RECURRENCE OF THE RANDOM PROCESS GOVERNED WITH THE FRACTIONAL LAPLACIAN AND THE CAPUTO TIME DERIVATIVE
RICORRENZA DI PROCESSI ALEATORI CON IL LAPLACIANO FRAZIONARIO E DERIVATA TEMPORALE DI CAPUTO

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Abstract. We are addressing a parabolic equation with fractional derivatives in time and space that governs the scaling limit of continuous-time random walks with anomalous diffusion. For these equations, the fundamental solution represents the probability density of finding a particle released at the origin at time 0 at a given position and time. Using some estimates of the asymptotic behaviour of the fundamental solution, we evaluate the probability of the process returning infinite times to the origin in a heuristic way. Our calculations suggest that the process is always recurrent.

Sunto. Ci occupiamo di un'equazione parabolica con derivate frazionarie in tempo e in spazio che governa il limite scalato di passeggiate aleatorie a tempo continuo con diffusione anomala. Per queste equazioni, la soluzione fondamentale rappresenta la probabilità di trovare una particella liberata all'origine al tempo 0 in una data posizione a un certo tempo. Utilizzando alcune stime sul comportamento asintotico della soluzione fondamentale, calcoliamo la probabilità del processo di ritornare infinite volte nell'origine. Il nostro metodo suggerisce che il processo sia sempre ricorrente.

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1. Introduction

Let us take into account the fractional Laplacian of order s with $0 < s < 2$ in $\mathbb{R}^n$ defined by

$$(-\Delta)^{s/2}u(x) := P.V. \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+s}}\, dy$$

where $P.V.$ stands for principal value. For the time derivative, we consider the Caputo time derivative, defined as

$$\partial_t^\alpha u(t) := \begin{cases} \frac{1}{\Gamma(1-\alpha)} \int_0^t u'(\tau)(t-\tau)^{-\alpha}\, d\tau & \text{for } 0 < \alpha < 1 \\ u'(t) & \text{for } \alpha = 1, \\ \frac{1}{\Gamma(2-\alpha)} \int_0^t u''(\tau)(t-\tau)^{1-\alpha}\, d\tau & \text{for } 1 < \alpha < 2. \end{cases} \tag{1}$$

Our aim is to discuss some properties of the random process governed by the equation

$$\partial_t^\alpha u(t, x) + (-\Delta)^{s/2}u(t, x) = 0 \qquad \text{in } \mathbb{R}^+ \times \mathbb{R}^n. \tag{2}$$

Equation (2) is linked to a fractional diffusion process if and only if the associated fundamental solution is non-negative and can be interpreted as a spatial probability density function evolving in time. These conditions are satisfied for an arbitrary dimension if $0 < \alpha \leq 1$ and $0 < s < 2$ and additionally for $1 < \alpha < 2$ and $\alpha \leq s < 2$ in the one-dimensional case, see [8] and the reference therein. In particular, the properties of positivity and integrability of the fundamental solutions were derived, providing scaling invariants and the production of entropy from these processes [8, 13]. Notice also that, for all the other ranges of parameters, the fundamental solution changes sign since the behaviour of the equation is closer to a wave equation rather than a heat equation. The fact that Caputo time derivatives for $1 < \alpha < 2$ interpolates between parabolic and hyperbolic equations was also pointed out in [4].

In the spirit of the work [1], we use a simple PDE approach to study the recurrence and transiency property of the random process governed by equation (2); namely, we analyse the behaviour in time of the walker's probability density at the origin, which corresponds to the starting site of the random walk. We say that the random process is recurrent if it visits its starting position infinitely often with probability one and transient otherwise.

The paper is organised as follows. In Section 2, we recall the notation. In Section 3, we recall the principal properties of the probability distribution of the random process related to equation (2). We also recall the decay in time of the classical solutions to (2) in Section 4, in order to give some insight on the structure of the equation and the behaviour of solutions. In Section 5, we present for the first time some calculations indicating that this type of fractional random processes are recurrent. However, due to the subtleties in this framework and the heuristic method, we cannot state that the process is recurrent. Moreover, the interpretation of this result might be different from the one of classic random walks and we discuss its meaning in the Conclusions.

