THE SUB-FINSLER BERNSTEIN PROBLEM IN $\mathbb{H}^1$
IL PROBLEMA DI BERNSTEIN SUBFINSLERIANO IN $\mathbb{H}^1$

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Abstract. This is a note based on the paper [32] written in collaboration with M. Ritoré. The purpose of this note is to present and discuss the Bernstein type problems in the sub-Finsler Heisenberg group $\mathbb{H}^1$. We give a general idea of the state of the art and we prove that a complete, stable, (X, Y)-Lipschitz surface is a vertical plane.

Sunto. Queste note sono basate sull'articolo [32] scritto in collaborazione con M. Ritoré. Lo scopo di queste note è quello di presentare e discutere alcuni problemi di tipo Bernstein nel gruppo di Heisenberg $\mathbb{H}^1$ subfinsleriano. Forniamo un'idea generale dello stato dell'arte e proviamo che una superficie (X, Y)-lipschitziana è un piano verticale.

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1. Introduction

In the 1915 N.S. Bernstein [4] established his celebrated theorem concerning entire minimal graphs

Theorem 1.1. *Assume $u \in C^2(\mathbb{R}^2)$ solves the minimal surface equation*

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \tag{1.1}$$

then u is linear.

The original proof provided by Bernstein was based on a topological argument, that is difficult to generalized to higher dimension. We have to wait the 1962 when Fleming in

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[23] showed by the monotonicity formula that a set that is minimum of the perimeter in $\mathbb{R}^n$ has to be an hypercone or an hyperplane. Therefore if we are able to show that all minimizing hypercones in $\mathbb{R}^n$ are planes we solve the Bernstein's problem. In the case $n = 2$ this approach provides a straightforward proof of Theorem 1.1. Indeed, since a minimal cone is foliated by straight lines one principal curvature is zero and also the other principal curvature vanishing since the the sum of the principal curvature vanishing by the minimal surface equation. Then we have that the cone is a plane. Following the approach proposed by Fleming, E. De Giorgi [17] in 1965 solved the Bernstein's problem for $n = 3$ and F. J. Almgren [1] in 1966 for $n = 4$. Later on J. Simons [46] showed that all stable minimal cones are planes for $n \leqslant 7$. Finally E. Bombieri, E. De Giorgi and E. Gusti [5] established a counterexample, nowadays known as the Simons hypercone, for the Bernstein's problem in dimension $n = 8$.

When we consider an oriented surface $S \subset \mathbb{R}^3$ we say that S solves the Finsler Bernstein's problem if S minimizes the Finsler area functional

$$A_\phi(S) = \int_S \phi(\nu) dS,$$

where ν unit normal, ϕ is 1-homogenous positive C^2 on $\mathbb{S}^2$ and dS the Riemannian area element. Then the Euler-Lagrange equation associated to A_ϕ is given by

$$\phi_{ij}(\nu) II_{ij} = 0, \tag{1.2}$$

for $i, j = 1, 2$ and where II is the second fundamental form and ϕ_{ij} for $i, j = 1, 2$ are the second partial derivatives of ϕ. If we assume that $\{\phi < 1\}$ is strictly convex body then the minimal surface equation (1.2) is an elliptic equation of the second order that coincides with (1.1) when ϕ is the Euclidean norm. Contrary to what happens for graphs, where a solution of (1.2) is automatically a minimizer for the area functional by the convexity of the area functional, for surfaces we need to assume an extra conditions in order to solve the Bernstein's problem. In 1959 R. Osserman [38] showed that a simply connected complete minimal surface in the Euclidean 3-space whose spherical image *omits a whole neighborhood of a point* must be a plane. In 1961 H.B. Jenkins [35] generalized the result by R. Osserman [38] to the Finsler setting. The extra condition for surface considered by M. do Carmo and C. K. Peng [18] is the *stability* condition, namely that the second variation of the area functional is positive. In [18] they proved that *stable* oriented complete minimal surfaces in $\mathbb{R}^3$ are planes. Their result was generalized by D. Fischer-Colbrie and R. Schoen in 1980 [22] to 3-manifolds of non-negative scalar curvature. Finally non-existence of non-orientable complete stable minimal surfaces in $\mathbb{R}^3$ has been proved by A. Ros [43].

Variational problems related to the sub-Riemannian perimeter introduced by Capogna, Danielli and Garofalo [8] (see also Garofalo and Nhieu [30] and Franchi, Serapioni and Serra Cassano [25]) have recently received great interest, specially in the Heisenberg groups $\mathbb{H}^n$. In particular, Bernstein type problems, either for stable intrinsic graphs or for stable surfaces without singular points, have received a special attention, see for instance [10, 42, 14, 3, 15, 34, 27, 44, 28, 41, 12, 11, 37, 7, 29]. The monograph [9] provides a quite complete survey of progress on the subject.

The nature of the sub-Riemannian Bernstein's problem in Heisenberg group even for graphs is completely different from the Euclidean one. The first Heisenberg group $\mathbb{H}^1$ is the simply connected Lie group $(\mathbb{R}^3, *)$ endowed with the non-Euclidean product $*$ defined by (2.1). In this setting we have to distinguish between t-graphs $t = u(x, y)$ and intrinsic graphs that, after a rotation about the vertical axis, are given by $y = u(x, t)$.

On one hand the area functional for t-graphs is convex as in the Euclidean setting. Therefore the critical points of the area, called H-minimal graphs, are automatically minimizers for the area functional. However, since t-graphs admit singular points where the horizontal gradient vanishing their classifications is not an easy task. Thanks to a deep study of the singular set for C^2 surface in $\mathbb{H}^1$, J.-H. Cheng, J.-F. Hwang, A. Malchiodi, and P. Yang provided in Theorem A in [10] a classification of the C^2 entire H-minimal t-graphs. Afterwards, M. Ritoré and C. Rosales showed in [42] that area-stationary entire t-graphs of class C^2 are congruent to the hyperbolic paraboloid $u(x, y) = xy$ or to Euclidean planes. If we consider the huge class of Euclidean Lipschitz t-graphs, the previous classification does not hold true since there are several examples of area-minimizing surfaces of low regularity, see [41]. The complete classification for stable C^2 surfaces was established by A. Hurtado, M. Ritoré and C. Rosales in [34] where they showed that a complete, orientable, connected, *stable* area-stationary surface is congruent to $u(x, y) = xy$ or Euclidean planes. As well as in the Euclidean setting the stability condition is crucial in order to avoid H-minimal surfaces such as helicoids and catenoids.

