A DETOUR ON A CLASS OF NONLOCAL DEGENERATE OPERATORS
UNA PANORAMICA SU UNA CLASSE DI OPERATORI DEGENERI E NONLOCALI

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Abstract. We present some recent results on a class of degenerate operators which are modeled on the fractional Laplacian, converge to the truncated Laplacian, and are extremal among operators with fractional diffusion along subspaces of possibly different dimensions. In particular, we will recall basic properties of these operators, validity of maximum principles, and related phenomena.

Sunto. Presentiamo alcuni recenti risultati riguardanti una classe di operatori degeneri che sono costruiti a partire dal Laplaciano frazionario, convergono al Laplaciano troncato, e sono estremali tra operatori con una diffusione frazionaria lungo sottospazi che possono avere differente dimensione. In particolare, richiamiamo alcune proprietà base di questi operatori, la validità dei principi di massimo, e fenomeni correlati.

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1. Introduction: The truncated and fractional Laplacians

This note is intended as a survey of selected results for a class of nonlocal degenerate operators which have been first studied in [13] and then complemented in [12, 26]. We also refer to [20] and the recent work [3]. Since these operators are inspired by the more known truncated Laplacian, and the fractional Laplacian, we will start shortly reviewing main facts about these operators.

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Take a bounded set Ω of $\mathbb{R}^N$, and a function $u \in C^2(\Omega)$. We denote $\mathcal{V}_k$ the family of k-dimensional orthonormal sets on $\mathbb{R}^N$, and $\lambda_i(D^2u)$ the eigenvalues of D^2u in non-decreasing order. Then, for $1 \le k \le N$, we set

$$\begin{aligned} \mathcal{P}_k^+(D^2u)(x) &:= \sum_{i=N-k+1}^{N} \lambda_i(D^2u(x)) \\ &= \max\left\{ \sum_{i=1}^{k} \langle D^2u(x)\xi_i, \xi_i\rangle \,:\, \{\xi_i\}_{i=1}^k \in \mathcal{V}_k \right\}. \end{aligned} \tag{1}$$

For a proof of equality in (1), we refer the reader to [16].

Similarly one can define $\mathcal{P}_k^-$ as the sum of the first k eigenvalues, precisely

$$\begin{aligned} \mathcal{P}_k^-(D^2u)(x) &:= \sum_{i=1}^{k} \lambda_i(D^2u(x)) \\ &= \min\left\{ \sum_{i=1}^{k} \langle D^2u(x)\xi_i, \xi_i\rangle \,:\, \{\xi_i\}_{i=1}^k \in \mathcal{V}_k \right\}. \end{aligned}$$

Notice that the name *truncated* is justified by the fact that in case $k = N$, namely if all the eigenvalues of the Hessian are taken into account, then one has $\mathcal{P}_N^+(D^2u) = \mathcal{P}_N^-(D^2u) = \Delta u$, and we recover the Laplace operator. Also, it is immediate to see that $\mathcal{P}_k^+(X) = -\mathcal{P}_k^-(-X)$, hence there is a sort of duality between the two operators above. These operators have been initially introduced in connection with Riemannian manifolds, see for instance [27, 28]. We point out as an important remark that we can consider less regularity on the function u (lower or upper semicontinuity is enough) once we exploit the viscosity notion of solutions, for which we refer to [19], see also Definition 2.2 below.

Example 1.1. *Let us take as an example the function $u(x) = (1 - |x|^2)^2$. Then, straightforward computations show that if $x \neq 0$ then*

$$\lambda_i(D^2u(x)) = 4|x|^2 - 4, \quad i = 1, \dots, N-1$$

and

$$\lambda_N(D^2u(x)) = 12|x|^2 - 4.$$

Thus

$$\mathcal{P}_k^+(D^2u)(x) = (4k+8)|x|^2 - 4k, \qquad \text{and} \qquad \mathcal{P}_k^-(D^2u)(x) = 4k|x|^2 - 4k, \quad k < N.$$

To start with, notice that these operators are nonlinear and do not have a variational structure. However, $\mathcal{P}_k^{\pm}$ satisfy the following sub/super-additivity conditions

$$\mathcal{P}_k^-(Y) \le \mathcal{P}_k^{\pm}(X+Y) - \mathcal{P}_k^{\pm}(X) \le \mathcal{P}_k^+(Y).$$

Furthermore, they are degenerate elliptic, in the sense that, given X, Y two $N \times N$ real symmetric matrices,

$$\langle X\xi, \xi \rangle \le \langle Y\xi, \xi \rangle \text{ for any } \xi \in \mathbb{R}^N \quad \Rightarrow \quad \mathcal{P}_k^{\pm}(X) \le \mathcal{P}_k^{\pm}(Y).$$

This monotonicity condition on the operator extends the classical notion of ellipticity for linear operators, namely, a linear operator $L(X) = \operatorname{tr}(AX)$ is elliptic if A is positive definite. Actually, the truncated Laplacians are strongly degenerate if $k < N$. Indeed, one has

$$\mathcal{P}_k^-(X) = \mathcal{P}_k^-(X + v \otimes v),$$

where v is an eigenvector corresponding to the largest eigenvalue of X. A corresponding equality holds for $\mathcal{P}_k^+$.

The literature regarding these operators is really vast, as they gained an increasing interest in the last years. We refer the interested reader to the works [23, 16] as more classic references, and to [7, 8, 10, 11, 18, 9, 14, 24] and references therein for more recent advances. In the following sections, we will also recall more precisely some results regarding these operators, in order to better emphasize differences and similarities with respect to the class of operators we will take into account.

The other operator which will be a model in order to build a suitable class of nonlocal truncated Laplacians is the so-called fractional Laplacian. Let us fix $s \in (0,1)$, and take a function $u \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ (actually, less regularity can be imposed on the function, see [15]). The fractional Laplacian of order s of u in x is defined as

$$-(-\Delta)^s u(x) := \frac{1}{2} C_{N,s} \int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}}\, dy,$$

where $C_{N,s}$ is a dimensional constant, see [15, Equations (2.10) and (2.15)] for its explicit value. One can equivalently write

$$(-\Delta)^s u(x) = C_{N,s} \text{P.V.} \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{u(x) - u(y)}{|x-y|^{N+2s}}\, dy,$$

where P.V. stands for principal value. The fractional Laplacian arises for instance when modeling a random process that allows long jumps. The most important feature of this operator is that it is nonlocal, since the value of $(-\Delta)^s u$ at x depends on the value of u in the whole of $\mathbb{R}^N$. It can be proved that

$$\lim_{s\to 1}(-\Delta)^s u = -\Delta u, \quad \lim_{s\to 0}(-\Delta)^s u = u.$$

Example 1.2. *The function $u(x) := (1-|x|^2)_+^s$ has constant fractional Laplacian in B_1, more precisely*

$$(-\Delta)^s u(x) = C_{N,s}\frac{\omega_N}{2}B(s,1-s)$$

where ω_N is the volume of the N-dimensional sphere, and B is the special Beta function, see [15].

