its proof can be found in [10], as well as more details and references on problems without the A-R condition.

Theorem 3.3. *If hypotheses* (f_1)-(f_4) *hold, then problem* (5) *admits two non-trivial constant sign solutions.*

3.4. **Asimptotically p-linear problems.** In this subsection we consider problem

$$\begin{cases} (-\Delta)_p^s u = h(x,u) & \text{in } \Omega, \\ \mathscr{N}_{s,p} u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \end{cases} \tag{6}$$

where $h(x,t) = \lambda |t|^{p-2} t + g(x,t)$ with $g(x,t) = o(|t|^{p-1})$ as $t \to \infty$. In light of this, it is clear that $h(x,\cdot)$ is asimptotically p-linear at infinity, that is

$$\lim_{|t|\to\infty} \frac{h(x,t)}{|t|^{p-2}t} = \lambda$$

uniformly a.e. in Ω, for some $\lambda \in (0,\infty)$. We recall that in general a problem of this kind is said to be of resonant type if $\lambda \in \sigma(s,p)$. Otherwise, it it said to be of non-resonant type. However, we shall see that problem (6) is of resonant type when $\lambda + 1 \in \sigma(s,p)$. We assume that $h : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, with

$$H(x,t) = \int_0^t h(x,\tau)\, d\tau \text{ for all } (x,t) \in \Omega \times \mathbb{R},$$

and satisfying the following hypotheses:

- (h_1) $|h(x,t)| \leq a(1 + |t|^{r-1})$ a.e. in Ω and for all $t \in \mathbb{R}$, with $a > 0$ and $1 < r < p_s^*$,
- (h_2) $\lim_{|t|\to\infty} \frac{h(x,t)}{|t|^{p-2}t} = \lambda$ uniformly a.e. in Ω, with $\lambda > 0$,
- (h_3) there exist $\delta > 0$ and $\mu \in (0,p)$ such that

$$h(x,t)t > 0, \quad \text{for } x \in \Omega, \quad 0 < |t| < \delta,$$

$$\mu H(x,t) - h(x,t)t \geq 0 \ \text{ for } x \in \Omega, \quad |t| < \delta.$$

The first result that we have covers the non-resonant case:

Theorem 3.4. *If hypotheses* (h_1)-(h_3) *hold with* $\lambda \in (\lambda_k, \lambda_{k+1}) \setminus \sigma(s,p)$ *for some* $k \in \mathbb{N}$, *then problem* (6) *admits a nontrivial solution.*

In order to deal with the resonant case, we need to assume additional conditions to have compactness of critical sequences. For all $(x,t) \in \Omega \times \mathbb{R}$ we define

$$\mathcal{H}(x,t) := pH(x,t) - h(x,t)t$$

We have the following result:

Theorem 3.5. *If hypotheses* (h_1)-(h_3) *hold with* $\lambda + 1 \in \sigma(s,p)$*, and there exist* $k \in \mathbb{N}$*,* $h_0 \in L^1(\Omega)$ *such that one of the following holds:*

(i) $\lambda_k < \lambda + 1 \le \lambda_{k+1}$*,* $\mathcal{H}(x,t) \le -h_0(x)$ *a.e. in* Ω *and for all* $t \in \mathbb{R}$*, and*

$$\lim_{|t|\to\infty} \mathcal{H}(x,t) = -\infty$$

uniformly a.e. in Ω*,*

(ii) $\lambda_k \le \lambda + 1 < \lambda_{k+1}$*,* $\mathcal{H}(x,t) \ge h_0(x)$ *a.e. in* Ω *and for all* $t \in \mathbb{R}$*, and*

$$\lim_{|t|\to\infty} \mathcal{H}(x,t) = \infty$$

uniformly a.e. in Ω*.*

Then problem (6) *admits a nontrivial solution.*

The proofs of Theorems 3.4 and 3.5 can be found in [12].

4. Mixed problems

In this section, we first let $s \in (0,1)$ and α, $\beta \in [0,+\infty)$ with $\alpha + \beta > 0$, and we consider the mixed operator

$$-\alpha\Delta + \beta(-\Delta)^s.$$

In order to deal with problems involving such an operator, the functional space that we consider is

$$X_{\alpha,\beta} = X_{\alpha,\beta}(\Omega) := \begin{cases} H^1(\Omega) & \text{if } \beta = 0, \\ H^s_\Omega & \text{if } \alpha = 0, \\ H^1(\Omega) \cap H^s_\Omega & \text{if } \alpha\beta \ne 0, \end{cases}$$

where

$$H^s_\Omega := \left\{ u : \mathbb{R}^n \to \mathbb{R} \text{ s.t. } u \in L^2(\Omega) \text{ and } \iint_{\mathcal{Q}} \frac{|u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy < +\infty \right\},$$

and $\mathcal{Q}$ is the cross-shaped set on Ω given by

$$\mathcal{Q} := \big(\Omega \times \Omega\big) \cup \big(\Omega \times (\mathbb{R}^n \setminus \Omega)\big) \cup \big((\mathbb{R}^n \setminus \Omega) \times \Omega\big).$$