On the other hand, the situation for the intrinsic graphs is completely different since their associated area functional is not convex. Indeed D. Danielli, N. Garofalo, D.M. Nhieu in [14] discovered that the family of graphs $u_\alpha(x, t) = \frac{\alpha x t}{1+2\alpha x^2}$ for $\alpha > 0$ are area-stationary but *unstable*. In [36] R. Monti, F. Serra Cassano, D. Vittone established an example of area minimizing intrinsic graph of regularity $C^{0,1/2}(\mathbb{R}^2)$ that is an intrinsic cone, although in addition it is the t-graph of an entire $C^{1,1}$ function. Therefore the Euclidean threshold of dimension $n = 8$ fails in the sub-Riemannian setting. In [29] M. Galli and M. Ritoré proved that complete, oriented and stable area-stationary C^1 surface without singular points is a vertical plane. Finally, S. Nicolussi Golo and F. Serra Cassano [37] showed that an Euclidean Lipschitz stable area-stationary intrinsic graphs are vertical planes. Lately, R. Young [48] proved that a *ruled* area-minimizing entire intrinsic graph in $\mathbb{H}^1$ is a vertical plane by introducing a family of deformations of graphical strips based on variations of a vertical curve.

Recently, a left-invariant sub-Finsler perimeter has been considered on the Heisenberg groups, see [45, 40, 24, 31, 39]. A sub-Finsler structure is obtained from a left-invariant asymmetric norm $||\cdot||$ in the horizontal distribution of $\mathbb{H}^1$. Such a norm can be recovered from a convex set K contained in the horizontal plane at $0 \in \mathbb{H}^1$. Following the De Giorgi approach [16], in [40, 24] they define the K-perimeter testing with horizontal vector fields with K-norm small or less that one. When $K = D$, the closed unit disk centered at the origin of $\mathbb{R}^2$, the K-perimeter coincides with classical sub-Riemannian perimeter.

A quite natural question is whether Bernstein type results similar to the sub-Riemannian ones hold for the sub-Finsler perimeter. A positive answer to this problem was first given by [32] where the authors proved that, in the Heisenberg group $\mathbb{H}^1$ with a sub-Finsler structure, a complete, stable, (X, Y)-Lipschitz surface (that roughly speaking means "without singular points") is a vertical plane. This is a generalization of the sub-Riemannian result obtained by S. Nicolussi and F. Serra-Cassano in [37] and by M. Galli and M. Ritoré [29]. The aim

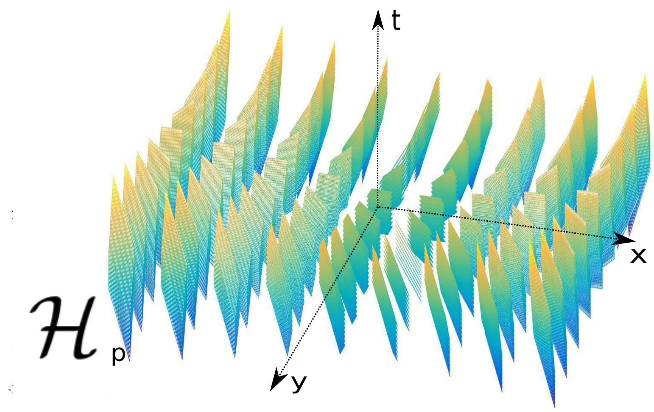

FIGURE 1. The contact distribution $\mathcal{H}$ in $\mathbb{H}^1$

of this paper is to provide an easy presentation of the proof of the result, see Theorem 6.3, obtain in [32].

2. Preliminaries

2.1. The Heisenberg group. We denote by $\mathbb{H}^1$ the first Heisenberg group, defined as the 3-dimensional Euclidean space $\mathbb{R}^3$ endowed with the product

$$(x, y, t) * (\bar{x}, \bar{y}, \bar{t}) = (x + \bar{x}, y + \bar{y}, t + \bar{t} + \bar{x}y - x\bar{y}). \tag{2.1}$$

A basis of left invariant vector fields is given by

$$X = \frac{\partial}{\partial x} + y\frac{\partial}{\partial t}, \qquad Y = \frac{\partial}{\partial y} - x\frac{\partial}{\partial t}, \qquad T = \frac{\partial}{\partial t}.$$

For $p \in \mathbb{H}^1$, the left translation by p is the diffeomorphism $L_p(q) = p * q$. The horizontal distribution $\mathcal{H}$ is the planar distribution generated by X and Y, which coincides with the kernel of the (contact) one-form $\omega = dt - ydx + xdy$.

We shall consider on $\mathbb{H}^1$ the auxiliary left-invariant Riemannian metric $g = \langle \cdot, \cdot \rangle$, so that $\{X, Y, T\}$ is an orthonormal basis at every point. Let D be the Levi-Civita connection associated to the Riemannian metric g. The following relations can be easily computed

$$\begin{aligned} &D_X X = 0, && D_Y Y = 0, && D_T T = 0 \\ &D_X Y = -T, && D_X T = Y, && D_Y T = -X \\ &D_Y X = T, && D_T X = Y, && D_T Y = -X. \end{aligned} \tag{2.2}$$

Setting $J(U) = D_U T$ for any vector field U in $\mathbb{H}^1$ we get $J(X) = Y$, $J(Y) = -X$ and $J(T) = 0$. Therefore $-J^2$ coincides with the identity when restricted to the horizontal distribution. The Riemannian volume of a set E is, up to a constant, the Haar measure of the group and is denoted by $|E|$. The integral of a function f with respect to the Riemannian measure is denoted by $\int f \, d\mathbb{H}^1$.

2.2. **The pseudo-hermitian connection.** The pseudo-hermitian connection ∇ is the only affine connection satisfying the following properties:

1. ∇ is a metric connection,
2. $\mathrm{Tor}(U, V) = 2\langle J(U), V\rangle T$ for all vector fields U, V.

The torsion tensor associated to ∇ is defined by

$$\mathrm{Tor}(U, V) = \nabla_U V - \nabla_V U - [U, V].$$

From this definition and Koszul formula, see formula (9) in the proof of Theorem 3.6 in [19], it follows easily that $\nabla X = \nabla Y = 0$ and $\nabla J = 0$. For a general discussion about the pseudo-hermitian connection see for instance [20, § 1.2]. Given a curve $\gamma : I \to \mathbb{H}^1$ we denote by ∇/ds the covariant derivative induced by the pseudo-hermitian connection along γ.