2. Basic concepts

We recall that equation (2) is derived as the scaling limit of a a continuous time random walk (CTRW) where both the random variables describing the length of the jumps and the waiting time between two consecutive jumps have an infinite expected value (see for example Section 4 in [15] or [12]).

In fact, by calling $\{Y_k\}_{k\in\mathbb{N}}$ the independent identically distributed (iid) variables giving the length of the jumps of a selected particle and $\{J_k\}_{k\in\mathbb{N}}$ the iid variables giving the time elapsing between two jumps, the position of a particle after n jumps is

$$S(n) := Y_1 + \cdots + Y_n$$

at the time

$$T(n) := J_1 + \cdots + J_n.$$

The process

$$X(t) = S(N(t)) = \sum_{n=1}^{N(t)} Y_n$$

with

$$N(t) = \max\{n : T(n) \leq t\}$$

is called a continuous time random walk (CTRW).

For classic random walks, one uses the Central Limit Theorem and the Renewal Theorem to show the convergence to the Brownian motion. For the convergence to the process related to the fractional diffusion, we refer to [15, Chapter 4]. In particular, in Section 4.4 the authors derive the limit process, and later in Section 4.5, it is shown that the governing equation is (2) in 1D. Thus, equation (2) is derived in a rigourous way and it is more than a simple generalisation of the governing equation of the Browian motion.

In the literature, some authors refer to the process governed by (2) as a random walk, even if it is the scaling limit of a CTRW. However, we highlight that the limit process can be very different than the CTRW itself. For example, if we start in 1D roughly with jump lengths of finite variance and waiting times with finite expectations, then CTRW in $(t, x)-$space describing the position x at time t is a step function, whereas its scaling limit is the Brownian motion having almost surely continuous but nowhere differential sample paths. This is why we refer to a random process rather than a random walk.

Now, we recall the definition of fundamental solution. Here, $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space and $\mathcal{S}'(\mathbb{R}^n)$ is its dual, which corresponds to the space of tempered distributions. Then, we have the following definition for the fundamental solution.

Definition 2.1. *The function* $\phi : \mathbb{R}^+ \times \mathbb{R}^n \to \mathbb{R}$ *is called a fundamental solution of* (2) *if* $\phi(t, \cdot)$ *solves* (2) *in the sense* $\mathcal{S}'(\mathbb{R}^n)$ *for all* $t > 0$ *and*

$$\lim_{t \to 0^+} \phi(t, x) = \delta_0(x) \qquad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

together with

$$\lim_{t \to 0^+} \partial_t \phi(t, x) = 0 \qquad \text{in } \mathcal{S}'(\mathbb{R}^n)$$

if $1 < \alpha < 2$.

The solution used here is quite weak, but this is necessary to allow the existence of a solution (see [8]). In fact, for some range of parameters, the fundamental solutions are singular at $x = 0$ not only at $t = 0$ but also for larger times. However, they are always integrable, as recalled in the forthcoming Lemma 3.1.

Our method relies on the properties of the fundamental solution $G(t, x)$ of (2), which in general cannot be expressed in terms of elementary functions. However, the asymptotics is known, which is enough for our purposes together with the positivity of the fundamental solution guaranteeing the probabilistic interpretation.

