THE BREZIS-NIRENBERG PROBLEM FOR MIXED LOCAL AND NONLOCAL OPERATORS
IL PROBLEMA DI BREZIS-NIRENBERG PER OPERATORI MISTI DI TIPO LOCALE-NONLOCALE

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Abstract. In this note we present some existence results, in the spirit of the celebrated paper by Brezis and Nirenberg (CPAM, 1983), for a *perturbed critical problem* driven by a mixed local and nonlocal linear operator. We develop an existence theory, both in the case of linear and superlinear perturbations; moreover, in the particular case of linear perturbations we also investigate the mixed Sobolev inequality associated with this problem, detecting the optimal constant, which we show that is never achieved.

Sunto. In questa nota presentiamo alcuni risultati di esistenza, nello spirito del noto lavoro di Brezis e Nirenberg (CPAM, 1983), per un *problema critico perturbato* associato ad un operatore misto di tipo locale-nonlocale. I risultati presentati riguardano sia il caso di perturbazioni lineari, sia il caso di perturbazioni non lineari; nel caso particolare di perturbazioni lineari studiamo anche la disuguaglianza di tipo Sobolev associata al problema, individuandone la costante ottimale e mostrando che essa non è mai assunta.

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1. Introduction

The present note is based on the talk titled *A Brezis-Nirenberg type result for mixed local and nonlocal operators*, given by the Author during the conference "Nonlocal and Nonlinear Partial Differential Equations" (September 08-09, 2022, University of Bologna).

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In its turn, this talk was based on the recent paper [7], which is a joint work with Serena Dipierro, Enirco Valdinoci and Eugenio Vecchi.

Let $n \geq 3$ and $s \in (0,1)$ be fixed. Given a bounded open set $\varnothing \neq \Omega \subseteq \mathbb{R}^n$, we aim to investigate the existence of solutions to the *perturbed critical problem*

$$(\mathrm{P})_{\lambda,p} \qquad \begin{cases} \mathcal{L}u = -\Delta u + (-\Delta)^s u = u^{\frac{n+2}{n-2}} + \lambda u^p & \text{in } \Omega, \\ u \gneqq 0 & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $(-\Delta)^s$ is defined (up to a multiplicative constant) as

$$(-\Delta)^s u(x) := 2\,\mathrm{P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy.$$

As regards the *parameters* λ, p, we assume that

(i) $\lambda \in \mathbb{R}$ (but we will soon restrict to the case $\lambda > 0$);

(ii) $p = 1$ (and in this case we will talk about *linear perturbation*) or $1 < p \leq \frac{n+2}{n-2}$ (and in this case we will talk about *superlinear perturbation*).

As we will see in detail in Section 2, an appropriate functional setting for the study of problem $(\mathrm{P})_{\lambda,p}$ is given by the *Hilbert space* $\mathcal{X}^{1,2}(\Omega)$ defined as follows

$$\mathcal{X}^{1,2}(\Omega) = \big\{ u \in H^1(\mathbb{R}^n) : \, u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \Omega \text{ and } u\big|_\Omega \in H^1_0(\Omega) \big\}.$$

On this space $\mathcal{X}^{1,2}(\Omega)$, the functional

$$\begin{aligned} \mathcal{J}_{\lambda,p}(u) := \frac{1}{2}\Big(\int_\Omega |\nabla u|^2\,dx + \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \Big) \\ - \frac{n-2}{2n} \int_\Omega |u|^{\frac{2n}{n-2}}\,dx - \frac{\lambda}{p+1} \int_\Omega |u|^{p+1}\,dx \qquad (u \in \mathcal{X}^{1,2}(\Omega)) \end{aligned}$$

is (well-defined and) of class C^1, and the non-negative critical points of $\mathcal{J}$ are *precisely* the solutions to $(\mathrm{P})_{\lambda,p}$. On the other hand, since the exponent

$$2^* = \frac{2n}{n-2}$$

is nothing but the critical exponent in the (local) Sobolev Embedding Theorem, the embedding $\mathcal{X}^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ is continuous but not compact, and we cannot proceed by direct minimization to prove the existence of a solution to problem $(\mathrm{P})_{\lambda,p}$.

In other words, we can see the *criticality* of problem $(\mathrm{P})_{\lambda,p}$ as the 'PDE counterpart' of the *lack of compactness* in the embedding $\mathcal{X}^{1,2}(\Omega) \hookrightarrow L^{2^*}(\Omega)$ (associated with the *critical exponent* 2^*), which prevents the functional $\mathcal{J}_{\lambda,p}$ to satisfy the Palais-Smale Condition.

To the best of our knowledge, the study of boundary-value problems involving critical exponents traces back to the seminal paper [16] by Brezis and Nirenberg, where the Authors obtain *optimal conditions* on the parameter λ for the solvability of what is now referred to as the *Brezis-Nirenberg problem*, namely,

$$(\star) \qquad \begin{cases} -\Delta u = u^{\frac{n+2}{n-2}} + \lambda u^p & \text{in } \Omega \subseteq \mathbb{R}^n, \\ u \gneqq 0 & \text{in } \Omega, \\ u \equiv 0 & \text{on } \partial\Omega, \end{cases}$$

(note that $(\star)$ is the purely local analog of our problem $(\mathrm{P})_{\lambda,p}$). Since then, critical boundary-value problems have been extensively studied under many different aspects and in various contexts: for *linear and quasilinear local operators* (see, for instance, the non-exhaustive list of papers [2–4, 15, 17, 18, 20, 21, 23, 26, 28, 33–35, 37, 39, 50, 51, 53, 54] and the references therein); for *linear and quasilinear nonlocal operators* (see, e.g., [5, 44, 48, 49]); for *higher-order local operators* (see, e.g., [6, 27, 38, 40, 45]); for *differential operators in non-Euclidean settings* (see, e.g., [41, 42] for the context of Carnot groups).

It is interesting to observe that, *independently of the setting*, the major part of the results concerning critical Dirichlet problems is obtained by suitable adaptations of the original argument by Brezis and Nirenberg. As we will describe in Section 3, such an argument is *purely variational*, and the main ingredient seems to be the *explicit knowledge* of the minimizers in the classical Sobolev Inequality, that is,

$$\mathcal{S}_n \|u\|^2_{L^{2^*}(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} |\nabla u|^2 \, dx \qquad \forall\ u \in C_0^\infty(\mathbb{R}^n)$$

(the so-called *Aubin-Talenti functions*); as a matter of fact, the results in [44] show that it may suffice to know the *asymptotic behavior* of these minimizers, and not their explicit expression (which can be very difficult to obtain in other settings).