2.3. **Sub-Finsler norms.** Given a convex set $K \subset \mathbb{R}^2$ with $0 \in \mathrm{int}(K)$ its associated asymmetric Minkowski norm is given by

$$||u||_K = \inf\{\lambda \geqslant 0 \,:\, u \in \lambda K\}.$$

The dual norm is given by $||\cdot||_{K,*} = \sup_{||v||_K \leqslant 1}\langle u, v\rangle$. Then we define a left-invariant norm $||\cdot||_K$ on the horizontal distribution of $\mathbb{H}^1$ by means of the equality

$$(||fX + gY||_K)(p) = ||(f(p), g(p))||,$$

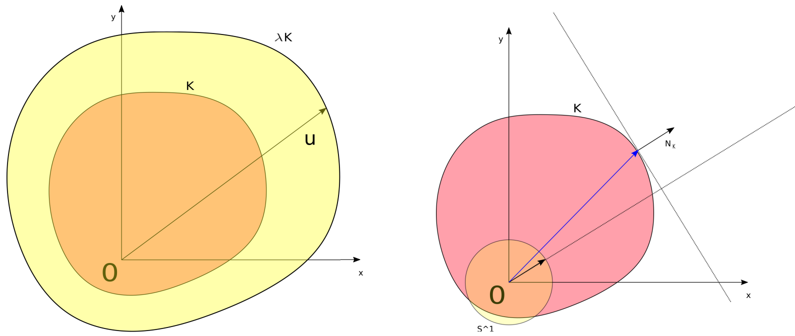

FIGURE 2. The Minkowski norm relative to K on the plane and the map π_K of a vector in $\mathbb{S}^1$.

for any $p \in \mathbb{H}^1$.

If the boundary of K is of class C^ℓ, $\ell \geqslant 2$, and the geodesic curvature of ∂K is strictly positive, we say that K is of class C^ℓ_+. When K is of class C^2_+, the outer Gauss map N_K is a diffeomorphism from ∂K to $\mathbb{S}^1$ and the map

$$\pi_K(fX + gY) = N_K^{-1}\left(\frac{(f,g)}{\sqrt{f^2+g^2}}\right),$$

defined for nowhere vanishing horizontal vector fields $U = fX + gY$, satisfies

$$||U||_{K,*} = \langle U, \pi_K(U)\rangle.$$

See § 2.3 in [40].

2.4. **Sub-Finsler perimeter.** Here we summarize some of the results contained in subsection 2.4 in [40].

Given a convex set $K \subset \mathbb{R}^2$ with $0 \in \text{int}(K)$, the norm $||\cdot||_K$ defines a perimeter functional: given a measurable set $E \subset \mathbb{H}^1$ and an open subset $\Omega \subset \mathbb{H}^1$, we say that E has locally finite K-perimeter in Ω if for any relatively compact open set $V \subset \Omega$ we have

$$|\partial E|_K(V) = \sup\left\{\int_E \text{div}(U)\, d\mathbb{H}^1 : U \in \mathcal{H}_0^1(V), ||U||_{K,\infty} \leqslant 1\right\} < +\infty,$$

where $\mathcal{H}_0^1(V)$ is the space of horizontal vector fields of class C^1 with compact support in V, and $||U||_{K,\infty} = \sup_{p\in V} ||U_p||_K$. The integral is computed with respect to the Riemannian measure $d\mathbb{H}^1$ of the left-invariant Riemannian metric g. When $K = D$, the closed unit disk centered at the origin of $\mathbb{R}^2$, the K-perimeter coincides with classical sub-Riemannian perimeter.

If K, K' are bounded convex bodies containing 0 in its interior, there exist constants $\alpha, \beta > 0$ such that

$$\alpha||x||_{K'} \leqslant ||x||_K \leqslant \beta||x||_{K'}, \quad \text{for all } x \in \mathbb{R}^2,$$

and it is not difficult to prove that

$$\tfrac{1}{\beta}|\partial E|_{K'}(V) \leqslant |\partial E_K|(V) \leqslant \tfrac{1}{\alpha}|\partial E|_{K'}(V).$$

As a consequence, E has locally finite K-perimeter if and only if it has locally finite K'-perimeter. In particular, any set with locally finite K-perimeter has locally finite sub-Riemannian perimeter.

Riesz Representation Theorem implies the existence of a $|\partial E|_K$-measurable vector field ν_K so that for any horizontal vector field U with compact support of class C^1 we have

$$\int_\Omega \operatorname{div}(U)\, d\mathbb{H}^1 = \int_\Omega \langle U, \nu_K\rangle\, d|\partial E|_K.$$

In addition, ν_K satisfies $|\partial E|_K$-a.e. the equality $||\nu_K||_{K,*} = 1$, where $||\cdot||_{K,*}$ is the dual norm of $||\cdot||_K$.

Given two convex sets $K, K' \subset \mathbb{R}^2$ containing 0 in their interiors, we have the following representation formula for the sub-Finsler perimeter measure $|\partial E|_K$ and the vector field ν_K

$$|\partial E|_K = ||\nu_{K'}||_{K,*}|\partial E|_{K'}, \quad \nu_K = \frac{\nu_{K'}}{||\nu_{K'}||_{K,*}}.$$

Indeed, for the closed unit disk $D \subset \mathbb{R}^2$ centered at 0 we know that in the Euclidean Lipschitz case $\nu_D = \nu_h$ and $|N_h| = ||N_h||_{D,*}$, where N is the *outer* unit normal and N_h is the projection of N onto the distribution $\mathcal{H}$. Hence we have

$$|\partial E|_K = ||\nu_h||_{K,*} d|\partial E|_D, \quad \nu_K = \frac{\nu_h}{||\nu_h||_{K,*}}.$$

Here $|\partial E|_D$ is the standard sub-Riemannian measure. Moreover, $\nu_h = N_h/|N_h|$ and $|N_h|^{-1}d|\partial E|_D = dS$, where dS is the standard Riemannian measure on S. Hence we get, for a set E with Euclidean Lipschitz boundary S

$$|\partial E|_K(\Omega) = \int_{S\cap\Omega} ||N_h||_{K,*}\, dS, \tag{2.3}$$

where dS is the Riemannian measure on S, obtained from the area formula using a local Lipschitz parameterization of S, see Proposition 2.14 in [25]. It coincides with the 2-dimensional Hausdorff measure associated to the Riemannian distance induced by g. We stress that here N is the *outer* unit normal. This choice is important because of the lack of symmetry of $||\cdot||_K$ and $||\cdot||_{K,*}$. Moreover when $S=\partial E\cap\Omega$ is a Euclidean Lipschitz surface the K-perimeter coincides with the area functional

$$A_K(S)=\int_S ||N_h||_{K,*}\,dS.$$

2.5. **Surfaces in $\mathbb{H}^1$.** Following [2, 25] we provide the following definition.