3. Positivity and asymptotics of the fundamental solution

It is possible to compute $G(t,x)$ by using the Fourier transform on (2), obtaining

$$\partial_t^\alpha \widehat{u}(t,\cdot) + |\cdot|^s \widehat{u}(t,\cdot) = 0 \quad \text{in } \mathcal{S}'(\mathbb{R}^n) \tag{3}$$

for all $t > 0$ [16]. It is known that the Mittag-Leffler function $E_\alpha(-|\cdot|^\beta t^\alpha)$ defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\alpha k)}, \quad z \in \mathbb{C},$$

is a solution of (3) [7, Chapter 4]. Then, it remains to find a function satisfying the initial condition(s) and whose Fourier transform is the Mittag-Leffler function. There are several ways in the literature to write the formula for the fundamental solution of (2) given in terms of the Fox H-function, see e.g. [6, 8, 9, 10]. In [8], the formulation is

$$G(t,x) = \pi^{-n/2}|x|^{-n} H_{23}^{21}\left(2^{-s}t^{-\alpha}|x|^s \,\Big|\, {(1,1),\quad (1,\alpha) \atop (n/2,s/2),\ (1,1),\ (1,s/2)}\right),$$

where H_{23}^{21} is a Fox H-function. See Appendix A.2 in [9] for more details on the Fox functions.

A first property of G we need is that

$$\int_{\mathbb{R}^n} G(t,x)dx = (2\pi)^{n/2}\widehat{G}(1,0) = E_\alpha(0) = 1,$$

as one would expect from a probability density.

The second property we wish for in a probability density function is positivity. However, this property is not always valid. Theorem 1 of [8] summarizes the cases of positivity for the fundamental solutions, which we have also already recalled in the introduction.

Theorem 3.1. *The fundamental solution $G(t,x)$ of the problem* (2)

(a) *is positive, if either $\alpha \in (0,1], s \in (0,2]$ and $n \geq 1$, or $\alpha \in (1,2)$, $s \in [\alpha,2]$ and $n = 1$;*

(b) *changes sign in the following cases of the parameters:*

  (i) $n \geq 2$, $\alpha \in (1,2)$ *and* $s \in (0,2]$*;*

  (ii) $n = 1$, $\alpha \in (1,2)$ *and* $s < \alpha$.

We also prove here a technical lemma that we need in the next section. This lemma is based on the asymptotic estimates for the fundamental solution given e.g. in [8, Lemma 1] (the same estimates can be found from some other articles, too). For completeness, we give all the cases for $\alpha \in (0,2)$ and $s \in (0,2]$; however, later we will apply the result only for the ranges of parameters for which the fundamental solution is positive.

Lemma 3.1. *Let us take $\rho \in (0,1)$ and $t \geq 1$. Then, the following holds for some positive constants c_1, c_2 independent of t and x:*

(1) *if $s > n$ and $0 < \alpha < 2$ or if $\alpha = 1$, then*
$$\int_{B_\rho} G(t,x)dx \in \left[c_1\, \rho^n t^{-\alpha n/s}, c_2\, \rho^n t^{-\alpha n/s}\right].$$
(2) *if $s = n$ and $0 < \alpha < 2$ and $\alpha \neq 1$, then*
$$\int_{B_\rho} G(t,x)dx \in \left[c_1\, \left(\alpha\rho^n \log(t) + \zeta(\rho)\right) t^{-\alpha n/s}, c_2\, \left(\rho^n \alpha \log(t) + \zeta(\rho)\right) t^{-\alpha n/s}\right],$$
where
$$\zeta(\rho) = \rho^n(2 - \log(\rho^n)).$$
(3) *if $s < n$ and $0 < \alpha < 2$ and $\alpha \neq 1$, then*
$$\int_{B_\rho} G(t,x)dx \in \left[c_1\, \rho^s t^{-\alpha}, c_2\, \rho^s t^{-\alpha}\right].$$

Proof. Notice that for $x \in B_\rho$ with $\rho \in (0,1)$ and $t \geq 1$ we have that $R = |x|^s t^{-\alpha} < 1$. Thus, we can apply the asymptotic estimates of point (i) in Lemma 1 of [8]. We distinguish the three different cases.