All that being said, we now spend a few words about the content of this note. While we refer to Sections 3 and 4 for the precise statement of the main results (see Theorem 3.2

for the linear case $p = 1$ and Theorem 4.1 for the superlinear case $p > 1$), here we content ourselves with a brief discussion. As already mentioned at the beginning, our main aim is to study the solvability of the *critical Dirichlet problem* $(\mathrm{P})_{\lambda,p}$. More precisely, we try to follow and adapt the approach by Brezis and Nirenberg [16] mentioned above in order to extend the main results in [16] to the *mixed local and nonlocal linear operator*

$$\mathcal{L} = -\Delta + (-\Delta)^s, \qquad s \in (0,1).$$

The investigation of operators of mixed order is a very topical subject of investigation, arising naturally in several fields, for instance as the superposition of different types of stochastic processes such as a classical random walk and a Lévy flight, which has also interesting application in the study of optimal animal foraging strategies, see, e.g., [25].

From the technical point of view, these operators offer quite relevant challenges caused by the combination of nonlocal difficulties with the lack of invariance under scaling; the contemporary investigation has specifically focused on a large number of problems in the *existence and regularity theory* (see, e.g., the series of papers [1, 9, 12–14, 19, 22, 29–32, 36, 43, 47, 52]), in *symmetry and classification results* (see, e.g., [8, 10, 11]), etc.

As regards the main results in this note, the lack of invariance under scaling of the operator $\mathcal{L}$ turns out to be the major difficulty in trying to follow the approach in [16], especially in the linear case $p = 1$. In fact, since $\mathcal{L}$ possesses a *well-behaved* variational structure, several technical results which compose the approach in [16] can be easily extended to our context; however, we will see in Section 3 that the lack of a scale-invariance causes the *non-existence* of the minizers in the *mixed Sobolev-type inequality*

$$(1) \qquad \mathcal{S}_n \|u\|^2_{L^{2^*}(\mathbb{R}^n)} \leq \int_{\mathbb{R}^n} |\nabla u|^2\,dx + \iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \qquad (u \in C_0^\infty(\mathbb{R}^n))$$

naturally associated with $(\mathrm{P})_{\lambda,1}$. Since the explicit knowledge of the minimizers in (1) (or, at least, the knowledge of their asymptotic behavior) is the main ingredient in the Brezis and Nirenberg approach, this non-existence phenomenon constitutes a serious obstruction, which needs to be circumvented (see Section 3 for the details). As a result, we obtain that problem $(\mathrm{P})_{\lambda,1}$ does not admit any solution both in the range of 'small' and 'large' values of λ, but it does possess solutions for an 'intermediate' regime of values of λ; it is worth mentioning that the restriction to values of λ 'sufficiently far' from $\lambda = 0$ for finding solutions of this type of problems is a common occurrence also in the local scenario in *low dimensions*, see in particular the case $n = 3$ in [16, Cor. 2.4]

As concerns the superlinear case $p > 1$, instead, the non-existence of the minimizers in the mixed Sobolev-type inequality (1) has less impact; in fact, the *superlinear growth* of the term λu^p allows us to adapt the variational argument exploited in [16] (and later extended to the purely nonlocal context in [5]), leading to Theorem 4.1. The unique difference between our result and the corresponding ones in the purely local/nonlocal setting is that, even if the dimension n is large, *we cannot establish* the existence of solutions to problem $(\mathrm{P})_{\lambda,p}$ for λ close to zero (but only for all λ large enough, see Remark 4.1); it is in this discrepancy that the lack of a scale-invariance of $\mathcal{L}$ plays its rôle.

Plan of the paper. We conclude the Introduction with a short plan of the paper. In Section 2 we briefly introduce the functional setting for the study of problem $(\mathrm{P})_{\lambda,p}$, and we give some qualitative properties of the (possible) solution to this problem (independently of the value of p). In Section 3 we study problem $(\mathrm{P})_{\lambda,p}$ in the linear case $p = 1$, and in Section 4 we consider the superlinear case $p > 1$.

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2. Preliminaries

The aim of this section is twofold: firstly, we introduce the appropriate functional setting for the study of problem $(\mathrm{P})_{\lambda,p}$; secondly, we state some 'qualitative' properties for any (possible) solution of this problem (independently of p).

The functional setting. Given an arbitrary open set $\varnothing \neq \mathcal{O} \subseteq \mathbb{R}^n$ (not necessarily bounded), we introduce on the functions space $C_0^\infty(\mathcal{O})$ the *inner product*

$$B_s(u,v) := \int_{\mathbb{R}^n} \langle \nabla u, \nabla v \rangle \, dx + \iint_{\mathbb{R}^{2n}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}} \, dx \, dy,$$

and the associated *mixed local and nonlocal norm*

$$\rho_s(u) := B_s(u,u) = \Bigg(\int_{\mathbb{R}^n} |\nabla u|^2\,dx + \underbrace{\iint_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^2}\,dx\,dy}_{=:\,[u]_s^2}\Bigg)^{1/2}.$$

Then, we indicate by $\mathcal{X}^{1,2}(\mathcal{O})$ the *Hilbert space* obtained by taking the (metric) completion of $C_0^\infty(\mathcal{O})$ with respect to this norm $\rho_s(\cdot)$, that is,

$$\mathcal{X}^{1,2}(\mathcal{O}) := \overline{C_0^\infty(\mathcal{O})}^{\,\rho_s(\cdot)}.$$

In order to better understand the *nature* of the space $\mathcal{X}^{1,2}(\mathcal{O})$ (and to highlight his rôle in the study of problem $(\mathrm{P})_{\lambda,p}$), we distinguish two cases.

(i) *$\mathcal{O}$ is bounded.* In this case we first recall the following inequality, which expresses the *continuous embedding* of $H^1(\mathbb{R}^n)$ into $H^s(\mathbb{R}^n)$ (see, e.g., [24, Proposition 2.2]): there exists a constant $\mathbf{c} = \mathbf{c}(s) > 0$ such that, for every $u \in C_0^\infty(\Omega)$, one has

$$[u]_s^2 \leq \mathbf{c}(s)\|u\|^2_{H^1(\mathbb{R}^n)} = \mathbf{c}(s)\big(\|u\|^2_{L^2(\mathbb{R}^n)} + \|\nabla u\|^2_{L^2(\mathbb{R}^n)}\big). \tag{2}$$

This, together with the classical *Poincaré inequality*, implies that $\rho_s(\cdot)$ and the full H^1-norm in $\mathbb{R}^n$ are *actually equivalent* on the space $C_0^\infty(\mathcal{O})$, and hence

$$\begin{aligned}\mathcal{X}^{1,2}(\mathcal{O}) &= \overline{C_0^\infty(\mathcal{O})}^{\,\|\cdot\|_{H^1(\mathbb{R}^n)}} \\ &= \{u \in H^1(\mathbb{R}^n) : \, u|_{\mathcal{O}} \in H_0^1(\mathcal{O}) \text{ and } u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \mathcal{O}\}.\end{aligned}$$

In particular, we see that the functions in $\mathcal{X}^{1,2}(\mathcal{O})$ naturally satisfy the nonlocal Dirichlet condition prescribed in problem $(\mathrm{P})_{\lambda,p}$.