Definition 2.1 ($\mathbb{H}$-regular surfaces)**.** A real continuous function f defined on an open set $\Omega\subset\mathbb{H}^1$ is of class $C^1_{\mathbb{H}}(\Omega)$ if the distributional derivative $\nabla_{\mathbb{H}}f=(Xf,Yf)$ is represented by a continuous function.

We say that $S\subset\mathbb{H}^1$ is an $\mathbb{H}$-regular surface if for each $p\in\mathbb{H}^1$ there exist a neighborhood U and a function $f\in C^1_{\mathbb{H}}(U)$ such that $\nabla_{\mathbb{H}}f\neq 0$ and $S\cap U=\{f=0\}$. In other words, the horizontal unit normal

$$\nu_h=\frac{\nabla_{\mathbb{H}}f}{|\nabla_{\mathbb{H}}f|}$$

is a non-vanishing continuous function. Notice that $Z=-J(\nu_h)$ is tangent to S and is horizontal.

Following [47] we provide the following definition.

Definition 2.2. A set $S\subset\mathbb{H}^1$ is an (X,Y)-Lipschitz surface if for each $p\in S$ there exist a neighborhood $U_p\subset\mathbb{H}^1$, a Lipschitz function $f:U\to\mathbb{R}$ such that

$$S\cap U=\{f=0\}$$

and

$$Xf\geqslant l\quad\text{a.e. on } U\qquad\text{or}\qquad Yf\geqslant l\quad\text{a.e. on } U$$

for a suitable $l>0$.

Given a vertical plane $P\subset\mathbb{H}^1$, and a function u defined on a domain $D\subset P$, we denote by $\mathrm{Gr}(u)$ the *intrinsic graph* of u, defined as the Riemannian normal graph of the function

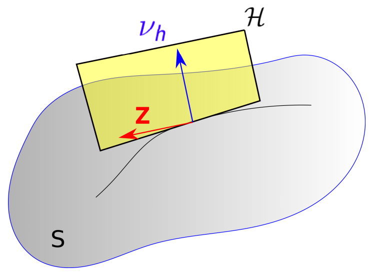

FIGURE 3. The horizontal foliation of a (X, Y)-Lipschitz surface S

u. Since the Riemannian unit normal to P is the restriction of a unitary left-invariant vector field X_P, the intrinsic graph of u is given by

$$\mathrm{Gr}(u) = \left\{ \exp_p \left(u(p)\, X_P(p)\right) : p \in D \right\}.$$

where exp is the exponential map on the Riemannian manifold $(\mathbb{H}^1, g)$. Clearly, when u is a Euclidean Lipschitz function we say that $\mathrm{Gr}(u)$ is an intrinsic graph of a Euclidean Lipschitz function. In the following we state a characterization theorem for (X, Y)-Lipschitz surfaces, for further details see [32, Theorem 2.3] or [47, Theorem 3.2] .

Theorem 2.3. *A set $S \subset \mathbb{H}^1$ is a (X, Y)-Lipschitz surface if and only if S is locally the intrinsic graph of a Euclidean Lipschitz function.*

3. Intrinsic graph of a Euclidean Lipschitz function on a vertical plane in $\mathbb{H}^1$

We denote by $\mathrm{Gr}(u)$ the *intrinsic* graph (Riemannian normal graph) of the Lipschitz function $u : D \to \mathbb{R}$, where D is a domain in a vertical plane P. Using Euclidean rotations about the vertical axis $x = y = 0$, that are isometries of the Riemannian metric g, we may assume that P is the plane $\{y = 0\}$, then $X_P = Y$. Since the vector field Y is a unit normal to this plane, the intrinsic graph $\mathrm{Gr}(u)$ is given by $\{\exp_p(u(p)Y_p) : p \in D\}$, where exp is the exponential map of g, and can be parameterized by the map

$$\Phi^u(x, t) = (x, u(x, t), t - xu(x, t)),$$

for $(x, 0, t) \in D$. Notice that $\Phi^u(x, t) = (x, 0, t) * (0, u(x, t), 0)$, where $*$ is the Heisenberg product defined in 2.1. For further details, we refer the reader to [26].

Given the intrinsic graph $\mathrm{Gr}(u)$ of a Euclidean Lipschitz function defined on some domain D of the vertical plane P, we know by Rademacher's Theorem that u is $\mathcal{H}^2$-a.e. differentiable on D, where $\mathcal{H}^2$ is the 2-dimensional Euclidean Hausdorff measure on D. Assuming $P = \{y = 0\}$, and given a differentiability point $(x_0, 0, t_0)$ of u, the tangent plane of $\mathrm{Gr}(u)$ is well defined at $\Phi^u(x_0, t_0)$ and so it is the normal vector field N. The tangent plane to any differentiability point of $S = \mathrm{Gr}(u)$ is generated by the vectors

$$\Phi^u_x = (1, u_x, -u - xu_x) = X + u_x Y - 2uT,$$

$$\Phi^u_t = (0, u_t, 1 - xu_t) = u_t Y + T$$

and the characteristic direction is given by $Z = \tilde{Z}/|\tilde{Z}|$ where

$$\tilde{Z} = X + (u_x + 2uu_t)Y. \tag{3.1}$$

A unit normal to S is given by $N = \tilde{N}/|\tilde{N}|$ where

$$\tilde{N} = \Phi^u_x \times \Phi^u_t = (u_x + 2uu_t)X - Y + u_t T \tag{3.2}$$

and $\mathrm{Jac}(\Phi^u) = |\Phi^u_x \times \Phi^u_t| = |\tilde{N}|$. Therefore the horizontal projection of the unit normal to S is given by $N_h = \tilde{N}_h/|\tilde{N}|$, where $\tilde{N}_h = (u_x + 2uu_t)X - Y$. Notice that N is never vertical. At differentiability points of $\mathrm{Gr}(u)$ we define