1. By the estimate for $s > n$ and $0 < \alpha < 2$ or if $\alpha = 1$, we get that for some positive constants C_1 and C_2 we have
$$G(t,x) \in [C_1 t^{-\alpha n/s}, C_2 t^{-\alpha n/s}].$$
Then, by integration we get
$$\int_{B_\rho} G(t,x)dx \in [c_1 \rho^n t^{-\alpha n/s}, c_2 \rho^n t^{-\alpha n/s}]$$
with $c_1 = w_n C_1$ and $c_2 = w_n C_2$ where w_n is the measure of the unitary ball in dimension n.

2. For $s = n$ and $\alpha \in (0,2)$ with $\alpha \neq 1$, we have that

$$G(t,x) \in [C_1 t^{-\alpha}(|\log(|x|^n t^{-\alpha})| + 1), C_2 t^{-\alpha}(|\log(|x|^n t^{-\alpha})| + 1)] \tag{4}$$

for some positive C_1, C_2. Observe that, for $|x| < 1$ and $t > 1$

$$\begin{aligned}\int_{B_\rho} |\log(|x|^n t^{-\alpha})| dx &= \int_{B_\rho} |\log(t^{-\alpha}) + \log(|x|^n)| dx \\ &= -w_n \rho^n \log(t^{-\alpha}) - \int_{B_\rho} \log(|x|^n) dx\end{aligned}$$

where w_n is the measure of the unitary ball in dimension n. Now, by passing to spherical coordinates and then changing variable, we get

$$\begin{aligned}\int_{B_\rho} |\log(|x|^n t^{-\alpha})| dx &= -w_n \rho^n \log(t^{-\alpha}) - n w_n \int_0^\rho r^{n-1} \log(r^n) dr \\ &= w_n \alpha \rho^n \log(t) - w_n \int_0^{\rho^n} \log(\sigma) d\sigma \\ &= w_n \alpha \rho^n \log(t) - w_n(\rho^n \log(\rho^n) - \rho^n).\end{aligned}$$

It follows that

$$\int_{B_\rho} t^{-\alpha}(|\log(|x|^n t^{-\alpha})| + 1) = w_n t^{-\alpha}(\alpha \rho^n \log(t) - \rho^n \log(\rho^n) + \rho^n + \rho^n) \tag{5}$$

So, by choosing $c_1 = w_n C_1$, $c_2 = w_n C_2$, from (4) and (5) we get the expression in point (2).

3. The estimates for $s < n$ and $0 < \alpha < 2$ give

$$\int_{B_\rho} G(t,x) dx \in \left[C_1 t^{-\alpha} \int_{B_\rho} |x|^{-n+s} dx, C_2 t^{-\alpha} \int_{B_\rho} |x|^{-n+s} dx \right] = \left[c_1 t^{-\alpha} \rho^s, c_2 t^{-\alpha} \rho^s \right],$$

where $c_1 = w_n C_1$ and $c_2 = w_n C_2$. □

4. Decay in time of the norm of the solutions to classic and fractional heat equations

For completeness, we recall here some ideas and results on the decay of the solutions of equation (2) in bounded domains that were studied in [2]. Even though these results do not apply directly to our case, we think that they connect and somehow justify the recurrence properties that we find in Section 5.

We consider now a bounded domain $\Omega \subset \mathbb{R}^n$, $p \in [1, +\infty)$, and $u_0 \in L^p(\Omega)$ a nonnegative, not identically 0 initial datum. We focus on the solutions of the following Dirichlet problem:

$$\begin{cases} (\lambda_1 \partial_t^\alpha + \lambda_2 \partial_t) u + (-\Delta)^{s/2} u = 0, & \text{for all } x \in \Omega,\ t > 0, \\ u(x,t) = 0, & \text{for all } x \in \mathbb{R}^n \setminus \Omega,\ t > 0, \\ u(x,0) = u_0(x), & \text{for all } x \in \mathbb{R}^n, \end{cases} \tag{6}$$

for either $\lambda_1 = 1$ and $\lambda_2 = 0$ (which corresponds to the case of the Caputo time derivative) or $\lambda_1 = 0$ and $\lambda_2 = 1$ (which corresponds to the classical time derivative).