Besides the already cited [15], one can take a look at the survey [21] and references therein, where many standard and not-so-standard results can be found, as well as examples and applications.

2. A class of nonlocal degenerate operators

In [13] a new class of nonlinear nonlocal operators has been introduced, with the aim of providing a suitable nonlocal analog of the truncated Laplacians. We first recall the definition given there, and then we extend it to a larger class of operators.

Let $u \in L^\infty(\mathbb{R}^N) \cap C^2(\Omega)$, fix $\xi \in \mathbb{R}^N$, and $s \in (0,1)$, and define

$$\mathcal{I}_\xi u(x) := C_s P.V.\int_{-\infty}^{+\infty} \frac{u(x+\tau\xi)-u(x)}{|\tau|^{1+2s}}\,d\tau,$$

where $C_s > 0$ is a normalizing constant and P.V. stands for principal value. Notice that this operator, which is basically a 1-dimensional fractional Laplacian in the space generated by ξ, acts as a $2s$ fractional derivative of u in the direction ξ. We substitute the second order derivatives appearing in the definition of the truncated Laplacian (1) with $\mathcal{I}_\xi u$, and we set

$$\mathcal{I}_k^+ u(x) := \sup\left\{\sum_{i=1}^k \mathcal{I}_{\xi_i} u(x) \colon \{\xi_i\}_{i=1}^k \in \mathcal{V}_k\right\}$$

and

$$\mathcal{I}_k^- u(x) := \inf\left\{\sum_{i=1}^k \mathcal{I}_{\xi_i} u(x) \colon \{\xi_i\}_{i=1}^k \in \mathcal{V}_k\right\}.$$

An important property is that, choosing C_s suitably, if $s \to 1$ then $\mathcal{I}_k^\pm u \to \mathcal{P}_k^\pm u$, see [13]. We recall that in the particular case $k = 1$, these operators were also considered in the papers [20, 3].

One may wonder what happens if we take, in place of $\mathcal{I}_\xi$, operators which are n-dimensionally nonlocal, where $1 \le n \le N$. Precisely,

$$\mathcal{J}_V u(x) := C_{n,s} P.V. \int_V \frac{u(x+z) - u(x)}{|z|^{n+2s}}\, d\mathcal{H}^n(z),$$

where V is a n-dimensional subspace of $\mathbb{R}^N$, $\mathcal{H}^n$ is the n-dimensional Hausdorff measure, and $C_{n,s} > 0$ suitably chosen. We point out that if $V = \langle \xi_1, \dots, \xi_n\rangle$ then

$$\mathcal{J}_V u(x) = C_{n,s} P.V. \int_{\mathbb{R}^n} \frac{u(x + \sum_{i=1}^n \tau_i \xi_i) - u(x)}{\left(\sum_{i=1}^n \tau_i^2\right)^{\frac{n+2s}{2}}}\, d\tau_1 \dots d\tau_n.$$

Clearly, in case $n = 1$, $\mathcal{J}_V$ reduces to $\mathcal{I}_\xi$.

We are now ready to introduce the operators we will consider in this survey.

Definition 2.1. *Choose* $1 \le \ell \le N$. *Let* $1 \le k_1 \le \dots \le k_\ell \le N$ *such that*

$$k := \sum_{j=1}^{\ell} k_j$$

satisfies $1 \le k \le N$. *Let us denote*

$$\begin{aligned} \sup{}^{k_1,\dots,k_\ell} &:= \sup_{\{\xi_j^1\}_{j=1}^{k_1} \in \mathcal{V}_{k_1}} \sup_{\{\xi_j^2\}_{j=1}^{k_2} \in \mathcal{V}_{k_1,k_2}} \dots \sup_{\{\xi_j^\ell\}_{j=1}^{k_\ell} \in \mathcal{V}_{k_1,\dots,k_\ell}} \\ &= \sup\left\{ \text{"argument" s.t. } \{\xi_j^1\}_{j=1}^{k_1} \in \mathcal{V}_{k_1},\ \{\xi_j^2\}_{j=1}^{k_2} \in \mathcal{V}_{k_1,k_2},\ \dots,\ \{\xi_j^\ell\}_{j=1}^{k_\ell} \in \mathcal{V}_{k_1,\dots,k_\ell}\right\}, \end{aligned} \tag{2}$$

where $\mathcal{V}_{k_1}$ *is the collection of all* k_1*-orthonormal sets of* $\mathbb{R}^N$, *and* $\mathcal{V}_{k_1,\dots,k_t}$, $t \ge 2$, *represents the collection of all* k_t*-orthonormal sets of* $\mathbb{R}^N$ *which are orthogonal to the space generated by the vectors* ξ_j^s, *with* $j = 1, \dots, k_s$ *and* $s = 1, \dots, t-1$.

Let also

$$V_i := \langle \xi_j^i \rangle_{j=1}^{k_i}. \tag{3}$$

Then for $u \in L^\infty(\mathbb{R}^N) \cap C^2(\Omega)$ we define

$$\mathcal{K}^+_{k_1,\dots,k_\ell} u(x) := \sup{}^{k_1,\dots,k_\ell} \sum_{i=1}^{\ell} \mathcal{J}_{V_i} u(x).$$

Analogously, we define $\mathcal{K}^-_{k_1,\dots,k_\ell}$ taking the infimum in place of the supremum.