(ii) *$\mathcal{O}$ is unbounded.* In this case, even if the *embedding inequality* (2) is still satisfied, the Poincaré inequality *does not hold*; hence, the norm $\rho_s(\cdot)$ is no more equivalent to the full H^1-norm in $\mathbb{R}^n$, and $\mathcal{X}^{1,2}(\mathcal{O})$ *is not a subspace of* $H^1(\mathbb{R}^n)$.

On the other hand, by the classical *Sobolev inequality* we infer the existence of a constant $\mathcal{S} = \mathcal{S}_n > 0$, *independent of the open set* $\mathcal{O}$, such that

$$\mathcal{S}_n\|u\|^2_{L^{2^*}(\mathbb{R}^n)} \leq \|\nabla u\|^2_{L^2(\mathbb{R}^n)} \leq \rho(u)^2 \qquad \text{for every } u \in C_0^\infty(\mathcal{O}). \tag{3}$$

From this, we deduce that every Cauchy sequence in $C_0^\infty(\mathcal{O})$ (with respect to the norm $\rho_s(\cdot)$) is also a Cauchy sequence *in the space* $L^{2^*}(\mathbb{R}^n)$; as a consequence, since the functions in $C_0^\infty(\mathcal{O})$ identically vanish out of Ω, we obtain

$$\mathcal{X}^{1,2}(\mathcal{O}) = \{u \in L^{2^*}(\mathbb{R}^n) :\, u \equiv 0 \text{ a.e. in } \mathbb{R}^n \setminus \mathcal{O},\, \nabla u \in L^2(\mathbb{R}^n) \text{ and } [u]_s < \infty\}.$$

In particular, when $\mathcal{O} = \mathbb{R}^n$ we have

$$\mathcal{X}^{1,2}(\mathbb{R}^n) = \{u \in L^{2^*}(\mathbb{R}^n) :\, \nabla u \in L^2(\mathbb{R}^n) \text{ and } [u]_s < \infty\}.$$

Remark 2.1. *We stress that the* mixed Sobolev-type inequality (3) *holds* for every open set $\Omega \subseteq \mathbb{R}^n$ *(bounded or not); thus, we always have*

$$\mathcal{X}^{1,2}(\mathcal{O}) \hookrightarrow L^{2^*}(\Omega).$$

Furthermore, by exploiting the density of $C_0^\infty(\mathcal{O})$ *in* $\mathcal{X}^{1,2}(\mathcal{O})$*, we can extend inequality* (3) *to* every function $u \in \mathcal{X}^{1,2}(\mathcal{O})$*, thereby obtaining*

$$\mathcal{S}_n \|u\|^2_{L^{2^*}(\mathbb{R}^n)} \leq \rho(u)^2 = \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + [u]_s^2 \quad \textit{for every } u \in \mathcal{X}^{1,2}(\mathcal{O}).$$

Some general results. Now we have introduced the relevant functional setting, we proceed by stating some general results concerning any possible solution of problem $(\mathrm{P})_{\lambda,p}$; we refer to [7] for all the omitted proofs.

To begin with, we give the following definition.

Definition 2.1. *We say that a function* $u \in \mathcal{X}^{1,2}(\Omega)$ *is a* weak solution *to* $(\mathrm{P})_{\lambda,p}$ *if*

(1) $u \geq 0$ *a.e. in* Ω *and* $|\{x \in \Omega :\, u(x) > 0\}| > 0$*;*
(2) *for every* $v \in \mathcal{X}^{1,2}(\Omega)$ *one has*

$$B_s(u,v) = \int_\Omega \left(u^{\frac{n+2}{n-2}} v + \lambda u^p v\right) dx.$$

Here and in what follows, we indicate by $|\cdot|$ *the* n*-dimensional Lebesgue measure of a measurable subset of* $\mathbb{R}^n$*.*

We then have the following theorem.

Theorem 2.1 (See [7, Thm. 4.3])**.** *Assume that there exists a solution* $u_0 \in \mathcal{X}^{1,2}(\Omega)$ *to problem* $(\mathrm{P})_{\lambda,p}$ *(for some* $\lambda \in \mathbb{R}$ *and for some* $p \in [1, 2^* - 1)$*). Then, we have:*

(1) $u_0 \in L^\infty(\mathbb{R}^n)$;

(2) *if, in addition,* $\lambda \geq 0$, *one has* $u_0 > 0$ *a.e. in* Ω.

Using the previous theorem, we can prove the following *non-existence result.*

Theorem 2.2 (See [7, Thm. 1.3]). *Let* $\lambda \leq 0$, *and assume that* $\Omega \subseteq \mathbb{R}^n$ *is* star-shaped. *Then, there do not exist solutions to* $(\mathrm{P})_{\lambda,p}$, *whatever the exponent* $p \in [1, 2^* - 1)$.

Proof. By contradiction, let us suppose that there exists solution u_0 to problem $(\mathrm{P})_{\lambda,p}$. From Theorem 2.1, we know that $u_0 \in L^\infty(\mathbb{R}^n)$; moreover, setting

$$f(x,t) := |t|^{2^*-2}t + \lambda|t|^{p-1}t \qquad (\text{with } t \in \mathbb{R}),$$

we see that u_0 solves

$$\begin{cases} \mathcal{L}u = f(x,u) & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

and the nonlinearity f satisfies the following properties:

(i) $f \in C^{0,1}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R})$;

(ii) for every $x \in \Omega$ and every $t \in \mathbb{R}$, we have

$$\frac{n-2}{2}\, t f(x,t) \geq n \int_0^t f(x,\tau)\, d\tau.$$

In view of these facts, and since Ω is star-shaped, we are entitled to apply [46, Thm. 1.3], ensuring that $u_0 \equiv 0$ a.e. in Ω. This is in contradiction with the fact that $u_0 \gneqq 0$ in Ω, and the proof is complete. $\square$

In view of the previous theorem, from now on we assume that $\lambda > 0$

3. The linear case

The aim of this section is to study the solvability of problem $(\mathrm{P})_{\lambda,p}$ in the case of *linear perturbations*, that is, when $p = 1$. Hence, we consider the problem

$$(\mathrm{P})_{\lambda,1} \qquad \begin{cases} \mathcal{L}u = u^{\frac{n+2}{n-2}} + \lambda u & \text{in } \Omega, \\ u \gneqq 0 & \text{in } \Omega, \\ u \equiv 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