We state the following result, which applies the estimates of Theorem 1.1 in [5] and of and Theorem 1.2 in [2] to the fractional Laplacian; the hypothesis for the application of the estimates can be found in Theorem 1.6 in [5]. These decays were also mentioned in the table in [1].

Theorem 4.1. *Suppose u is a smooth solution of (6) with $u_0 \in L^p(\Omega)$ for some $p \in (1, +\infty)$. Then there exists some constants C_1, C_2 depending on Ω, s, $||u_0||_{L^p(\Omega)}$, p and α such that*

(1) *if $\lambda_1 = 0$ and $\lambda_2 = 1$, we have that*

$$||u(\cdot,t)||_{L^p(\Omega)} \leq C_2 e^{-\frac{t}{C_1}} \qquad \textit{for all } t > 0.$$

(2) *if $\lambda_1 = 1$ and $\lambda_2 = 0$, we have that*

$$||u(\cdot,t)||_{L^p(\Omega)} \leq \frac{C_2}{1 + t^\alpha} \qquad \textit{for all } t > 0.$$

This result states that the norm of the solution of the Dirichlet problem in a bounded domain has exponential decay in time if the time derivative is classical, despite the presence of fractional diffusion in space, while for the Caputo time derivative the decay is only of power-law type.

This behaviour is understood by the following heuristics. We look at the solutions of the equation

$$\partial_t v(t) = -v(t), \quad v(0) = 1,$$

which is clearly the exponential $v(t) = e^{-t}$, and of the equation

$$\partial_t^\alpha v_\alpha(t) = -v_\alpha(t), \quad v_\alpha(0) = 1,$$

which is explicit in the terms of the Mittag-Leffler function (see [14]) and has asymptotic decay

$$v_\alpha(t) \sim \frac{1}{t^\alpha}.$$

As mentioned in [5], for a suitable radius $R > 0$ the ball B_R has the first eigenvalue equal to 1, so that the eigenvalue problem

$$\begin{cases} (-\Delta)^{s/2}\phi = \phi & \text{in } B_R, \\ \phi = 0 & \text{in } \mathbb{R}^n \setminus B_R, \\ ||\phi||_{L^\infty(B_R)} = 1. \end{cases}$$

has a positive solution ϕ.

Then, $u(x,t) = v(t)\phi(x)$ is a solution of (6) for $u_0 = \phi$ and $\lambda_1 = 0$ (that is, for classical time derivative). Moreover, $u(x,t)$ has an exponential decay in time. On the other side, we notice that $u(x,t) = v_\alpha(t)\phi(x)$ is a solution of (6) for $u_0 = \phi$ and $\lambda_2 = 0$ (that is, for Caputo time derivative) and has a polynomial decay.

5. Recurrence of the random process associated to (2)

In this section, we study the recurrence of the random process associated to (2), under some hypothesis on the correlation structure of the process. The method used is similar to the one in [1]. Some changes are due to the fact that the fundamental solution of (2) is singular at $x = 0$ when $n > 1$, thus it is not bounded, as it was in the case of [1]. Thus, we have to use the estimates of Lemma 3.1.

Let us fix $\rho > 0$ and $M > 0$ and take the sequence $t_k = Mk$ for $k \in \mathbb{N}$. Let us call B_ρ the ball of radius ρ centred at the origin and consider the quantity

$$q_k(\rho) := \int_{\mathbb{R}^n \setminus B_\rho} G(t_k, x)dx.$$

So, by the probabilistic interpretation, this is the probability that a particle released at the origin at time $t = 0$ is found outside B_ρ at time t_k.