$$\nu_h = \frac{N_h}{|N_h|} = \frac{(u_x + 2uu_t)X - Y}{\sqrt{1 + (u_x + 2uu_t)^2}},$$

and the vector field Z by

$$Z = -J(\nu_h),$$

which is tangent to S and horizontal. An orthonormal basis at the tangent space of $\mathrm{Gr}(u)$ at the differentiable point is obtained by adding to Z the vector

$$E = \langle N, T\rangle \nu_h - |N_h| T. \tag{3.3}$$

Remark 3.1. Let $\gamma(s) = (x, t)(s)$ be a Lipschitz curve in D then

$$\Gamma(s) = (x, u(x, t), t - xu(x, t))(s) \subset \mathrm{Gr}(u)$$

is also Lipschitz and

$$\Gamma'(s) = x' X + (x' u_x + t' u_t) Y + (t' - 2ux') T$$

a.e. in s. In particular horizontal curves in $\mathrm{Gr}(u)$ satisfy the ordinary differential equation

$$t' = 2u(x,t)x'. \tag{3.4}$$

From (2.3), the sub-Finsler K-area for a Euclidean Lipschitz surface S is

$$A_K(S) = \int_S \|N_h\|_{K,*} dS,$$

where $\|N_h\|_{K,*} = \langle N_h, \pi(N_h)\rangle$ with $\pi = (\pi_1, \pi_2) = \pi_K$ and dS is the Riemannian area measure. Therefore when we consider the intrinsic graph $S = \mathrm{Gr}(u)$ we obtain

$$\begin{aligned} A(\mathrm{Gr}(u)) &= \int_D \langle \tilde{N}_h, \pi(\tilde{N}_h)\rangle \, dxdt \\ &= \int_D (u_x + 2uu_t)\pi_1(u_x + 2uu_t, -1) - \pi_2(u_x + 2uu_t, -1)\, dxdt. \end{aligned}$$

Observe that the K-perimeter of a set was defined in terms of the *outer* unit normal. Hence we are assuming that S is the boundary of the *epigraph* of u.

Given $v \in C_0^\infty(D)$, a straightforward computation shows that

$$\frac{d}{ds}\Big|_{s=0} A(\mathrm{Gr}(u+sv)) = \int_D (v_x + 2vu_t + 2uv_t) M dxdt, \tag{3.5}$$

where

$$M = F(u_x + 2uu_t), \tag{3.6}$$

and F is the function

$$F(x) = \pi_1(x,-1) + x\frac{\partial \pi_1}{\partial x}(x,-1) - \frac{\partial \pi_2}{\partial x}(x,-1). \tag{3.7}$$

Since $(u_x + 2uu_t)$ is continuous and π is at least C^1 the function M is continuous.

3.1. **The local bi-Lipschitz homeomorphism.** Let $\Gamma(s)$ be a characteristic curve passing through p in $\mathrm{Gr}(u)$. Let $\gamma(s)$ be the projection of $\Gamma(s)$ onto the xt-plane. By composition with a left-translation we may assume that $(0,0) \in D$ is the projection of p to the xt-plane. We parameterize γ by $s \to (s, t(s))$. By Remark 3.1 the curve $s \to (s, t(s))$ satisfies the ordinary differential equation $t' = 2u$. For ε small enough, Picard-Lindelöf's

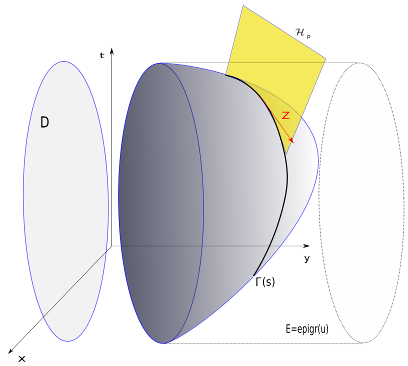

FIGURE 4. An intrinsic graph on the plane $\{y = 0\}$

theorem implies the existence of $r > 0$ and a solution $t_\varepsilon :]-r, r[\to \mathbb{R}$ of the Cauchy problem

$$\begin{cases} t'_\varepsilon(s) = 2u(s, t_\varepsilon(s)), \\ t_\varepsilon(0) = \varepsilon. \end{cases} \tag{3.8}$$

We define $\gamma_\varepsilon(s) = (s, t_\varepsilon(s))$ so that $\gamma_0 = \gamma$. By Lemma 2.6 in [32] we obtain that $G(\xi, \varepsilon) = (\xi, t_\varepsilon(\xi))$ is a biLipschitz homeomorphisms where the determinant of the Jacobian of G is given by $\partial t_\varepsilon(s)/\partial\varepsilon > C > 0$ for each $s \in]-r, r[$ and a.e. in ε .

Remark 3.2. Notice that

$$t''_\varepsilon(s) = 2(u_x + 2uu_t).$$

4. Foliation by straight lines of area-stationary surfaces

Definition 4.1. Let $S \subset \mathbb{H}^1$ be a (X, Y)-Lipschitz surface. We say S is area-stationary if, for any C^1 vector field U with compact support such that $\text{supp}(U) \cap \partial S = \emptyset$, and associated one-parameter group of diffeomorphisms $\{\varphi_s\}_{s\in\mathbb{R}}$, we have

$$\frac{d}{ds}\Big|_{s=0} A_K(\varphi_s(S)) = 0.$$

Theorem 4.2. *Let $K \in C^2_+$ be a convex body with $0 \in \text{int}(K)$. Let $S \subset \mathbb{H}^1$ be an area-stationary (X, Y)-Lipschitz surface. Then the surface S is an $\mathbb{H}$-regular surface foliated by horizontal straight lines.*

Proof. Here we recall the main steps of the proof, for further details see [32] . Let $p \in S$. Since S is (X,Y)-Lipschitz, by Theorem 2.3, there exist an open ball $B_r(p)$ and a Lipschitz function $u : D \to \mathbb{R}$ such that $S \cap B_r(p) = \mathrm{Gr}(u)$ where $\mathrm{Gr}(u) = \{(x, u(x,y), t - xu(x,t)) \in \mathbb{H}^1 : (x,t) \in D\}$. By providing more details, after a rotation about the vertical axis we may assume that there exists $l > 0$ such that $Yf(q) \geqslant l > 0$ for every point of differentiability close enough to p. By the Implicit function Theorem for Lipschitz functions [13, page 255] there exists an open neighborhood $D \subset \{y = 0\}$ of the projection of p on $\{y = 0\}$ and a Euclidean Lipschitz function $u : D \to \mathbb{R}$ such that the S is locally an intrinsic Euclidean Lipschitz graph over the vertical plane $\{y = 0\}$.