We observe that $X_{\alpha,\beta}$ is a Hilbert space with respect to the scalar product

$$\begin{aligned}(u,v)_{X_{\alpha,\beta}} &:= \int_\Omega u(x)v(x)\,dx + \alpha\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx \\ &\quad + \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy,\end{aligned}$$

for every $u, v \in X_{\alpha,\beta}$.

We also define the seminorm

$$[u]^2_{X_{\alpha,\beta}} := \frac{\alpha}{2}\int_\Omega |\nabla u(x)|^2\,dx + \frac{\beta}{4}\iint_{\mathcal{Q}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy. \tag{7}$$

Definition 4.1. *Let $f \in L^2(\Omega)$. We say that $u \in X_{\alpha,\beta}$ is a weak solution of*

$$-\alpha\Delta u + \beta(-\Delta)^s u = f \qquad \text{in } \Omega$$

with (α,β)-Neumann condition if

$$\alpha\int_\Omega \nabla u(x)\cdot\nabla v(x)\,dx + \frac{\beta}{2}\iint_{\mathcal{Q}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy = \int_\Omega f(x)\,v(x)\,dx, \tag{8}$$

for every $v \in X_{\alpha,\beta}$.

We remark that, formally, the external condition in (8) can be detected by taking v with $v = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$ (which produces a normal derivative prescription along $\partial\Omega$) and then by taking $v = 0$ in $\overline{\Omega}$ (which produces a nonlocal prescription in $\mathbb{R}^n \setminus \overline{\Omega}$): that is, formally, the external condition in (8) can be written in the form

$$\begin{cases} \mathscr{N}_s u = 0 & \text{in } \mathbb{R}^n \setminus \overline{\Omega}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{9}$$

where ν is the exterior normal to Ω, the first condition in (9) being dropped when $\alpha = 0$, the second condition in (9) being dropped when $\beta = 0$. We call this conditions the (α,β)-Neumann condition for short.

More details and references on this mixed operator can be found in [3].

4.1. **The eigenvalue problem.** The first problem that we consider in this section involves a weighted eigenvalue equation, so we let $m : \Omega \to \mathbb{R}$ and consider

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u = \lambda m u & \text{in } \Omega, \\ \text{with } (\alpha,\beta)\text{-Neumann condition.} \end{cases} \tag{10}$$

According to (8) the notion of solution in (10) is in the weak sense in the space $X_{\alpha,\beta}$.

Definition 4.2. *We say that $u \in X_{\alpha,\beta}$ is an eigenfunction of* (10) *associated to λ if it is a weak solution of* (10)*, that is*

$$\alpha \int_\Omega \nabla u(x) \cdot \nabla v(x)\, dx + \frac{\beta}{2} \iint_{\mathcal{Q}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\, dx\, dy = \lambda \int_\Omega m(x)u(x)v(x)\, dx,$$

for every $v \in X_{\alpha,\beta}$ and some $\lambda \in \mathbb{R}$.

To deal with the integrability condition of the weight m, it is convenient to consider the following "critical" exponent:

$$\underline{q} := \begin{cases} \dfrac{n}{2} & \text{if } \beta = 0 \text{ and } n > 2, \\ \dfrac{n}{2s} & \text{if } \beta \neq 0 \text{ and } n > 2s, \\ 1 & \text{if } \beta = 0 \text{ and } n \leq 2, \text{ or if } \beta \neq 0 \text{ and } n \leq 2s. \end{cases} \tag{11}$$

As customary, the exponent $2_s^* := \frac{2n}{n-2s}$ denotes the fractional Sobolev critical exponent for $n > 2s$. Similarly, the exponent $2^* := \frac{2n}{n-2}$ denotes the classical Sobolev critical exponent for $n > 2$.

Furthermore, we suppose that

$$m \in L^q(\Omega), \quad \text{for some } q \in \big(\underline{q}, +\infty\big],$$

where $\underline{q}$ is given in (11).