As already mentioned in the Introduction, in obtaining our main result concerning $(\mathrm{P})_{\lambda,1}$ (see Theorem 3.2 below) we are much indebted with the ideas in the beautiful paper by Brezis and Nirenberg [16], where the Authors study the *purely local* analog of problem $(\mathrm{P})_{\lambda,1}$. More precisely, they consider the problem

$$(\star) \qquad \begin{cases} -\Delta u = u^{2^*-1} + \lambda u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u \equiv 0 & \text{in } \partial\Omega, \end{cases}$$

obtaining the following results:

(a) if $n \geq 4$, then there exists a solution to $(\star)$ *if and only if* $0 < \lambda < \lambda_1$,

(b) if $n = 3$ and Ω is a ball, then there exists a solution to $(\star)$ *if and only if*

$$\frac{\lambda_1}{4} < \lambda < \lambda_1.$$

(here, λ_1 is the *first Dirichlet eigenvalue of* $-\Delta$ *in* Ω). Due to its relevance in our argument, we now briefly describe the approach by Brezis and Nirenberg.

Roughly put, the key ingredients for the proof of (a)-(b) are the following.

(I) Given a non-void open set $\mathcal{O} \subseteq \mathbb{R}^n$, the *best constant in the Sobolev inequality* is independent of $\mathcal{O}$ and depends only on n: more precisely, we have

$$\begin{aligned} &\inf\left\{\|\nabla u\|^2_{L^2(\mathcal{O})} : u \in C_0^\infty(\mathcal{O}) \cap \mathcal{M}\right\} \\ &\qquad = \inf\left\{\|\nabla u\|^2_{L^2(\mathbb{R}^n)} : u \in C_0^\infty(\mathbb{R}^n) \cap \mathcal{M}\right\} =: \mathcal{S}_n. \end{aligned}$$

where $\mathcal{M}$ is the unit sphere in $L^{2^*}(\mathbb{R}^n)$, that is,

$$\mathcal{M} := \{u \in L^{2^*}(\mathbb{R}^n) : \|u\|_{L^{2^*}(\mathbb{R}^n)} = 1\}.$$

Moreover, if $\mathcal{O}$ is *bounded*, then $\mathcal{S}_n$ is *never achieved*; if, instead, $\mathcal{O} = \mathbb{R}^n$, then $\mathcal{S}_n$ is achieved by the family of functions (Aubin-Talenti functions)

$$U_{t,x_0}(x) = c_t\big(t^2 + |x - x_0|^2\big)^{\frac{2-n}{2}} \qquad (x_0 \in \mathbb{R}^n,\, t > 0),$$

where $c_t > 0$ is a suitable normalization constant (depending on t).

(II) Consider the functional $\mathcal{Q}_\lambda$ defined as follows

$$\mathcal{Q}_\lambda(u) := \|\nabla u\|^2_{L^2(\Omega)} - \lambda \|u\|^2_{L^2(\Omega)} \qquad \big(u \in H^1_0(\Omega),\, \lambda > 0\big),$$

and let $\mathcal{S}_n(\lambda) := \inf\{\mathcal{Q}_\lambda(u) : u \in H^1_0(\Omega) \cap \mathcal{M}\} > -\infty$. Then,

$$\big(\mathcal{S}_n(\lambda) \text{ is achieved and } S_n(\lambda) > 0\big) \implies \exists \text{ a solution of } (\star).$$

We explicitly stress that, by definition, we have $\mathcal{S}_n(\lambda) > 0 \Leftrightarrow 0 < \lambda < \lambda_1$.

(III) If $\mathcal{S}_n(\lambda) < \mathcal{S}_n$ $(= \mathcal{S}_n(0))$, then $\mathcal{S}_n(\lambda)$ is achieved.

(IV) For every $\lambda > 0$, we have $\mathcal{S}_n(\lambda) < \mathcal{S}_n$.

It is worth mentioning that the proof of (IV) follows by choosing as a *competitor function* (in the minimization problem defining $\mathcal{S}_\lambda$) the map $u_t = v_t/\|v_t\|_{L^{2^*}(\mathbb{R}^n)}$, where

$$v_t(x) = \frac{\phi(x)}{(t^2 + |x|^2)^{\frac{n-2}{2}}} \qquad (t > 0)$$

and $\phi \in C^\infty_0(\Omega)$ is suitable cut-off function. This gives (at least for $n \geq 4$)

$$\mathcal{Q}_\lambda(u_t) = \begin{cases} \mathcal{S}_n - \lambda C t^2 + O(t^{n-2}) & \text{if } n \geq 5, \\ \mathcal{S}_n - \lambda C t^2 \log|t| + O(t^2) & \text{if } n = 4, \end{cases}$$

and thus $\mathcal{S}_\lambda \leq \mathcal{Q}_\lambda(u_t) < \mathcal{S}_n$ if $t \ll 1$.

We then try to follow this *scheme* (I)-to-(IV) for the study of $(\mathrm{P})_{\lambda,1}$.

(I) A mixed Sobolev-type inequality. In trying to adapt the above *scheme* (I)-to-(IV) to our mixed context, the first step is to study the *mixed Sobolev-type inequality*

$$\mathbf{c}(s, n, \mathcal{O})\rho_s(u)^2 \leq \|u\|^2_{L^{2^*}(\mathbb{R}^n)} \qquad \big(u \in \mathcal{X}^{1,2}(\mathcal{O})\big),$$

which reflects the *continuous embedding* $\mathcal{X}^{1,2}(\mathcal{O}) \hookrightarrow L^{2^*}(\mathbb{R}^n)$. We define

$$\mathcal{S}_{n,s}(\mathcal{O}) := \inf\big\{\rho_s(u)^2 : u \in C^\infty_0(\mathcal{O}) \cap \mathcal{M}\big\} = \inf\big\{\rho_s(u)^2 : u \in \mathcal{X}^{1,2}(\mathcal{O}) \cap \mathcal{M}\big\}.$$

Our main result in this context is the following.

Theorem 3.1 (See [7, Thm.s 1.1 and 1.2]). *Let $\varnothing \neq \mathcal{O} \subseteq \mathbb{R}^n$ be an arbitrary open set. Then, the following assertions hold.*

(1) $\mathcal{S}_{n,s}(\mathcal{O}) = \mathcal{S}_n$.

(2) $\mathcal{S}_{n,s}(\mathcal{O})$ *is* never achieved *in* $\mathcal{X}^{1,2}(\mathcal{O})$ *(even if* $\mathcal{O} = \mathbb{R}^n$*).*

We explicitly stress that assertion (2) in Theorem 3.1 highlights a first *discrepancy* between the purely local/nonlocal setting and our *mixed local and nonlocal* setting.