By Section 3.1 we consider the biLipschitz homeomorphisms $G(\xi, \varepsilon) = (\xi, t_\varepsilon(\xi))$ where $t_\varepsilon(s)$ solves (3.8). Any function φ defined on D can be considered as a function of the variables (ξ, ε) by making $\tilde{\varphi}(\xi, \varepsilon) = \varphi(\xi, t_\varepsilon(\xi))$. Since the function G is C^1 with respect to ξ we have

$$\frac{\partial \tilde{\varphi}}{\partial \xi} = \varphi_x + t'_\varepsilon\, \varphi_t = \varphi_x + 2u\varphi_t.$$

Furthermore, by [21, Theorem 2 in Section 3.3.3] or [33, Theorem 3], we may apply the change of variables formula for Lipschitz maps. Assuming that the support of v is contained in a sufficiently small neighborhood of $(0,0)$, we can express the integral (3.5) as

$$\int_I \Big(\int_{-r}^{r} \big((\frac{\partial \tilde{v}}{\partial \xi} + 2\tilde{v}\, \tilde{u}_t) \tilde{M} d\xi \big) d\varepsilon = 0, \tag{4.1}$$

where I is a small interval containing 0. Putting $\tilde{v}h/(t_{\varepsilon+h} - t_\varepsilon)$ instead of $\tilde{v}$ in (4.1), where h is a small enough parameter and letting $h \to 0$ we obtain

$$\int_I \left(\int_{-r}^{r} \frac{\partial \tilde{v}}{\partial \xi} \tilde{M}\, d\xi \right) d\varepsilon = 0. \tag{4.2}$$

Let $\eta : \mathbb{R} \to \mathbb{R}$ be a positive function compactly supported in I and for $\rho > 0$ we consider the family $\eta_\rho(x) = \rho^{-1}\eta(x/\rho)$, that weakly converge to the Dirac delta distribution. Putting the test functions $\eta_\rho(\varepsilon)\psi(\xi)$ in (4.2) and letting $\rho \to 0$ we get

$$\int_{-r}^{r} \psi'(\xi) \tilde{M}(\xi, 0)\, d\xi = 0, \tag{4.3}$$

for each $\psi \in C_0^\infty((-r,r))$. Since F is C^1 and the distributional derivatives of a Lipschitz function belongs L^∞ we gain that M defined in (3.6) is $L^\infty(D)$. In particular we have that M belongs $L^1_{\rm loc}(D)$, thus also $\tilde M$ belongs to $L^1_{\rm loc}((-r,r))$. By Lemma [6, Lemma 8.1] we get that $\tilde M$ is constant a.e. in ξ. Therefore M is constant a.e. in s. By Lemma 3.2 in [31] F is a C^1 invertible function, therefore also $g(s)=(u_x+2uu_t)_{\Gamma(s)}=F^{-1}(M)$ is constant a.e. in s. Let $c\in\mathbb{R}$ be the previous constant. By construction we know that $t'_\varepsilon(s)$ is a Lipschitz function, therefore the function $h(s)=\frac{t'_\varepsilon(s)}{2}-cs$ is also Lipschitz in s. Since $t''_\varepsilon(s)=2(u_x+2uu_t)$ we gain that $h'(s)=0$ a.e. in s. Since h is Lipschitz we gain that $h(s)$ is constant for each s and $h'(s)=0$ for each s. Therefore $g(s)$ is constant for each s. This shows that horizontal normal given by

$$\nu_h=\frac{(u_x+2uu_t)X-Y}{\sqrt{1+(u_x+2uu_t)^2}} \tag{4.4}$$

is constant along the characteristic curves, thus also $Z=-J(\nu_h)$ is constant. Hence the characteristic curves of S are straight lines. Moreover $t_\varepsilon(s)$ is a polynomial of the second order given by

$$t_\varepsilon(s)=\varepsilon+a(\varepsilon)s+b(\varepsilon)s^2 \tag{4.5}$$

where $a(\varepsilon)=u(0,\varepsilon)$ that is Lipschitz continuous and $b(\varepsilon)=(u_x+2uu_t)(0,\varepsilon)=(u_x+2uu_t)(s,\varepsilon)$. Furthermore, choosing $s>0$ we can easily prove that $b(\varepsilon)$ is also a Lipschitz function function in ε. Hence in particular the horizontal normal ν_h given by (4.4) is continuous, then the regular part S is an $\mathbb{H}$-regular surface. □

5. The second variation formula and the Codazzi equation

In this Section, we recall the second variation formula obtained in [32]. Notice that a similar proof was obtained in the sub-Riemannian setting in [29] for surfaces of class C^1, but the proof here is more delicate, since the surface is only (X,Y)-Lipschitz.

Theorem 5.1. *Let $K\in C^2_+$ be a convex body with $0\in\text{int}(K)$. Let $S\subset\mathbb{H}^1$ be an area-stationary (X,Y)-Lipschitz surface. Let U be an horizontal C^2 vector field compactly supported on S and associated one-parameter group of diffeomorphisms $\{\varphi_s\}_{s\in\mathbb{R}}$. Then the second variation of the sub-Finsler area induced by U is given by*

$$\frac{d^2}{ds^2}\Big|_{s=0}A_K(\varphi_s(S))=\int_S\left(Z(f)^2+qf^2\right)\frac{|N_h|}{\kappa(\pi_K(\nu_h))}dS, \tag{5.1}$$

where

$$q = 4\left(Z\left(\frac{\langle N,T\rangle}{|N_h|}\right) - \frac{\langle N,T\rangle^2}{|N_h|^2}\right),$$

κ is the positive curvature of the boundary ∂K and $f = \langle U, \nu_h\rangle$.

Here we show that the quantity q defined in the second variation is well-posed, indeed $\langle N,T\rangle/|N_h|$ solves a Codazzi type equations along the characteristic curves.