By the positivity of G and (3), we have that

$$0 \le q_k(\rho) \le 1.$$

Now, let us define

$$q(\rho) := \prod_{k=1}^{+\infty} q_k(\rho) \in [0,1].$$

If the events

$$\{A_k := \{\text{“The particle is outside } B_\rho \text{ at time } t_k\text{”}\}\}_{k\in\mathbb{N}}$$

are independent, then $q(\rho)$ is the probability that the particle is outside B_ρ for the sequence of times $\{t_k\}_{k\in\mathbb{N}}$; so, $q(\rho)$ gives us some indication on the recurrence and transiency of the process.

Unfortunately, the correlation structure of the events A_k for the process governed by (2) is not known and may be hard to obtain due to the quite complicated structure of the stochastic process corresponding to (2). For the case of the classic Brownian motion, the events A_k are dependent, but their dependence becomes weaker and weaker for large gaps between the instants t_k. On the other hand, the peculiarity of the Caputo derivative (defined in (1)) is that it depends on the function at all times in the past. However, the dependency becomes less strong as events are more distant in time, that is, for large M. Thus, in our analysis, we can expect the error between the probability $P(\bigcap_{k\in\mathbb{N}} A_k)$ and $q(\rho)$ to become smaller and smaller as $M \to \infty$.

Notice also that we choose to sample the process with a linear sequence of times t_k. This choice in [1] leads to the correct prediction in the cases where the true behaviour is known.

Now we set

$$p_k(\rho) = 1 - q_k(\rho) = \int_{B_\rho} G(t_k,\rho)dx.$$

Recall that by Lemma 3.1, we get a result of the type

$$\int_{B_\rho} G(t_k,\rho)dx \in [c_1 f(t_k,\rho), c_2 f(t_k,\rho)]$$

for some constants positive c_1, c_2 and some function f that depends on the parameters α, s, and n. Then we have that

$$\begin{aligned}\log(q(\rho)) &= \log\left(\prod_{k=1}^{+\infty} q_k(\rho)\right)\\ &= \sum_{k=1}^{+\infty}\log(q_k(\rho))\\ &\in \left[\sum_{k=1}^{+\infty}\log(1-c_2 f(t_k,\rho)), \sum_{k=1}^{+\infty}\log(1-c_1 f(t_k,\rho))\right].\end{aligned}$$

We distinguish three different cases according to the ones in Lemma 3.1 to study the convergence of the series

$$\sum_{k=1}^{+\infty}\log(1-cf(t_k,\rho)). \tag{7}$$

We exclude the case $\alpha = 1$ since it was already considered in [1].

If $s > n$, we recall that the fundamental solution is positive for all $n \geq 1$ and $\alpha \in (0,1)$, and moreover for $1 < \alpha \leq s$ for $n = 1$. Moreover, the first case of Lemma 3.1 applies. Then, recalling the Taylor expansion, the convergence of the series in (7) can be reduced to the one of the series

$$-c\sum_{k=1}^{+\infty}\frac{\rho^n}{(Mk)^{\alpha n/s}} = -c\frac{\rho^n}{M^{\alpha n/s}}\sum_{k=1}^{+\infty}\frac{1}{k^{\alpha n/s}} = \begin{cases} -\bar{c}\rho^n & \text{if } n > s/\alpha,\\ -\infty & \text{if } n \leq s/\alpha.\end{cases}$$

However, notice that for $\alpha \in (0,1)$ we have $n < s < s/\alpha$, while for $n = 1$ and $\alpha \in (1,2)$, the condition on s, $s \geq \alpha$ gives that $s/\alpha \geq 1 = n$. Thus, if $s > n$, we only have the case $n \leq s/\alpha$ and since $\log(q(\rho)) = -\infty$,

$$q(\rho) = 0 \qquad \text{if } s > n. \tag{8}$$

If instead $s = n$ and $\alpha \neq 1$, the fundamental solution is positive for $\alpha \in (0,1)$ for every $n \geq 1$. Notice that for $n = s = 1$, the condition on α is $\alpha \leq s = 1$, so no value of $(1,2)$ is acceptable. Then, by the second estimate in Lemma 3.1, in this case the convergence of the series in (7) is equivalent to the one of