We first have an existence result

Proposition 4.1. *Suppose that m^+, $m^- \not\equiv 0$ and that*

$$\int_\Omega m(x)\, dx \neq 0.$$

Then, problem (10) *admits two unbounded sequences of eigenvalues:*

$$\cdots \leq \lambda_{-2} \leq \lambda_{-1} < \lambda_0 = 0 < \lambda_1 \leq \lambda_2 \leq \cdots \quad .$$

In particular, if

$$\int_\Omega m(x)\,dx < 0,$$

then

$$\lambda_1 = \min_{u \in X_{\alpha,\beta}} \left\{ [u]^2_{X_{\alpha,\beta}} \;\; s.t. \;\; \int_\Omega m(x)u^2(x)\,dx = 1 \right\}$$

where we use the notation in (7)*. If instead*

$$\int_\Omega m(x)\,dx > 0,$$

then

$$\lambda_{-1} = -\min_{u \in X_{\alpha,\beta}} \left\{ [u]^2_{X_{\alpha,\beta}} \;\; s.t. \;\; \int_\Omega m(x)u^2(x)\,dx = -1 \right\}.$$

In addition, the first positive eigenvalue λ_1 has the following properties:

Proposition 4.2. *Suppose that* $m^+ \not\equiv 0$ *and*

$$\int_\Omega m(x)\,dx < 0.$$

Then, the first positive eigenvalue λ_1 *of* (10) *is simple, and the first eigenfunction* e *can be taken such that* $e \geq 0$*.*

A similar statement holds if $m^- \not\equiv 0$ *and*

$$\int_\Omega m(x)\,dx > 0.$$

The proofs of Propositions 4.1 and 4.2 can be found in [3].

4.2. **Logistic equation.** In this subsection we deal with a problem involving the mixed operator and a logistic equation. For details and references on the biological framework see [4]. We first set $s \in (0,1)$, α, $\beta \in [0,+\infty)$, with $\alpha+\beta > 0$, $m : \Omega \to \mathbb{R}$, $\mu : \Omega \to \big[\underline{\mu}, +\infty\big)$, with $\underline{\mu} > 0$, $\tau \in [0,+\infty)$ and $J \in L^1(\mathbb{R}^n, [0,+\infty))$ with

$$J(x) = J(-x)$$

and

$$\int_{\mathbb{R}^n} J(x)\, dx = 1,$$

we consider the mixed order logistic equation

$$-\alpha\Delta u + \beta(-\Delta)^s u = (m - \mu u)u + \tau\ J \star u \qquad \text{in } \ \Omega,$$

where

$$J \star u(x) := \int_\Omega J(x-y)\, u(y) dy.$$

The problem that we consider is the following

$$\begin{cases} -\alpha\Delta u + \beta(-\Delta)^s u = (m - \mu u)u + \tau\ J \star u & \text{in } \ \Omega, \\ \text{with } (\alpha,\beta)\text{-Neumann condition.} \end{cases} \tag{12}$$

Recalling the definition of $\underline{q}$ in (11), we have an existence result

Theorem 4.1. *Assume that*

$$\begin{aligned} &m \in L^q(\Omega), \textit{ for some } q \in \big(\underline{q}, +\infty\big] \\ &\textit{and } (m+\tau)^3\mu^{-2} \in L^1(\Omega). \end{aligned}$$

Then, there exists a nonnegative solution of (12) *which can be obtained as a minimum of an energy functional.*

The next two results give information on the solution found in Theorem 4.1, depending on the integral of m.

Theorem 4.2. *Assume that*

$$\begin{aligned} &m \in L^q(\Omega), \textit{ for some } q \in \big(\underline{q}, +\infty\big] \\ &\textit{and } (m+\tau)^3\mu^{-2} \in L^1(\Omega). \end{aligned}$$

Then,

(i) *if $m \equiv 0$ and $\tau = 0$, then the only solution of* (12) *is the one identically zero;*

(ii) *if*

$$\int_\Omega \big(m(x) + \tau\, J \star 1(x)\big)\, dx > 0$$

and

$$\mu \in L^1(\Omega),$$

then (12) *admits a nonnegative solution* $u \not\equiv 0$.

Theorem 4.3. *Assume that* $m \in L^q(\Omega)$, *for some* $q \in \big(\underline{q}, +\infty\big]$, *and* $(m+\tau)^3\mu^{-2} \in L^1(\Omega)$. *Then,*

(i) *if* $m \le -\tau$, *then the only solution of* (12) *is the one identically zero;*

(ii) *if* $m^+ \not\equiv 0$, $\mu \in L^1(\Omega)$,

$$\int_\Omega m(x)\, dx < 0,$$

and

(7)
$$\lambda_1 - 1 < \tau \int_\Omega (J \star e(x))e(x)\, dx,$$

then (12) *admits a nonnegative solution* $u \not\equiv 0$.