Remark 2.1. *One has $\mathcal{K}^+_{k_1,\dots,k_\ell}(-u) = -\mathcal{K}^-_{k_1,\dots,k_\ell} u$.*

Remark 2.2. *In what follows, we give a proof of (2). Let us only show the case $\ell = 2$, the general case follows by iteration. Let*

$$A := \sup_{\{\xi^1_j\}_{j=1}^{k_1} \in \mathcal{V}_{k_1}} \sup_{\{\xi^2_j\}_{j=1}^{k_2} \in \mathcal{V}_{k_1,k_2}} f(\xi^1_1, \dots, \xi^1_{k_1}, \xi^2_1, \dots, \xi^2_{k_2})$$

and

$$B := \sup_{\{\xi^1_j\}_{j=1}^{k_1} \in \mathcal{V}_{k_1}, \{\xi^2_j\}_{j=1}^{k_2} \in \mathcal{V}_{k_1,k_2}} f(\xi^1_1, \dots, \xi^1_{k_1}, \xi^2_1, \dots, \xi^2_{k_2})$$

where

$$f(\xi^1_1, \dots, \xi^1_{k_1}, \xi^2_1, \dots, \xi^2_{k_2}) = \mathcal{J}_{V_1} u(x) + \mathcal{J}_{V_2} u(x).$$

We first notice that $A \le B$. Indeed, for any choice of $\{\xi^1_j\}_{j=1}^{k_1} \in \mathcal{V}_{k_1}$, $\{\xi^2_j\}_{j=1}^{k_2} \in \mathcal{V}_{k_1,k_2}$, one has

$$f(\xi^1_1, \dots, \xi^1_{k_1}, \xi^2_1, \dots, \xi^2_{k_2}) \le B.$$

Hence, taking the supremum first with respect to $\{\xi^2_j\}_{j=1}^{k_2} \in \mathcal{V}_{k_1,k_2}$ with fixed $\{\xi^1_j\}_{j=1}^{k_1} \in \mathcal{V}_{k_1}$, and then with respect to $\{\xi^1_j\}_{j=1}^{k_1} \in \mathcal{V}_{k_1}$, one gets $A \le B$.

Let us assume $A < B$. Then there exists C such that $A < C < B$. By definition of B there exist $\{\bar\xi^1_j\}_{j=1}^{k_1} \in \mathcal{V}_{k_1}$, $\{\bar\xi^2_j\}_{j=1}^{k_2} \in \mathcal{V}_{k_1,k_2}$ such that

$$f(\bar\xi^1_1, \dots, \bar\xi^1_{k_1}, \bar\xi^2_1, \dots, \bar\xi^2_{k_2}) > C.$$

This implies

$$\sup_{\{\xi^2_j\}_{j=1}^{k_2} \in \mathcal{V}_{k_1,k_2}} f(\bar\xi^1_1, \dots, \bar\xi^1_{k_1}, \xi^2_1, \dots, \xi^2_{k_2}) > C > A.$$

This contradicts the definition of A.

Remark 2.3. *Notice that*

$$\mathcal{K}^{+}_{k_1,\dots,k_\ell}u(x) = \sup_{\{\xi_j\}_{j=1}^{k}\in\tilde{\mathcal{V}}_k} \sum_{i=1}^{\ell} \mathcal{J}_{\tilde{V}_i}u(x),$$

where

$$\tilde{V}_i = \langle \xi_j \rangle_{j\in\mathcal{A}_i},$$

$$\mathcal{A}_i = \left\{ \sum_{j=0}^{i-1} k_j + 1, \dots, \sum_{j=0}^{i} k_j \right\},\ i = 1,\dots,\ell, \tag{4}$$

and $\tilde{\mathcal{V}}_k$ *is the set of all ordered* k*-uples of orthonormal vectors in* $\mathbb{R}^N$*. Here we set for notational convenience* $k_0 := 0$*.*

Remark 2.4. *By* (2)*, one has, for instance in the case* $\ell = 3$*,*

$$\mathcal{K}^{+}_{k_1,k_2,k_3}u(x) = \sup_{\{\xi^1_j\}_{j=1}^{k_1}\in\mathcal{V}_{k_1}} \Big(\mathcal{J}_{V_1}u(x) + \sup_{\{\xi^2_j\}_{j=1}^{k_2}\in\mathcal{V}_{k_1,k_2}} \Big(\mathcal{J}_{V_2}u(x) + \sup_{\{\xi^3_j\}_{j=1}^{k_3}\in\mathcal{V}_{k_1,k_2,k_3}} \mathcal{J}_{V_3}u(x) \Big)\Big).$$

Remark 2.5. *We point out that if we choose the normalization constants* $C_{k_i,s}$ *such that*

$$\frac{C_{k_i,s}|\mathcal{S}^{k_i-1}|}{4k_i(1-s)} \to 1 \qquad as\ s \to 1^-, \tag{5}$$

where $|\mathcal{S}^{k_i-1}|$ *is the volume of the* k_i *dimensional sphere, then*

$$\mathcal{K}^{\pm}_{k_1,\dots,k_\ell}u(x) \to \mathcal{P}^{\pm}_k u(x) \qquad as\ s \to 1^-, \tag{6}$$

see also [13, Lemma 6.1]*. Actually, these constants can be explicitly given, see* [15, Equations (2.10) and (2.15)]*. In order to see* (6)*, let us fix* $\varepsilon > 0$*. By definition of supremum, there exist* $\{\xi^i_j\}_{j=1}^{k_i}$ *with* $i = 1,\dots,\ell$ *such that*

$$\begin{aligned}\mathcal{K}^{+}_{k_1,\dots,k_\ell}u(x) - \mathcal{P}^{+}_k u(x) - \varepsilon &\le \sum_{i=1}^{\ell} \mathcal{J}_{V_i}u(x) - \mathcal{P}^{+}_k u(x)\\ &\le \sum_{i=1}^{\ell} \mathcal{J}_{V_i}u(x) - \sum_{i=1}^{\ell}\sum_{j=1}^{k_i} \langle D^2u(x)\xi^i_j, \xi^i_j\rangle.\end{aligned}$$

By [13, Lemma 6.1] *we know that for any* $\varepsilon > 0$ *fixed there exists a* $\delta > 0$ *such that*

$$\mathcal{J}_{V_i}u(x) \le O(\varepsilon) + \frac{C_{k_i,s}|\mathcal{S}^{k_i-1}|}{4k_i(1-s)}\delta^{2-2s} \sum_{j=1}^{k_i} \langle D^2u(x)\xi^i_j, \xi^i_j\rangle + C_{k_i,s}O(\delta^{-2s}).$$

Assuming (5), *we get*

$$\lim_{s\to 1}\mathcal{K}^+_{k_1,\dots,k_\ell}u(x)-\mathcal{P}^+_k u(x)\le C\varepsilon.$$

A reverse inequality can be found analogously, hence we obtain the conclusion. Similar arguments show the convergence for $\mathcal{K}^-_{k_1,\dots,k_\ell}$.