Proposition 5.2. *Let S be a complete oriented area-stationary (X,Y)-Lipschitz surface. Then along any arc-length parametrization geodesic $\bar\gamma_\varepsilon(s)$, the function $\langle N,T\rangle/|N_h|(\bar\gamma_\varepsilon(s))$ satisfies the ordinary differential equation (5.4) for a.e. ε. Furthermore, $\langle N,T\rangle/|N_h|(\bar\gamma_\varepsilon(s))$ is smooth in s for a.e. ε.*

Proof. Let p in S. Since S is (X,Y)-Lipschitz, Theorem 2.3 implies the existence of an open ball $B_r(p)$ and of a Lipschitz function $u : D \to \mathbb{R}$ such that $S \cap B_r(p) = \mathrm{Gr}(u)$. Following Section 3 the unit normal is given by $N = \tilde N/|\tilde N|$, where $\tilde N$ is define in (3.2).

Let $\Gamma(s)$ be a characteristic curve passing through p in $\mathrm{Gr}(u)$. Let $\gamma(s)$ be the projection of $\Gamma(s)$ onto the xt-plane, and $(0,0) \in D$. We parameterize γ by $s \to (s, t(s))$. By Remark 3.1 the curve $s \to (s, t(s))$ satisfies the ordinary differential equation $t' = 2u$ and

$$\Gamma'(s) = X + (u_x + 2uu_t)Y.$$

Since we have

$$\nu_h = \frac{(u_x + 2uu_t)X - Y}{\sqrt{1 + (u_x + 2uu_t)^2}} \quad \text{and} \quad Z = -J(\nu_h)$$

we get that $Z = -\Gamma'(s)/|\Gamma(s)|$. Let $t_\varepsilon(s)$ be the solution of (3.8) and $\gamma_\varepsilon(s) = (s, t_\varepsilon(s))$. Since S is area-stationary we have that $(u_x + 2uu_t)$ is constant along $\gamma_\varepsilon(s)$. Moreover

$$t''_\varepsilon(s) = 2(u_x + 2uu_t)(\gamma_\varepsilon(s)) = 2b(\varepsilon) = 2(u_x + 2uu_t)(0, \varepsilon)$$

is constant as a a function of s. Thus we have

$$t_\varepsilon(s) = \varepsilon + a(\varepsilon)s + b(\varepsilon)s^2, \tag{5.2}$$

where $a(\varepsilon) = u(0, \varepsilon)$. Choosing $s > 0$ in (5.2) we can easily prove that $b(\varepsilon)$, that a priori is only continuous, is also a Lipschitz function. By equation (7) in [37, Theorem 3.7] we have

$$\frac{\partial}{\partial\varepsilon}\frac{\partial}{\partial s} t_\varepsilon(s) = \frac{\partial}{\partial s}\frac{\partial}{\partial\varepsilon} t_\varepsilon(s) \tag{5.3}$$

a.e. in ε, where the equality has to be interpreted in the sense of distributions. Putting (3.8) in the left hand side of (5.3) and applying the chain rule for Lipschitz functions (see [37, Remark 3.6]) we get

$$2u_t(s,t_\varepsilon(s))(1+a'(\varepsilon)s+b'(\varepsilon)s^2)=(a'(\varepsilon)+2b'(\varepsilon)s)$$

a.e. in ε. Therefore we get

$$u_t(s,t_\varepsilon(s))=\frac{\frac{a'(\varepsilon)}{2}+b'(\varepsilon)s}{(1+a'(\varepsilon)s+b'(\varepsilon)s^2)},$$

a.e. in ε, since by Lemma 3.9 in [32] we have $\partial t_\varepsilon/\partial\varepsilon>0$ a.e. in ε. Since we have $Z=-\Gamma'(s)/|\Gamma(s)|$ we consider $\tilde\gamma_\varepsilon(s)=\gamma_\varepsilon(-s)$. By Lemma 5.3 we have that

$$u_t(\tilde\gamma_\varepsilon(s))=\frac{\frac{a'(\varepsilon)}{2}-b'(\varepsilon)s}{(1-a'(\varepsilon)s+b'(\varepsilon)s^2)}$$

solves the equation (5.4) with initial condition $y(0)=a'(\varepsilon)/2$ and $y'(0)=\frac{a'(\varepsilon)^2}{2}-b'(\varepsilon)$ for a.e. ε. Moreover we have

$$t_\varepsilon(-s)=\varepsilon-a(\varepsilon)s+b(\varepsilon)s^2$$

For each ε fixed we have $b(\varepsilon)=(u_x+2uu_t)(\tilde\gamma_\varepsilon)$ is constant, let

$$\bar\gamma_\varepsilon(s)=\tilde\gamma_\varepsilon\left(s/\sqrt{1+b(\varepsilon)^2}\right)$$

be an arc-length parametrization of $\tilde\gamma_\varepsilon$. Then Remark 4.2 in [32] shows that also

$$\langle N,T\rangle/|N_h|(\bar\gamma_\varepsilon)=\frac{u_t}{\sqrt{1+(u_x+2uu_t)^2}}(\bar\gamma_\varepsilon)$$

is a solution of (5.4) a.e. in ε. □

Lemma 5.3. *Given $a,b\in\mathbb{R}$, the only solution of equation*

$$y''-6y'y+4y^3=0 \tag{5.4}$$

about the origin with initial conditions $y(0)=a$, $y'(0)=b$, is

$$y_{a,b}(s)=\frac{a-(2a^2-b)s}{1-2as+(2a^2-b)s^2}. \tag{5.5}$$

Moreover, we have

$$y_{a,b}^2(s)-y'_{a,b}(s)=\frac{a^2-b}{(1-2as+(2a^2-b)s^2)^2} \tag{5.6}$$

If $y_{a,b}$ is defined for every $s \in \mathbb{R}$ then either $a^2 - b > 0$ or $y_{a,b} \equiv 0$.

6. The sub-Finsler Bernstein's problem for (X, Y)-Lipschitz surfaces

Definition 6.1. We say that a complete oriented area-stationary surface $S \subset \mathbb{H}^1$ is stable if

$$\int_S \left(Z(f)^2 + 4 \left(Z \left(\frac{\langle N, T\rangle}{|N_h|} \right) - \frac{\langle N, T\rangle^2}{|N_h|^2} \right) f^2 \right) \frac{|N_h|}{\kappa(\pi(\nu_h))} dS \geqslant 0 \tag{6.1}$$

holds for any continuous function f on S with compact support such that $Z(f)$ exists and is continuous.

The following lemma is proven in [3, page 45].