Proof. (1) Since $\rho_s(u) \geq \|\nabla u\|_{L^2(\mathcal{O})}$ for every $u \in C_0^\infty(\mathcal{O})$, we clearly have

$$\mathcal{S}_{n,s}(\mathcal{O}) \geq \mathcal{S}_n.$$

To prove the reverse inequality, by the *translation-invariance* of $\mathcal{S}_{n,s}(\mathcal{O})$ we assume that $x_0 = 0 \in \mathcal{O}$, and we let $r > 0$ be such that $B_r(0) \subseteq \mathcal{O}$. We now observe that, given any $u \in C_0^\infty(\mathbb{R}^n) \cap \mathcal{M}$, there exists $k_0 = k_0(u) \in \mathbb{N}$ such that

$$\mathrm{supp}(u) \subseteq B_{kr}(0) \quad \text{for every } k \geq k_0;$$

as a consequence, setting $u_k := k^{\frac{n-2}{2}} u(kx)$ (for $k \geq k_0$), we readily see that

- $\mathrm{supp}(u_k) \subseteq B_r(0) \subseteq \mathcal{O}$;
- $\|\nabla u_k\|_{L^2(\mathbb{R}^n)} = \|\nabla u\|_{L^2(\mathbb{R}^n)}$ and $\|u_k\|_{L^{2^*}(\mathbb{R}^n)} = \|u\|_{L^{2^*}(\mathbb{R}^n)} = 1$.

By definition $\mathcal{S}_{n,s}(\mathcal{O})$ we then find that, for every $k \geq k_0$,

$$\mathcal{S}_{n,s}(\mathcal{O}) \leq \rho_s(u_k)^2 = \|\nabla u\|^2_{L^2(\mathbb{R}^n)} + k^{2s-2}[u]_s^2.$$

From this, letting $k \to \infty$, we obtain

$$\mathcal{S}_{n,s}(\mathcal{O}) \leq \|\nabla u\|^2_{L^2(\mathbb{R}^n)}.$$

By the arbitrariness of $u \in C_0^\infty(\mathcal{O}) \cap \mathcal{M}$ and the fact that $\mathcal{S}_n$ *is independent of* $\mathcal{O}$ *and depends only on* n, we finally infer that

$$\mathcal{S}_{n,s}(\mathcal{O}) \leq \inf\left\{\|\nabla u\|^2_{L^2(\mathbb{R}^n)} :\, u \in C_0^\infty(\mathbb{R}^n) \cap \mathcal{M}\right\} = \mathcal{S}_n,$$

and hence $\mathcal{S}_{n,s}(\mathcal{O}) = \mathcal{S}_n$.

Proof. (2) Arguing by contradiction, let us suppose that $\mathcal{S}_n$ is achieved by some function $u_0 \in \mathcal{X}^{1,2}(\mathcal{O}) \cap \mathcal{M}$, that is, $\|u_0\|_{L^{2^*}(\mathbb{R}^n)} = 1$ and $\rho_s(u_0)^2 = \mathcal{S}_n$.

Taking into account that $\mathcal{X}^{1,2}(\mathcal{O}) \subseteq \mathcal{D}_0^{1,2}(\mathcal{O})$, we infer that

$$\mathcal{S}_n \leq \|\nabla u_0\|^2_{L^2(\mathbb{R}^n)} \leq \|\nabla u_0\|^2_{L^2(\mathbb{R}^n)} + [u_0]_s^2 = \rho_s(u_0)^2 = \mathcal{S}_n,$$

from which we derive that $[u_0]_s = 0$. As a consequence, the function u_0 must be *constant in* $\mathbb{R}^n$, but this is contradiction with the fact that $\|u_0\|_{L^{2^*}(\mathbb{R}^n)} = 1$. $\square$

(II)-(III) The functional $\mathcal{Q}_\lambda$. Despite this discrepancy with the purely local case, we continue to follow the *scheme* (I)-to-(IV) by Brezis and Nirenberg.

To this end, we introduce the functional

$$\mathcal{Q}_\lambda(u) := \rho_s(u)^2 - \lambda\|u\|^2_{L^2(\mathbb{R}^n)} \qquad \big(u \in \mathcal{X}^{1,2}(\Omega)\big),$$

and we define

$$\mathcal{S}_n(\lambda) := \inf\big\{\mathcal{Q}_\lambda(u) : \, u \in \mathcal{X}^{1,2}(\Omega) \cap \mathcal{M}\big\}.$$

Remark 3.1. *Using Theorem 3.1 and Hölder's inequality, we have*

$$\begin{aligned}\mathcal{Q}_\lambda(u) &\geq \inf\big\{\rho^2(u) : \, u \in \mathcal{X}^{1,2}(\Omega) \cap \mathcal{M}\big\} - \lambda\|u\|^2_{L^2(\mathbb{R}^n)} \\ &\geq \mathcal{S}_n - \lambda\|u\|^2_{L^2(\mathbb{R}^n)} \geq \mathcal{S}_n - \lambda|\Omega|^{4/n} \qquad \forall\, u \in \mathcal{X}^{1,2}(\Omega) \cap \mathcal{M}.\end{aligned}$$

As a consequence, we obtain

$$\mathcal{S}_n(\lambda) \geq \mathcal{S}_n - \lambda|\Omega|^{4/n} > -\infty \quad \textit{for every } \lambda > 0.$$

By *arguing exactly* as in [16], and by taking into account that the best constant in our mixed Sobolev-type inequality is *precisely* $\mathcal{S}_n$, we can prove the following results.

Lemma 3.1. *Assume that $\lambda > 0$ is such that*

$$\mathcal{S}_n(\lambda) < \mathcal{S}_n = \mathcal{S}_n(0).$$

Then, $\mathcal{S}_n(\lambda)$ is achieved.

Lemma 3.2 (See [7, Lem. 4.5])**.** *Assume that $\lambda > 0$ is such that $\mathcal{S}_n(\lambda) > 0$ and $\mathcal{S}_n(\lambda)$* is achieved*, that is, there exists some function $w = w_\lambda \in \mathcal{X}^{1,2}(\Omega) \cap \mathcal{M}$ such that*

$$\mathcal{Q}_\lambda(w) = \mathcal{S}_n(\lambda) > 0.$$

Then, there exists a solution to $(\mathrm{P})_{\lambda,1}$.