Example 2.1. *Notice that if* $\ell=1$, *then* $k=k_1$, *and by definition*

$$\sup{}^{k_1}=\sup_{\{\xi^1_j\}_{j=1}^{k_1}\in\mathcal{V}_{k_1}}.$$

This class of operators has been denoted in [13] *as follows:*

$$\mathcal{J}^+_k u(x):=\sup\left\{\mathcal{J}_V u(x)\colon V=\langle\xi_1,\dots,\xi_k\rangle,\{\xi_i\}_{i=1}^k\in\mathcal{V}_k\right\}$$

$$\mathcal{J}^-_k u(x):=\inf\left\{\mathcal{J}_V u(x)\colon V=\langle\xi_1,\dots,\xi_k\rangle,\{\xi_i\}_{i=1}^k\in\mathcal{V}_k\right\}.$$

In particular, if $\ell=1$, *and* $k_1=1$, *then* $\mathcal{K}^\pm_1=\mathcal{J}^\pm_1=\mathcal{I}^\pm_1$, *whereas if* $\ell=1$, *and* $k_1=N$, *then* $\mathcal{K}^\pm_N=\mathcal{J}^\pm_N=-(-\Delta)^s$.

Also, if $k_1=\dots=k_\ell=1$, *then* $k=\ell$ *and*

$$\sup{}^{1,\dots,1}\sum_{i=1}^{\ell}\mathcal{J}_{V_i}u(x)=\sup_{\{\xi_j\}_{j=1}^k\in\mathcal{V}_k}\sum_{i=1}^{\ell}\mathcal{J}_{\langle\xi_i\rangle}u(x),$$

so that $\mathcal{K}^\pm_{1,\dots,1}=\mathcal{I}^\pm_k$. *In particular, if* $\ell=N$, *then* $\mathcal{K}^\pm_{1,\dots,1}=\mathcal{I}^\pm_N$.

We finally notice that if $\ell\neq 1$, *then* $\mathcal{K}^\pm_{k_1,\dots,k_\ell}$ *does not coincide with the fractional Laplacian, even if* $k=N$.

Example 2.2. *Let* $\ell=3$, *and* $k_1=1,k_2=1,k_3=3$. *Then,* $k=5$, *and*

$$\mathcal{K}^+_{1,1,3}u(x)=\sup_{|\xi^1_1|=1}\ \sup_{\substack{|\xi^2_1|=1\\ \xi^2_1\perp\xi^1_1}}\ \sup_{\substack{\{\xi^3_1,\xi^3_2,\xi^3_3\}\in\mathcal{V}_3\\ \xi^3_j\perp\langle\xi^1_1,\xi^2_1\rangle}}\Big(\mathcal{J}_{\langle\xi^1_1\rangle}u(x)+\mathcal{J}_{\langle\xi^2_1\rangle}u(x)+\mathcal{J}_{\langle\xi^3_1,\xi^3_2,\xi^3_3\rangle}u(x)\Big).$$

When dealing with degenerate operators, regularity issues may arise, and the natural notion of solutions turns out to be that of viscosity solutions. Below, we give the definition, see [1, 2], adapted to our context. For definitions and main properties of viscosity solutions in the classical local framework we refer to the survey [19].

Definition 2.2. *Let $f \in C(\Omega \times \mathbb{R})$. We say that $u \in L^\infty(\mathbb{R}^N) \cap LSC(\Omega)$ (respectively $USC(\Omega)$) is a (viscosity) supersolution (respectively subsolution) to*

$$\mathcal{K}^+_{k_1,\dots,k_\ell} u + f(x, u(x)) = 0 \text{ in } \Omega \tag{7}$$

if for every point $x_0 \in \Omega$ and every function $\varphi \in C^2(B_\rho(x_0))$, $\rho > 0$, such that x_0 is a minimum (resp. maximum) point to $u - \varphi$, one has

$$\mathcal{K}(u, \varphi, x_0, \rho) + f(x_0, u(x_0)) \leq 0 \quad (\text{resp. } \geq 0) \tag{8}$$

where

$$\mathcal{K}(u,\varphi,x_0,\rho) = \sup{}^{k_1,\dots,k_\ell} \sum_{i=1}^{\ell} C_{k_i,s} \Bigg\{ P.V. \int_{B_\rho(0)} \frac{\varphi(x_0 + \sum_{j=1}^{k_i} \tau_j \xi_j^i) - \varphi(x_0)}{(\sum_{j=1}^{k_i} \tau_j^2)^{\frac{k_i+2s}{2}}} \, d\tau_1 \dots d\tau_{k_i}$$
$$+ \int_{B_\rho(0)^c} \frac{u(x_0 + \sum_{j=1}^{k_i} \tau_j \xi_j^i) - u(x_0)}{(\sum_{j=1}^{k_i} \tau_j^2)^{\frac{k_i+2s}{2}}} \, d\tau_1 \dots d\tau_{k_i} \Bigg\}.$$

We say that a continuous function u is a solution of (7) *if it is both a supersolution and a subsolution of* (7)*. We analogously define viscosity sub/super solutions for the operator $\mathcal{K}^-_{k_1,\dots,k_\ell}$, taking the infimum in place of the supremum.*

Remark 2.6. *In the definition of supersolution above we can assume without loss of generality that $u > \varphi$ in $B_\rho(x_0) \setminus \{x_0\}$, and $\varphi(x_0) = u(x_0)$, see also Remark 2.6 in* [12].

Remark 2.7. *The operators $\mathcal{K}^\pm_{k_1,\dots,k_\ell}$ satisfy the following ellipticity condition: if $\psi_1, \psi_2 \in C^2(B_\rho(x_0)) \cap L^\infty(\mathbb{R}^N)$ for some $\rho > 0$ are such that $\psi_1 - \psi_2$ has a maximum in x_0, then*

$$\mathcal{K}^\pm_{k_1,\dots,k_\ell} \psi_1(x_0) \leq \mathcal{K}^\pm_{k_1,\dots,k_\ell} \psi_2(x_0).$$

Indeed, if $\psi_1(x_0) - \psi_2(x_0) \geq \psi_1(x) - \psi_2(x)$, for all $x \in B_\rho(x_0)$ then

$$\psi_1\left(x_0 + \sum_{j=1}^{k_i} \tau_j \xi_j^i\right) - \psi_1(x_0) \leq \psi_2\left(x_0 + \sum_{j=1}^{k_i} \tau_j \xi_j^i\right) - \psi_2(x_0),$$

which yields the conclusion.