In light of (4.3), a natural question consists in quantifying the size of the first eigenvalue.

To address this problem, since the eigenvalue $\lambda_1 = \lambda_1(m)$ depends on m, it is convenient to consider an optimization problem for λ_1 in terms of three structural parameters of m, namely its minimum, its maximum and its average, in order to detect under which conditions on these parameters the first eigenvalue can be made conveniently small. More precisely, given $\overline{m}$, $\underline{m} \in (0, +\infty)$ and $m_0 \in (-\underline{m}, 0)$ we consider the class of resources

$$\mathscr{M} = \mathscr{M}(\overline{m}, \underline{m}, m_0) := \Big\{ m \in L^\infty(\Omega) \;\text{ s.t. }\; \inf_\Omega m \ge -\underline{m}, \quad \sup_\Omega m \le \overline{m}, \\ \int_\Omega m(x)\, dx = m_0|\Omega| \quad \text{and} \quad m^+ \not\equiv 0 \Big\}.$$

We will also consider the smallest possible first eigenvalue among all the resources in $\mathscr{M}$, namely we set

$$\underline{\lambda} := \inf_{m \in \mathscr{M}} \lambda_1(m).$$

When we want to emphasize the dependence of $\underline{\lambda}$ on the structural quantities $\overline{m}$, $\underline{m}$ and m_0 that characterize $\mathscr{M}$, we will adopt the explicit notation $\underline{\lambda}(\overline{m}, \underline{m}, m_0)$.

The first two results cover the case $n \ge 2$.

Theorem 4.4. *Let $n \geq 2$. Then,*

$$\lim_{\boldsymbol{m} \nearrow +\infty} \underline{\lambda}(\boldsymbol{m}, \underline{m}, m_0) = 0.$$

Theorem 4.5. *Let $n \geq 2$. Then,*

$$\lim_{\boldsymbol{m} \nearrow +\infty} \underline{\lambda}(\overline{m}, \boldsymbol{m}, m_0) = 0.$$

For the case $n = 1$ we have different situations depending on the coefficients α and β, as we can see in the following

Theorem 4.6. *Let $n = 1$, $\alpha > 0$ and $\beta \geq 0$. Then, for any $\underline{m} > 0$ and $m_0 \in (-\underline{m}, 0)$,*

$$\underline{\lambda}(\boldsymbol{m}, \underline{m}, m_0) \geq C$$

for every $\boldsymbol{m} > 0$, for some $C = C(\underline{m}, m_0, \alpha, \beta, \Omega) > 0$, and

$$\lim_{\boldsymbol{m} \searrow 0} \underline{\lambda}(\boldsymbol{m}, \underline{m}, m_0) = +\infty.$$

Moreover, for any $\overline{m} > 0$ and $m_0 < 0$,

$$\underline{\lambda}(\overline{m}, \boldsymbol{m}, m_0) \geq C$$

for every $\boldsymbol{m} > -m_0$, for some $C = C(\overline{m}, m_0, \alpha, \beta, \Omega) > 0$.

Theorem 4.7. *Let $n = 1$, $\alpha = 0$ and $\beta > 0$.*

If $s \in (1/2, 1)$, then, for any $\underline{m} > 0$ and $m_0 \in (-\underline{m}, 0)$

$$\underline{\lambda}(\boldsymbol{m}, \underline{m}, m_0) \geq C$$

for every $\boldsymbol{m} > 0$, for some $C = C(\underline{m}, m_0, \alpha, \beta, \Omega) > 0$, and

$$\lim_{\boldsymbol{m} \searrow 0} \underline{\lambda}(\boldsymbol{m}, \underline{m}, m_0) = +\infty.$$

Moreover, for any $\overline{m} > 0$ and $m_0 < 0$

$$\underline{\lambda}(\overline{m}, \boldsymbol{m}, m_0) \geq C$$

for every $\boldsymbol{m} > -m_0$, for some $C = C(\overline{m}, m_0, \alpha, \beta, \Omega) > 0$.

If $s \in (0, 1/2]$, then,

$$\lim_{\boldsymbol{m} \nearrow +\infty} \underline{\lambda}(\boldsymbol{m}, \underline{m}, m_0) = 0,$$

and

$$\lim_{\boldsymbol{m}\nearrow+\infty} \underline{\lambda}(\overline{m},\boldsymbol{m},m_0)=0.$$

The proofs of the results in this subsection, as well as more details on the matter, can be found in [3].

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