Lemma 6.2. *Let $A, B \in \mathbb{R}$ be such that $A^2 \leqslant 2B$ and set $h(s) := 1 + As + Bs^2/2$. If*

$$\int_{\mathbb{R}} \phi'(s)^2 h(s) ds \geqslant (2B - A^2) \int_{\mathbb{R}} \phi(s)^2 \frac{1}{h(s)} ds$$

for each $\phi \in C^1_0(\mathbb{R})$ then $2B = A^2$.

Theorem 6.3 (Bernstein's theorem)**.** *Let $S \subset \mathbb{H}^1$ be a complete, stable, (X, Y)-Lipschitz surface. Then S is a vertical plane.*

Proof. First of all we have that S is an $\mathbb{H}$-regular surface by Theorem 4.2. Let p in S. Since S is (X, Y)-Lipschitz, by Theorem 2.3, there exist an open ball $B_r(p)$ and a Lipschitz function $u : D \to \mathbb{R}$ such that $S \cap B_r(p) = \mathrm{Gr}(u)$ where $\mathrm{Gr}(u) = \{(x, u(x, y), t - xu(x, t)) \in \mathbb{H}^1 \ : \ (x, t) \in D\}$. Let $(0, 0) \in D$ be the projection of p to the xt-plane. On D we consider the coordinates around $(0, 0)$ furnished by $G(s, \varepsilon)$ defined in Section 3.1. Let I be a small interval containing 0, then $\varepsilon \in I$ and $s \in]-r, r[$. Since S is complete by the Hopf-Rinow Theorem each geodesic (in particular the straight lines in the Z-direction) can be indefinitely extended along any direction, thus the open interval $]-r, r[$ extend to $\mathbb{R}$. Notice that $\bar{\gamma}_\varepsilon(s)$ is the integral curve of Z, thus $Z(f) = \partial_s(f)$. Hence, taking into account that $(u_x + 2uu_t)(s)$ is constant along $\bar{\gamma}_\varepsilon$ and equal to $b(\varepsilon)$, the stability condition (6.1) is equivalent to

$$\int_I \int_{\mathbb{R}} \left((\partial_s f)^2 - 4 \left(\frac{\langle N, T\rangle^2}{|N_h|^2} - \partial_s \left(\frac{\langle N, T\rangle}{|N_h|} \right) \right) f^2 \right) \frac{\partial t_\varepsilon}{\partial \varepsilon} \frac{\sqrt{1 + b(\varepsilon)^2}}{\kappa(\pi(\nu_h))} \, ds \, d\varepsilon \geqslant 0, \tag{6.2}$$

for any continuous function f on S with compact support such that $Z(f)$ exists and is continuous.

Since $\langle N,T\rangle/|N_h|$ solves the equation (5.4) with initial condition $y(0)=a'(\varepsilon)/2$ and $y'(0)=a'(\varepsilon)^2/2-b'(\varepsilon)$, by Lemma 5.3 we get

$$\frac{\langle N,T\rangle^2}{|N_h|^2}-\left(\frac{\langle N,T\rangle}{|N_h|}\right)'=\frac{b'(\varepsilon)-\frac{a'(\varepsilon)^2}{4}}{(1-a'(\varepsilon)s+b'(\varepsilon)s^2)^2}.$$

Therefore, computing $\partial t_\varepsilon/\partial\varepsilon$ from (5.2), we obtain that (6.2) is equivalent to

$$\int_I\int_{\mathbb{R}}\left((1-a'(\varepsilon)s+b'(\varepsilon)s^2)(\partial_s f)^2-\frac{4b'(\varepsilon)-a'(\varepsilon)^2}{(1-a'(\varepsilon)s+b'(\varepsilon)s^2)}f^2\right)\frac{\sqrt{1+b(\varepsilon)^2}}{\kappa(\pi(\nu_h))}\,ds\,d\varepsilon\geqslant 0.$$

Let $\eta:\mathbb{R}\to\mathbb{R}$ be a positive function compactly supported in $\mathbb{R}$ and for $\rho>0$ we consider the family $\eta_\rho(x)=\rho^{-1}\eta(x/\rho)$, that weakly converge to the Dirac delta distribution. Putting the test functions $\eta_\rho(x-\varepsilon)\psi(s)$, where $\psi\in C_0^1(\mathbb{R})$, in the previous equation and letting $\rho\to 0$ we get

$$\int_{\mathbb{R}}(1-a'(\varepsilon)s+b'(\varepsilon)s^2)(\psi'(s))^2ds\geqslant(4b'(\varepsilon)-a'(\varepsilon)^2)\int_{\mathbb{R}}\frac{\psi(s)^2}{(1-a'(\varepsilon)s+b'(\varepsilon)s^2)}\,ds,$$

for a.e. ε since $\kappa(\pi(\nu_h))$ is a positive constant along the horizontal straight lines for each ε (since ν_h is constant along such horizontal straight lines) and $\sqrt{1+b(\varepsilon)^2}$ is a positive constant on $\bar\gamma_\varepsilon$.

Setting $A=-a'(\varepsilon)$, $B=2b'(\varepsilon)$ and $h(s):=1+As+Bs^2/2$, we obtain

$$\int_{\mathbb{R}}h(s)\psi'(s)^2ds\geqslant(2B-A^2)\int_{\mathbb{R}}\frac{\psi^2(s)}{h(s)}ds$$

for each $\psi\in C_0^1(\mathbb{R})$. Assume that $2B-A^2\geqslant 0$ then by Lemma 6.2 we get that $2B=A^2$, then $4b'(\varepsilon)-a'(\varepsilon)^2=0$. Therefore by Lemma 5.3 we obtain $\langle N,T\rangle\equiv 0$, $a'(\varepsilon)=b'(\varepsilon)=0$ a.e. in ε. On the other hand, if $2B-A^2<0$ then directly by Lemma 5.3 we obtain $\langle N,T\rangle\equiv 0$, $a'(\varepsilon)=b'(\varepsilon)=0$ a.e. in ε. Hence $a(\varepsilon)$ and $b(\varepsilon)$ are constant functions in ε and

$$t_\varepsilon(s)=\varepsilon+as+bs^2,$$

for some constant $a,b\in\mathbb{R}$. Since $t'_\varepsilon(s)=2u(s,t_\varepsilon)=2\tilde u(s,\varepsilon)$ we get $\tilde u(s,\varepsilon)=a/2+bs$, thus $\tilde u$ is an affine function. Hence S is locally a strip contained in a vertical plane. A standard connectedness argument implies that each connected component of S is a vertical plane. □

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