Remark 2.8. *Notice that in the definition above we assumed $u \in L^\infty(\mathbb{R}^N)$, as this will be enough for our purposes, however, one can also consider unbounded functions u with a suitable growth condition at infinity, see* [13].

3. Continuity, maximum principles, and related problems

Notice that, assuming $u \in C^2(\Omega)$, the map $x \mapsto \mathcal{P}_k^{\pm} u(x)$ is continuous. This property is also verified by the class of fully nonlinear integro differential operators taken into account in the work by Caffarelli and Silvestre [17], see Lemma 4.2 there. As a particular case of the results in [17], also $(-\Delta)^s u$ is continuous if $u \in C^2(\Omega)$ and bounded in $\mathbb{R}^N$. However, the operators we are considering lack continuity, as we proved in [12, 26].

For instance, a counterexample for the case $\ell = k < N$, for which $\mathcal{K}^+_{k_1,\dots,k_\ell}$ reduces to $\mathcal{I}_k^+$, $k < N$, is given by the following

$$u(x) = \begin{cases} 0 & \text{if } |x| \leq 1 \text{ or } \langle x, e_N \rangle \leq 0 \\ -1 & \text{otherwise.} \end{cases}$$

Notice that this function is smooth in $B_1(0)$, however, it is not continuous in the whole space $\mathbb{R}^N$. Direct computations show that

$$\mathcal{I}_k^+ u(0) = 0.$$

On the other hand,

$$\limsup_{n \to +\infty} \mathcal{I}_k^+ u\left(\frac{1}{n} e_N\right) < 0$$

where $e_N = (0, \dots, 0, 1)$, see [12] for the details.

The general case follows by taking into account a similar function u, see [26], precisely

$$u(x) = \begin{cases} 0 & \text{if } |x| \leq 1, \text{ or } \exists i = 1, \dots, \ell \text{ s.t. } x \in \langle e_j \rangle_{j \in \mathcal{A}_i} \\ -1 & \text{otherwise,} \end{cases}$$

where $\mathcal{A}_i$ are defined in (4). Notice that $u \equiv 0$ if and only if $\ell = 1$ and $k_1 = N$, which is the case of the fractional Laplacian.

However, we gain continuity of the operators once we assume a *global* regularity assumption on the function. The proof is completely analogous to [12, Proposition 3.1].

Proposition 3.1. *Let $u \in C^2(\Omega) \cap L^\infty(\mathbb{R}^N)$, and consider the map*

$$\mathcal{K}^{\pm}_{k_1,\dots,k_\ell} u : x \in \Omega \mapsto \mathcal{K}^{\pm}_{k_1,\dots,k_\ell} u(x).$$

If $u \in LSC(\mathbb{R}^N)$ (respectively $USC(\mathbb{R}^N)$, $C(\mathbb{R}^N)$) then $\mathcal{K}^{\pm}_{k_1,\dots,k_\ell} u \in LSC(\Omega)$ (respectively $USC(\Omega)$, $C(\Omega)$).

Another interesting fact which is related to the lack of continuity is that the sup or inf in the definition of $\mathcal{K}^{\pm}_{k_1,\dots,k_\ell}$ are in general not attained under the only assumption $u \in C^2(\Omega) \cap L^\infty(\mathbb{R}^N)$. We refer to [12, 26] for some counterexamples. This is a difference with respect to the truncated Laplacians $\mathcal{P}^{\pm}_k$. Indeed, the minimum in the definition of $\mathcal{P}^{-}_k$ is attained by the eigenvectors corresponding to the first k eigenvalues, whereas the maximum in $\mathcal{P}^{+}_k$ is attained by the eigenvectors corresponding to the last k eigenvalues, see [16].

Continuity of the operator was crucial in [17] to obtain a comparison principle. Therefore, in order to prove it for the operators $\mathcal{K}^{\pm}_{k_1,\dots,k_\ell}$, we need to exploit a different approach. Recall that in the theory of viscosity solutions the comparison principle for second order operators requires the Jensen-Ishii's lemma, see [19], which in turn lies on a complex proof that uses tools from convex analysis. Here, instead, the proof, for which we refer to [12, 26], is completely self contained and uses only a straightforward calculation, somehow more similar to the case of first order local equations, where just the doubling variable technique is used. We also refer the reader to [20] for a different proof in case the domain is strictly convex.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and let $c(x), f(x) \in C(\Omega)$ be such that $\|c^+\|_\infty < \sum_{i=1}^{\ell} C_{k_i,s} \frac{1}{2s} (diam(\Omega))^{-2s}$. If $u \in USC(\overline{\Omega}) \cap L^\infty(\mathbb{R}^N)$ and $v \in LSC(\overline{\Omega}) \cap L^\infty(\mathbb{R}^N)$ are respectively sub and supersolution of*

$$\begin{cases} \mathcal{K}^{\pm}_{k_1,\dots,k_\ell} u + c(x)u = f(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

then $u \le v$ in Ω.

This comparison principle gives as an immediate corollary the validity of weak maximum/minimum principles for the operators $\mathcal{K}^{\pm}_{k_1,\dots,k_\ell}$, namely a sign propagation property is satisfied. However, it does not guarantee that the strong maximum/minimum principle are satisfied, and indeed we will see that a very complicated behavior arises, depending on whether we are considering $\mathcal{K}^{+}_{k_1,\dots,k_\ell}$ or $\mathcal{K}^{-}_{k_1,\dots,k_\ell}$, and also on the value of k. We recall that an operator $\mathcal{K}$ satisfies the strong minimum principle in Ω if

$$\mathcal{K}u \le 0 \text{ in } \Omega, \quad u \ge 0 \text{ in } \mathbb{R}^N \quad \Longrightarrow \quad u > 0 \text{ or } u \equiv 0 \text{ in } \Omega.$$

Notice that since we are dealing with nonlinear operators, maximum and minimum principles are different concepts, and it may happen that one holds, whereas the other one not. However, since $\mathcal{K}^+_{k_1,\dots,k_\ell}(-u) = -\mathcal{K}^-_{k_1,\dots,k_\ell}u$, there is a duality, and maximum principles for one operator correspond to minimum principles for the other one.