(IV) The inequality $\mathcal{S}_n(\lambda) < \mathcal{S}_n$ and the main result. Motivated by the analogy of these results with the purely local case, and continuing to follow the approach by Brezis and Nirenberg, we then try to prove that

$$\mathcal{S}_n(\lambda) < \mathcal{S}_n = \mathcal{S}_n(0) \qquad \forall\ \lambda > 0.$$

To this end, recalling that $\mathcal{S}_n$ *is never achieved* in our mixed Sobolev-type inequality (that is, there do not exist *mixed Aubin-Talenti-type functions*), it seems natural to use as a *competitor function* (in the definition of $\mathcal{S}_n(\lambda)$) the very same function used by Brezis and Nirenberg, that is, $u_t = v_t/\|v_t\|_{L^{2^*}(\mathbb{R}^n)}$, where

$$v_t(x) = \frac{\phi(x)}{(t^2+|x|^2)^{\frac{n-2}{2}}} \qquad (t>0).$$

Remark 3.2. *We explicitly stress that the choice of u_t as a competitor functions is motivated by the following two facts:*

(i) *the best constant in our mixed Sobolev-type inequality* is $\mathcal{S}_n$*;*

(ii) $\mathcal{S}_n$ is never achieved*, even in the whole space $\mathbb{R}^n$.*

Hence, there does not exist $w \in \mathcal{X}^{1,2}(\mathbb{R}^n) \cap \mathcal{M}$ *such that $\rho_s(w)^2 = \mathcal{S}_n$.*

Unfortunately, the use of u_t is not helpful: in fact, we get (at least for $n \geq 5$)

$$\mathcal{Q}_\lambda(u_t) = \mathcal{S}_n - \lambda C t^2 + O(t^{n-2}) + O(t^{2-2s}) = \mathcal{S}_n + O(t^{2-2s}),$$

but this does not allow to conclude that $\mathcal{Q}_\lambda(u_t) < \mathcal{S}_n$ when $t \ll 1$!

As a matter of fact, we have the following result.

Lemma 3.3 (See [7, Lem. 4.7 and Thm. 1.4])**.** *Let $\lambda_{1,s}$ denote the* first Dirichlet eigenvalue *of the operator $(-\Delta)^s$ in Ω, that is,*

$$\lambda_{1,s} = \inf\left\{[u]_s^2 :\ u \in C_0^\infty(\Omega) \text{ and } \|u\|_{L^2(\Omega)} = 1\right\} > 0.$$

Then, $\mathcal{S}_n(\lambda) = \mathcal{S}_n$ for all $0 < \lambda \leq \lambda_{1,s}$, and $\mathcal{S}_n(\lambda) = \mathcal{S}_n$ is not achieved*.*

In view of the above lemma, we *cannot follow* the approach by Brezis and Nirenberg up to the end when $0 < \lambda \leq \lambda_{1,s}$: in fact, in this regime we know that $\mathcal{Q}_\lambda$ does not posses minimizers, and so we are not able to prove the solvability of $(\mathrm{P})_{\lambda,1}$.

As a (partial) comfort from this disappointing find, we make the following observation: denoting by $\lambda_{1,\mathcal{L}}$ the *first Dirichlet eigenvalue of* $\mathcal{L}$ *in* Ω, that is,

$$\lambda_{1,\mathcal{L}} = \inf\big\{\rho_s(u)^2 : \, u \in C_0^\infty(\Omega) \text{ and } \|u\|_{L^2(\Omega)} = 1\big\} > \lambda_{1,s} > 0,$$

from the very definition of $\mathcal{S}_n(\lambda)$ it follows that

$$\mathcal{S}_n(\lambda) \leq 0 \text{ for every } \lambda \geq \lambda_{1,\mathcal{L}}.$$

As a consequence, since we also know that $\mathcal{S}_n(\lambda) = \mathcal{S}_n > 0$ for $0 < \lambda \leq \lambda_{1,s}$, we may hope to find a suitable *open set* $\mathcal{D} \subseteq (\lambda_{1,s}, \lambda_{1,\mathcal{L}})$ such that

$$0 < \mathcal{S}_n(\lambda) < \mathcal{S}_n \text{ for all } \lambda \in \mathcal{D}.$$

If this is the case, we can then follow the approach by Brezis and Nirenberg in order to prove the existence of a solution to problem $(\mathrm{P})_{\lambda,1}$ (at least for $\lambda \in \mathcal{D}$).

The above intuition can be made rigorous, and we obtain the following result.

Theorem 3.2 (See [7, Thm. 1.4])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open and bounded set. Then, there exists some $\lambda^* \in [\lambda_{1,s}, \lambda_{1,\mathcal{L}})$ such that problem $(\mathrm{P})_{\lambda,1}$ possesses at least one solution if*

$$\lambda^* < \lambda < \lambda_{1,\mathcal{L}}.$$

Moreover, the following facts hold:

(1) *there do not exist solutions to problem $(\mathrm{P})_{\lambda,1}$ if $\lambda \geq \lambda_{1,\mathcal{L}}$;*
(2) *for every $0 < \lambda \leq \lambda_{1,s}$ there do no exist solutions to problem $(\mathrm{P})_{\lambda,1}$ belonging to the closed ball $\mathcal{B} \subseteq L^{2^*}(\mathbb{R}^n)$ defined as*

$$\mathcal{B} := \big\{u \in L^{2^*}(\mathbb{R}^n) : \, \|u\|_{L^{2^*}(\mathbb{R}^n)} \leq \mathcal{S}_n^{(n-2)/4}\big\}.$$

Proof (sketch). It is not difficult to recognize that the map $(0,\infty) \ni \lambda \mapsto \mathcal{S}_n(\lambda)$ is (non-increasing and) *continuous* on $(0,\infty)$; hence, recalling that

- $\mathcal{S}_n(\lambda) = \mathcal{S}_n$ for every $0 < \lambda \leq \lambda_{1,s}$;
- $\mathcal{S}_n(\lambda) \geq 0$ for $\lambda \leq \lambda_{1,\mathcal{L}}$ and $\mathcal{S}_n(\lambda) \leq 0$ for $\lambda \geq \lambda_{1,\mathcal{L}}$;

it is easy to prove that there exists a *unique* $\lambda^* \in [\lambda_{1,s}, \lambda_{1,\mathcal{L}})$ such that

$$\mathcal{S}_n(\lambda) = \mathcal{S}_n \text{ for all } 0 < \lambda \leq \lambda^* \qquad \text{and} \qquad \mathcal{S}_n(\lambda) < \mathcal{S}_n \text{ for all } \lambda > \lambda^*.$$

Let now $\lambda \in (\lambda^*, \lambda_{1,\mathcal{L}})$ be fixed. Since $\mathcal{S}_n(\lambda) < \mathcal{S}_n$, we know that $\mathcal{S}_n(\lambda)$ is achieved by some function $w \in \mathcal{X}^{1,2}(\Omega) \cap \mathcal{M}$; in particular, since $\lambda < \lambda_{1,\mathcal{L}}$, we have

$$\mathcal{S}_n(\lambda) = \mathcal{Q}_\lambda(w) \geq (\lambda_{1,\mathcal{L}} - \lambda)\|w\|^2_{L^2(\mathbb{R}^n)} > 0,$$

and thus we can conclude that there exists a solution to problem $(\mathrm{P})_{\lambda,1}$.