$$-c\alpha\rho^n\frac{\log(M)}{M^\alpha}\sum_{k=1}^{+\infty}\frac{\log(k)}{k^\alpha} = -\infty$$

where the last equality holds because $\alpha \in (0,1)$. So, we have that

$$q(\rho) = 0 \qquad \text{if } s = n. \tag{9}$$

The last case is $s < n$. Here, the fundamental solution is positive for $\alpha \in (0,1)$ for every $n \geq 1$, while for $n = 1$ we have that the condition $\alpha \leq s < n = 1$ excludes any value of α in $(1,2)$. By applying the estimate in the third point of Lemma 3.1, we get that the convergence of the series in (7) is equivalent to

$$-\frac{c\rho^s}{M^\alpha} \sum_{k=1}^{+\infty} \frac{1}{k^\alpha} = -\infty$$

where again we have divergence since $\alpha \in (0,1)$. Thus,

$$q(\rho) = 0 \qquad \text{if } s < n. \tag{10}$$

Hence, by (8), (9), and (10), we have $q(\rho) = 0$ for all ranges of parameters for which the fundamental solution is positive. Under the simplifications discussed above, for any $\rho > 0$, we estimate the probability that a particle released at the origin at $t = 0$ lies outside B_ρ for all the times $t_k = Mk$, $k \in \mathbb{N}$, is 0. So, the method indicated that the random process related to (2) is always recurrent.

6. Conclusions

We analysed the recurrence of the random process governed by the equation (2), restricting our analysis to the ranges of parameters for which the fundamental solution to (2) has a probabilistic meaning, which are $\alpha \in (0,1]$ for all $n \in \mathbb{N}$ and additionally for $\alpha \in (1,2)$ under the condition $\alpha \leq s \leq 2$ for $n = 1$. Moreover, we supposed that the events $\{A_k := \{\text{“The particle is outside } B_\rho \text{ at time } t_k\text{”}\}\}_{k\in\mathbb{N}}$ are weakly correlated when the interval between two instants t_k's is large enough, which is very reasonable but not proven yet.

The case of $\alpha = 1$ and $s \in (0,2)$ was already studied in [1], where it was found that the random process is recurrent if $n \leq s$ and transient if $n > s$.

Surprisingly, when $\alpha \neq 1$, our calculation indicates that the random process is always recurrent. This might depend on several causes.

When the fundamental solution is not singular, that is for $0 < \alpha < 1$ and $s > n$, we automatically have that the dimension n must be 1 or 2. In this case, even for the Brownian motion, the random process is recurrent. Since the diffusion governed by equation (2) is expected to be slower, we also expect recurrence here.

The fundamental solution $G(t, x)$ is singular at the origin for all the other cases: this concentration at the origin causes the diffusion to be very slow. The diffusion is slower than the one produced by a process governed by a classical time derivative (i.e., a process with finite expected waiting time between two jumps), as pointed out in [5] and in [2] and recalled in Section 4. In the presence of the Caputo time derivative, the decay of the Lebesgue norm of the smooth solutions of the Dirichlet problem associated to (2) on a bounded domain follows a power law, while for the classic time derivative (that corresponds to $\alpha = 1$) the decay in time of the norm is exponential. Our computations show once again that the presence of the Caputo derivative changes dramatically the behaviour of the solutions. We also point out that the probability $p_k(\rho)$ of finding the particle in the ball B_ρ centred at the origin and of radius ρ at time t_k decays in time, as prescribed by Lemma 3.1. This means that the process does leave the neighbourhood of the origin and comes back infinitely often. However, we are not sure if this recurrence could be intended in the sense that the process comes back infinitely often to a generic set not containing the origin, since our calculation depends on the behaviour of the fundamental solution around the starting site of the random walk.

The findings of the paper may be viewed as suggestions or preliminary results. We believe that it is difficult to obtain such a result for the corresponding stochastic process, which in the fractional does not share the basic properties of the Brownian motion, which are needed in the proof of recurrence.

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