Let us first briefly review what happens for truncated and fractional Laplacians. It is well known that the fractional Laplacian satisfies the strong maximum (and minimum) principle, see [15, Theorem 2.3.3]. Actually, a stronger property holds, in the sense that

$$(-\Delta)^s u(x) \geq 0 \text{ in } \Omega, \quad u \geq 0 \text{ in } \mathbb{R}^N \;\Rightarrow\; u > 0 \text{ in } \Omega \text{ or } u \equiv 0 \text{ in } \mathbb{R}^N. \tag{9}$$

For the truncated Laplacians, the situation is deeply different. The operator $\mathcal{P}^-_k$, with $k < N$, satisfies the weak minimum principle, but not the strong minimum principle, see [10, Proposition 2.4]. On the other hand, the operator $\mathcal{P}^+_k$ satisfies the strong minimum principle. Indeed, if $\mathcal{P}^+_k(D^2u(x)) \leq 0$ in Ω, then in particular

$$k\,\lambda_{N-k+1}(D^2u(x)) \leq \sum_{i=N-k+1}^{N} \lambda_i(D^2u(x)) \leq 0$$

and since the eigenvalues are in non-decreasing order, $\lambda_i(D^2u(x)) \leq 0$ for any $i \leq N-k+1$. Thus,

$$\Delta u(x) = \sum_{i=1}^{N} \lambda_i(D^2u(x)) \leq 0$$

which yields the conclusion.

One finds a similar picture when considering the operators $\mathcal{K}^\pm_{k_1,\dots,k_\ell}$.

Theorem 3.2. *[[26]] The following conclusions hold.*

(i) *The operators $\mathcal{K}^-_{k_1,\dots,k_\ell}$, with $k < N$, do not satisfy the strong minimum principle in Ω.*

(ii) *The operators $\mathcal{K}^-_{k_1,\dots,k_\ell}$ with $k = N$ satisfy the strong minimum principle in Ω.*

(iii) *The operators $\mathcal{K}^+_{k_1,\dots,k_\ell}$ satisfy the following implication*

$$\mathcal{K}u(x) \leq 0 \text{ in } \Omega, \quad u \geq 0 \text{ in } \mathbb{R}^N \;\Rightarrow\; u > 0 \text{ in } \Omega \text{ or } u \equiv 0 \text{ in } \mathbb{R}^N.$$

In particular, they satisfy the strong minimum principle.

(iv) *Let $k < N$, or $k = N$ and $\ell > 1$. There exist functions u such that $\mathcal{K}^-_{k_1,\dots,k_\ell} u \le 0$ in Ω, $u \equiv 0$ in $\overline{\Omega}$, and $u \not\equiv 0$ in $\mathbb{R}^N \setminus \overline{\Omega}$, namely these operators do not satisfy the implication in (iii).*

Conclusion *(ii)* is the most delicate, and a careful analysis is needed, see [12, 26]. Conclusion *(i)* instead follows by taking φ a smooth bounded function of one variable which attains the minimum in a point of Ω, and setting $u(x) = \varphi(x_N)$. Conclusion *(iii)* is suggested by property (9) satisfied by the fractional Laplacian. In order to prove it, take u which satisfies the assumptions of the minimum principle, and assume there exists $x_0 \in \Omega$ such that $u(x_0) = 0$. Choose any orthonormal basis of $\mathbb{R}^N$ $\{\xi_1, \dots, \xi_N\}$. Thus,

$$0 \ge \mathcal{K}^+_{k_1,\dots,k_\ell} u(x_0) \ge \sum_{i=1}^{\ell} \mathcal{J}_{V_i} u(x_0) = \sum_{i=1}^{\ell} C_{k_i,s} P.V. \int_{\mathbb{R}^{k_i}} \frac{u(x_0 + \sum_{j\in\mathcal{A}_i} \tau_j \xi_j)}{|\tau|^{k_i+2s}}\, d\tau,$$

where $\mathcal{A}_i$ is defined in (4). Hence, since $u \ge 0$ in $\mathbb{R}^N$, we conclude that $u \equiv 0$ on every space $V_i + x_0$ for any $i = 1, \dots, \ell$. Since the directions ξ_i are arbitrary, we get $u \equiv 0$ on $\mathbb{R}^N$. Finally, a counterexample to show conclusion *(iv)* is given by the function

$$u(x) = \begin{cases} 0 & \text{if } \exists i = 1, \dots, \ell \text{ such that } d(x, \langle e_j \rangle_{j\in\mathcal{A}_i}) \le 1, \\ 1 & \text{otherwise,} \end{cases}$$

where $\mathcal{A}_i$ are defined in (4), and $d(x, A)$ is the distance from the point x to the space A.

Remark 3.1. *We wish to cite here the recent paper* [6]*, in which the geometry of the sets of minima for supersolutions of equations involving the operators $\mathcal{I}^\pm_k$ is characterized.*

Remark 3.2. *The comparison principle can be exploited in order to get uniqueness for the Dirichlet problem*

$$\begin{cases} \mathcal{K}^\pm_{k_1,\dots,k_\ell} u = f(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

where f is a bounded continuous function. Existence is guaranteed by the Perron method, in case Ω is a uniformly convex domain, namely it is the intersection of a family of balls of same radius, [1, 2, 12, 26].

Another related problem, which turns out to be affected by the nonlocality, is the validity of a Hopf type lemma. Whereas for truncated Laplacians [10] one has that any supersolution can be bounded from below by the distance function to the boundary, here, as in the case of the fractional Laplacian, for which we refer to [22], we need to compare the supersolutions with the distance to the power s. Precisely,

Proposition 3.2 ([12]). *Let Ω be a bounded C^2 domain, $k = N$, and let u satisfy*

$$\begin{cases} \mathcal{K}^-_{k_1,\dots,k_\ell} u \leq 0 & \textit{in } \Omega \\ u \geq 0 & \textit{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Assume $u \not\equiv 0$ in Ω. Then there exists a positive constant $c = c(\Omega, u)$ such that

$$u(x) \geq c\, d(x)^s \quad \forall x \in \overline{\Omega}, \tag{10}$$

where $d(x) = \inf_{y \in \partial\Omega} |x - y|$.