We then turn to prove assertions (1)-(2).

(1) Let $\psi_0 \in \mathcal{X}^{1,2}(\Omega)$ be a positive eigenfunction for $\mathcal{L}$ relative to $\lambda_{1,\mathcal{L}}$, and assume that there exists a solution $u \in \mathcal{X}^{1,2}(\Omega)$ to problem $(\mathrm{P})_{\lambda,1}$ (for some $\lambda > 0$).

Recalling that $u > 0$ a.e. in Ω, we then have

$$\lambda_{1,\mathcal{L}} \int_\Omega \psi_0 u\,dx = \mathcal{B}_s(u,\psi_0) = \int_\Omega (u^{2^*-1} + \lambda u)\psi_0\,dx > \lambda \int_\Omega u\psi_0\,dx,$$

from which we derive that one necessarily has $\lambda < \lambda_{1,\mathcal{L}}$.

(2) Let $0 < \lambda \leq \lambda_{1,s}$ be fixed. Arguing by contradiction, let us assume that there exists a solution $u \in \mathcal{X}^{1,2}(\Omega) \cap \mathcal{B}$ to problem $(\mathrm{P})_{\lambda,1}$. Then, setting

$$v := u/\|u\|_{L^{2^*}(\mathbb{R}^n)},$$

a direct computation shows that

$$\begin{aligned}\mathcal{S}_n(\lambda) \leq \mathcal{Q}_\lambda(v) &= \rho_s(v)^2 - \lambda\|v\|^2_{L^2(\mathbb{R}^n)} = \|u\|^{4/(n-2)}_{L^{2^*}(\mathbb{R}^n)} \\ &\leq \mathcal{S}_n = \mathcal{S}_n(\lambda) = \inf\big\{\mathcal{Q}_\lambda(w) :\, w \in \mathcal{X}^{1,2}(\Omega) \cap \mathcal{M}\big\},\end{aligned}$$

but this is contradiction with the fact that $\mathcal{S}_n(\lambda) = \mathcal{S}_n$ is not achieved.

This ends the proof. □

We end this section with some conclusive remarks and open problems.

(i) Our main theorem holds in every dimension $n \geq 3$, with no distinction between the cases $n = 3$ and $n \geq 4$; in this perspective, it is resemblant to the result proved by Brezis and Nirenberg in the purely local setting when $n = 3$.

(ii) Despite being somehow disappointing, the a-priori estimate on the L^{2^*}-norm of the solutions to problem $(\mathrm{P})_{\lambda,1}$ when $0<\lambda\leq\lambda_{1,s}$ (provided they exist) *is uniform with respect to* λ.

(iii) Unfortunately, we have no idea about the existence of solutions to problem $(\mathrm{P})_{\lambda,1}$ in the gap $\lambda_{1,s}<\lambda\leq\lambda^*$; actually, we do not even know whether

$$\lambda^*>\lambda_{1,s} \text{ or } \lambda^*=\lambda_{1,s}$$

(it may depends on the geometry of the set Ω).

(iv) Hopefully, something more about λ^* can be said for particular choices of the open set Ω (e.g., when Ω is a ball, and the solutions are radially symmetric, see [10]); however, the ODE associated with $\mathcal{L}$ is not easy-to-handle.

4. The superlinear case

Now we have studied the case of *linear perturbations*, we turn to investigate the solvability of problem $(\mathrm{P})_{\lambda,p}$. If compared with the *linear case* $p=1$, this superlinear case is rather standard and it does not present new strange/unexpected phenomena; our main result in this context is the following.

Theorem 4.1 (See [7, Thm. 1.5]). *Let $n\geq 3$ and $p\in(1,2^*-1)$ be fixed. We define*

$$\kappa_{s,n}:=\min\{2-2s,n-2\} \qquad \textit{and} \qquad \beta_{n,p}:=n-\frac{(n-2)(p+1)}{2}.$$

Then, the following facts hold.

(1) *If $\kappa_{s,n}>\beta_{p,n}$, then problem $(\mathrm{P})_{\lambda,p}$ admits a solution for every $\lambda>0$.*

(2) *If $\kappa_{s,n}\leq\beta_{p,n}$, then $(\mathrm{P})_{\lambda,p}$ admits a solution for λ sufficiently large.*

Remark 4.1. *We explicitly stress that a dichotomy similar to that in Theorem 4.1 is somehow* hidden *also in the following cases:*

(i) *in the purely local case, where we have $\kappa_{s,n}=n-2$ and*

$$\kappa_{s,n}>\beta_{p,n} \iff n>2+\frac{4}{p+1};$$

(ii) *in the purely nonlocal case, where we have* $\kappa_{s,n} = n - 2s$ *and*

$$\kappa_{s,n} > \beta_{p,n} \iff n > 2 + \frac{4s}{p+1}.$$

A first difference between our mixed setting and the purely local/nonlocal is that, in our context, $\kappa_{s,n}$ *depends on a suitable order relation between* n, s*, namely,*

$$n + 2s < 4 \text{ or } n + 2s \geq 4.$$

Most importantly, even if the ambient dimension n *is large* $($*so that* $\kappa_{s,n} = 2 - 2s)$*, the inequality* $\kappa_{s,n} > \beta_{p,n}$ *is equivalent to* $n > \theta_{s,p}$*, where*

$$\theta_{s,p} = 2 + \frac{4s}{p-1} \to \infty \quad \text{as } p \to 1^+;$$

thus, independently of how large the fixed dimension is*, we have*

$$\begin{aligned} p \sim 1 &\implies \kappa_{s,n} \leq \beta_{p,n} \\ &\implies \text{ existence of solutions to } (\mathrm{P})_{\lambda,p} \text{ only for } \lambda \gg 1 \end{aligned}$$

$($*and this is coherent with the linear case* $p = 1)$*. This is not true, e.g., in the purely local case: in fact, in this case we have*

$$\kappa_{s,n} > \beta_{p,n} \iff n > 2 + \frac{4}{p+1} =: \theta_p,$$

but $\theta_p < 4$ for every $p > 1$*. As a consequence, we obtain*

$$\begin{aligned} n \geq 4 &\implies \kappa_{s,n} > \beta_{s,n,p} \quad \text{for all } p > 1 \\ &\implies \text{ existence of solutions for every } \lambda > 0 \text{ and } p > 1. \end{aligned}$$

The proof of Theorem 4.1 is based on a suitable adaptation of the original argument by Brezis and Nirenberg [16] (and later extended to the purely nonlocal setting in [5]).