Notice that the conclusion is not true for the operators $\mathcal{K}^-_{k_1,\dots,k_\ell}$, $k < N$. Indeed, consider the function

$$u(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1 \end{cases}$$

fix $x \in B_1(0)$, and take $\{\xi_i\} \in \mathcal{V}_k$ such that $\langle x, \xi_i \rangle = 0$ for any $i = 1, \dots, k$. Hence

$$\left| x + \sum_{j \in \mathcal{A}_i} \tau_j \xi_j \right|^2 = |x|^2 + |\tau|^2 \geq |x|^2,$$

see (4), and using the radial monotonicity of u

$$\mathcal{K}^-_{k_1,\dots,k_\ell} u(x) \leq 0 \text{ in } B_1(0).$$

However, u clearly does not satisfy

$$u(x) \geq c\, d(x)^\gamma$$

for any positive constants c, γ.

As a consequence of Proposition 3.2, we immediately obtain the following

Corollary 3.1. *Let Ω be a bounded C^2 domain, and let u satisfy*

$$\begin{cases} \mathcal{K}^+_{k_1,\dots,k_\ell} u \leq 0 & \text{in } \Omega \\ u \geq 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases}$$

Assume $u \not\equiv 0$ in Ω. Then

$$u(x) \geq c\, d(x)^s$$

for some positive constant $c = c(\Omega, u)$.

4. Principal eigenvalues

In this section, we aim to briefly discuss existence and main properties of suitably defined principal eigenvalues, and their relation with the validity of maximum principles. It is well known that the operator $-\Delta - \lambda$ satisfies the maximum principle if and only if $\lambda < \lambda_1$, which is the principal eigenvalue of the Laplace operator, given by

$$\lambda_1 = \inf \left\{ \int_\Omega |\nabla u|^2, \quad u \in C_0^\infty(\Omega), \quad \int_\Omega u^2 = 1 \right\}.$$

However, when the problem lacks a variational formulation, as it is the case in our analysis, one has to give a different characterization of the principal eigenvalue. It turns out that the most convenient way to do so, is to exploit the formulation in [4]. In the case of fully nonlinear operators like the Pucci operators, this is done in [25], see also [5] for a class of p-homogeneous operators.

The case of truncated Laplacians has been treated in [10]. In our setting we have a similar situation as [10], hence we follow their approach and define

$$\mu^\pm_{k_1,\dots,k_\ell} = \sup \Big\{ \mu : \exists v \in LSC(\Omega) \cap L^\infty(\mathbb{R}^N), v > 0 \text{ in } \Omega, v \geq 0 \text{ in } \mathbb{R}^N, \\ \mathcal{K}^\pm_{k_1,\dots,k_\ell} v + \mu v \leq 0 \text{ in } \Omega \Big\},$$

As noticed in [10], see also [18], when considering degenerate operators the situation is delicate due to the possible lack of continuity up to the boundary. Thus, we need also to introduce the values

$$\bar{\mu}^{\pm}_{k_1,\dots,k_\ell} = \sup\Big\{\mu :\ \exists v \in LSC(\Omega)\cap L^\infty(\mathbb{R}^N),\ \inf_\Omega v > 0,\ v \ge 0 \text{ in } \mathbb{R}^N, \\ \mathcal{K}^{\pm}_{k_1,\dots,k_\ell} v + \mu v \le 0 \text{ in } \Omega\Big\}.$$

One has, by similar arguments as for the proof of the comparison principle,

Theorem 4.1 ([12, 26])**.** *The operators $\mathcal{K}^{\pm}_{k_1,\dots,k_\ell}(\cdot) + \mu\cdot$ satisfy the maximum principle for $\mu < \bar{\mu}^{\pm}_{k_1,\dots,k_\ell}$.*

However, this result is not completely satisfactory, as the natural threshold for validity of maximum principles should be $\mu^{\pm}_{k_1,\dots,k_\ell}$. It is immediate that $\bar{\mu}^{\pm}_{k_1,\dots,k_\ell} \le \mu^{\pm}_{k_1,\dots,k_\ell}$. Also, one proves [26] that $\bar{\mu}^{-}_{k_1,\dots,k_\ell} = \mu^{-}_{k_1,\dots,k_\ell} = +\infty$ for any $k < N$. Indeed, let $w(x) = e^{-\alpha|x|^2} > 0$ for $\alpha > 0$ and fix any $\mu > 0$. Notice that

$$\int_{\mathbb{R}^{k_i}} \frac{1 - e^{-\alpha\sum\tau_j^2}}{(\sum\tau_j^2)^{\frac{k_i+2s}{2}}}\, d\tau_1 \dots d\tau_{k_i} = \alpha^s \int_{\mathbb{R}^{k_i}} \frac{1 - e^{-\sum\tau_j^2}}{(\sum\tau_j^2)^{\frac{k_i+2s}{2}}}\, d\tau_1 \dots d\tau_{k_i}.$$

Hence, we obtain, choosing $\{\xi_j\} \in \mathcal{V}_k$ such that ξ_j is orthogonal to x for any j,

$$\mathcal{K}^{-}_{k_1,\dots,k_\ell} w(x) + \mu w(x) \le -\sum C_{k_i,s} e^{-\alpha|x|^2} \int_{\mathbb{R}^{k_i}} \frac{1 - e^{-\alpha\sum\tau_j^2}}{(\sum\tau_j^2)^{\frac{k_i+2s}{2}}}\, d\tau_1 \dots d\tau_{k_i} + \mu e^{-\alpha|x|^2} \le 0$$

if α is big enough.

Furthermore, in convex domains $\bar{\mu}^{\pm}_{k_1,\dots,k_\ell} = \mu^{\pm}_{k_1,\dots,k_\ell}$, see [12, Lemma 6.7]. However, it is in general an open problem whether these values coincide or not.

We finally recall some bounds for these generalized eigenvalues.

Proposition 4.1 ([26])**.** *One has*

(i) *If $B_{R_1} \subseteq \Omega$, then*

$$\bar{\mu}^{-}_{k_1,\dots,k_\ell} \le \frac{c_1}{R_1^{2s}} < +\infty$$

if $k = N$, and

$$\bar{\mu}^{+}_{k_1,\dots,k_\ell} \le \frac{c_1}{R_1^{2s}} < +\infty$$

for any $k_1, \dots, k_\ell$, where $c_1 > 0$.