Such an argument essentially relies on the *Mountain Pass Lemma*, and we sketch here below the key ingredients/steps (as usual, we refer to [7] for all the details).

(I) **(Solutions to** $(\mathrm{P})_{\lambda,p}$ **as unconstrained critical points)** Using the *Weak Maximum Principle for* $\mathcal{L}$ proved in [9], we easily obtain the following correspondence

$$\text{solutions to } (\mathrm{P})_{\lambda,p} \iff \text{(unconstrained) critical points of } \mathcal{J}_{\lambda,p},$$

where $\mathcal{J}_{\lambda,p}$ is the C^1-functional defined as follows

$$\mathcal{J}_{\lambda,p}(u) := \frac{1}{2}\rho(u)^2 - \frac{1}{2^*}\int_\Omega (u_+)^{2^*}\,dx - \frac{\lambda}{p+1}\int_\Omega (u_+)^{p+1}\,dx.$$

(II) **(Nice Mountain Pass geometry of $\mathcal{J}_{\lambda,p}$)** There exist $\alpha,\beta>0$ such that

(i) for any $u\in\mathcal{X}^{1,2}(\Omega)$ with $\rho_s(u)=\alpha$, we have $\mathcal{J}_{\lambda,p}(u)\geq\beta$;

(ii) there exists a positive function $e\in\mathcal{X}^{1,2}(\Omega)$ such that

$$\rho_s(e)>\alpha \text{ and } \mathcal{J}_{\lambda,p}(e)<\beta.$$

Moreover, for every positive function $u\in\mathcal{X}^{1,2}(\Omega)$, it holds that

$$\lim_{t\to 0^+}\mathcal{J}_{\lambda,p}(tu)=0.$$

(III) **($(\mathrm{PS})_c$ condition for $\mathcal{J}_{\lambda,p}$)** The functional $\mathcal{J}_{\lambda,p}$ satisfies the *Palais-Smale compacteness condition* at every level $c<c^*$, where

$$c_* := \frac{1}{n}(\mathcal{S}_n)^{n/2}.$$

(IV) **(Existence of a path with energy below c_*)** Consider the maps

$$v_t(x) = \frac{\phi(x)}{(t^2+|x|^2)^{\frac{n-2}{2}}} \qquad \text{and} \qquad u_t := v_t/\|v_t\|_{L^{2^*}(\mathbb{R}^n)}$$

(where $\phi\in C_0^\infty(\Omega)$ is a suitable cut-off function). Then,

(a) if $\kappa_{s,n}>\beta_{p,n}$, there exists $t>0$ such that

$$\sup_{\alpha\geq 0}\mathcal{J}_{\lambda,p}(\alpha u_t)<c^* \qquad \forall\ \lambda>0;$$

(b) if $\kappa_{s,n}\leq\beta_{p,n}$, there exist $t>0$ and $\lambda_0\gg 1$ such that

$$\sup_{\alpha\geq 0}\mathcal{J}_{\lambda,p}(\alpha u_t)<c^* \qquad \forall\ \lambda\geq\lambda_0.$$

Even if we refer to [7] for the detailed proofs of the above assertions (I)-to-(IV), here we highlight that the proof of assertion (IV) follows essentially from the *same estimate* we tried to use (without success) in the linear case: in fact, we have

$$\begin{aligned}
\mathcal{J}_{\lambda,p}(\alpha u_t) &\leq \frac{\alpha^2}{2}\big(\mathcal{S}_n + O(t^{n-2}) + O(t^{2-2s})\big) - \frac{\alpha^{2^*}}{2^*} - C\lambda t^{\beta_{p,n}}\alpha^{p+1} \\
&\leq \underbrace{\frac{\alpha^2}{2}\big(\mathcal{S}_n + Ct^{\kappa_{s,n}}\big) - \frac{\alpha^{2^*}}{2^*} - C\lambda t^{\beta_{p,n}}\alpha^{p+1}}_{=g(\alpha)},
\end{aligned}$$

and a direct computation shows that there exists $\bar{\alpha} = \bar{\alpha}_{t,\lambda} > 0$ such that

$$\begin{aligned}
\sup_{\alpha\geq 0} g(\alpha) = g(\bar{\alpha}) &= \frac{\bar{\alpha}^2}{2}\big(\mathcal{S}_n + Ct^{\kappa_{s,n}}\big) - \frac{\bar{\alpha}^{2^*}}{2^*} - C\lambda t^{\beta_{p,n}}\bar{\alpha}^{p+1} \\
&\leq \underbrace{\frac{1}{n}(\mathcal{S}_n)^{n/2}}_{=c_*} + Ct^{\kappa_{s,n}} - C\lambda t^{\beta_{p,n}}.
\end{aligned}$$

We now distinguish two cases.

(i) If $\kappa_{s,n} > \beta_{p,n}$, we readily see that

$$\sup_{\alpha\geq 0} g(\alpha) \leq c_* + Ct^{\kappa_{s,n}} - C\lambda t^{\beta_{p,n}} < c_*$$

provided that t is small enough *and for every* $\lambda > 0$.

(ii) If, instead, $\kappa_{s,n} \leq \beta_{p,n}$, we have

$$\begin{aligned}
\sup_{\alpha\geq 0} g(\alpha) &= \frac{\bar{\alpha}^2}{2}\big(\mathcal{S}_n + Ct^{\kappa_{s,n}}\big) - \frac{\bar{\alpha}^{2^*}}{2^*} - C\lambda t^{\beta_{p,n}}\bar{\alpha}^{p+1} \\
&\leq \frac{\bar{\alpha}^2}{2}\big(\mathcal{S}_n + Ct^{\kappa_{s,n}}\big) - \frac{\bar{\alpha}^{2^*}}{2^*}.
\end{aligned}$$

From this, since it easy to see that $\bar{\alpha}_{t,\lambda} \to 0$ as $\lambda \to \infty$, we get

$$0 \leq \sup_{\alpha\geq 0} \mathcal{J}_{\lambda,p}(\alpha u_t) \leq \frac{\bar{\alpha}^2}{2}\big(\mathcal{S}_n + Ct^{\kappa_{s,n}}\big) - \frac{\bar{\alpha}^{2^*}}{2^*} \to 0 \quad \text{as } \lambda \to \infty,$$

and this readily implies that

$$\sup_{\alpha\geq 0} \mathcal{J}_{\lambda,p}(\alpha u_t) < c_*,$$

provided that $t \ll 1$ *and* λ *is sufficiently large.*

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