(ii) *If* $\Omega \subseteq B_{R_2}$, *then*

$$0 < \frac{c_2}{R_2^{2s}} \sum_{i=1}^{\ell} k_i \omega_{k_i} \le \bar{\mu}_1^+ \sum_{i=1}^{\ell} k_i \omega_{k_i} \le \bar{\mu}^+_{k_1,\dots,k_\ell} \le \bar{\mu}^-_{k_1,\dots,k_\ell}$$

where $c_2 > 0$ *and* ω_{k_i} *is the volume of the* k_i *dimensional sphere.*

Actually, in case $\ell = k$, namely when the operators $\mathcal{I}_k^\pm$ are taken into account, one can also prove that the eigenvalues are ordered in the sense that

$$\bar{\mu}_1^+ \le \cdots \le \bar{\mu}_N^+ \le \bar{\mu}_N^-$$

see [12, Proposition 6.3].

A natural question is whether there exists an eigenfunction corresponding to these eigenvalues. The answer is very partial, even in the case of the truncated Laplacians, and relies on regularity estimates up to the boundary. We only treat the case of the operator $\mathcal{I}_1^\pm$, proving, see [12],

Theorem 4.2. *Let* Ω *be a uniformly convex domain, and let* $s > \frac{1}{2}$. *Then there exists a positive function* $\psi_1 \in C^{0,2s-1}(\overline{\Omega})$ *such that*

$$\begin{cases} \mathcal{I}_1^+ \psi_1 + \mu_1^+ \psi_1 = 0 & \text{in } \Omega \\ \psi_1 = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{11}$$

The crucial regularity result is the following

Proposition 4.2. *Let* u *satisfy*

$$\begin{cases} \mathcal{I}_1^+ u(x) = f(x) & \text{in } \Omega \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{12}$$

where Ω *is a uniformly convex domain. If* $s > \frac{1}{2}$, *then* u *is Hölder continuous of order* $2s-1$ *in* $\mathbb{R}^N$.

For the proof we refer to [12], see also the very recent paper [3] for a different approach. Notice that here the nonlocal nature of the problem plays a crucial role. Indeed, for the truncated Laplacian one gets Lipschitz regularity [10], whereas here the regularity depends on s and resembles what happens for the fractional Laplacian, see [21].

5. Representation formulas

We conclude this survey considering the question of whether it is possible, at least in the radial case, to find the optimal frame for $\mathcal{K}^{\pm}_{k_1,\dots,k_\ell}$. However, it turns out to be hard to give an answer in its full generality, and only partial results are known. In [13] the cases $\mathcal{I}^{\pm}_k$ and $\mathcal{J}^{\pm}_k$ are considered, and the following results are proved.

Proposition 5.1. *Assume $u(x) = \tilde{g}(|x|^2) \in C^2(\mathbb{R}^N \setminus \{0\}) \cap L^\infty(\mathbb{R}^N)$, and let $x \neq 0$.*

(i) *If $\tilde{g}, \tilde{g}''$ are convex, for all $N, k \in \mathbb{N}$ with $1 \leq k \leq N$ we have*

$$\mathcal{I}^+_k u(x) = \mathcal{I}_{\hat{x}} u(x) + (k-1)\mathcal{I}_{x^\perp} u(x),$$

where $x^\perp \in \langle\{\hat{x}\}\rangle^\perp$ with $\left|x^\perp\right| = 1$, and $\hat{x} = x/|x|$.

(ii) *If $\tilde{g}$ is convex, and $1 \leq k < N$, we have*

$$\mathcal{I}^-_k u(x) = k\mathcal{I}_{x^\perp} u(x),$$

where $x^\perp$ is as in the previous point.

(iii) *If $\tilde{g}''$ is convex, then*

$$\mathcal{I}^-_N u(x) = N\mathcal{I}_{\xi^*} u(x),$$

where $\xi^* \in \mathbb{R}^N$ is a unit vector such that $\langle \hat{x}, \xi^* \rangle = \frac{1}{\sqrt{N}}$.

Proposition 5.2. *Assume $1 < k < N$. Let $u(x) = \tilde{g}\left(|x|^2\right) \in C^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. If $\tilde{g}$ is convex, then*

(i) *one has*

$$\mathcal{J}^-_k u(x) = \mathcal{J}_V u(x),$$

where V is any k dimensional subspace which is orthogonal to x.

(ii) *one has*

$$\mathcal{J}^+_k u(x) = \mathcal{J}_V u(x),$$

where V is any k-dimensional subspace containing x.

Remark 5.1. *Some examples of functions $\tilde{g}$ satisfying the above assumption are $\tilde{g}(t) = \sqrt{t}^{-\gamma}$ with $\gamma \in (0,1)$, $\tilde{g}(t) = (a+\sqrt{t})^{-\gamma}$, $\tilde{g}(t) = (a+t)^{-\gamma}$, $\tilde{g}(t) = e^{-at}$ for $a > 0$ and $\gamma > 0$.*

Only partial results can be proved for more general operators of the form $\mathcal{K}^{\pm}_{k_1,\dots,k_\ell}$, precisely,

Lemma 5.1 ([26]). *Let $k<N$. Assume $u(x)=\tilde g(|x|^2)\in C^2(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$ such that $\tilde g$ is convex. Then for any $x\neq 0$*

$$\mathcal{K}^{-}_{k_1,\dots,k_\ell}u(x)=\sum_{i=1}^{\ell}\mathcal{J}_{W_i}u(x) \tag{13}$$

for any W_i of dimension k_i such that x is orthogonal to W_i.

Actually, by similar arguments as in Lemma 5.1 one can also treat operators of the form

$$\tilde{\mathcal{K}}_{k_1,\dots,k_\ell}u(x):=\sup_{\{\xi^1_j\}_{j=1}^{k_1}\in\mathcal{V}_{k_1}}\ \inf_{\{\xi^2_j\}_{j=1}^{k_2}\in\mathcal{V}_{k_1,k_2}}\dots\inf_{\{\xi^\ell_j\}_{j=1}^{k_\ell}\in\mathcal{V}_{k_1,\dots,k_\ell}}\sum_{i=1}^{\ell}\mathcal{J}_{V_i}u(x).$$

Indeed, we have

$$\tilde{\mathcal{K}}_{k_1,\dots,k_\ell}u(x)=(-\Delta)^s_{\mathbb{R}^{k_1}}u(x)+\sum_{i=2}^{\ell}\mathcal{J}_{W_i}u(x),$$

for any W_i of dimension k_i such that x is orthogonal to W_i. However, it does not seem trivial to understand the behavior of other mixed operators, precisely when more then one sup is involved, or when taking $\mathcal{K}^{-}_{k_1,\dots,k_\ell}$ if $k=N$, as in these cases a competition between different terms in the sum arises.

Notice that these representation formula are of interest in order to give Liouville type results, see [13, 26].

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