ON WEIGHTED SECOND ORDER ADAMS INEQUALITIES WITH NAVIER BOUNDARY CONDITIONS

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Abstract. We obtain some sharp weighted version of Adams' inequality on second order Sobolev spaces with Navier boundary conditions. The weights that we consider determine a supercritical exponential growth, except in the origin, and the corresponding inequalities hold on *radial* functions only. We also consider the problem of extremal functions, and we show that the sharp suprema are achieved, as in the unweighted classical case.

Sunto. Si dimostrano alcune versioni con pesi della disuguaglianza ottimale di Adams su spazi di Sobolev del secondo ordine con condizioni di Navier al bordo. I pesi considerati determinano una crescita esponenziale sopracritica, ad eccezione dell'origine, e le corrispondenti disuguaglianze sono valide solo per funzioni *radiali*. Viene affrontato anche il problema dell'esistenza di estremali, e si dimostra che le disuguaglianze ottimali sono assunte, come nel caso classico senza pesi.

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1. Introduction

Let $B_R \subset \mathbb{R}^n$ be the Euclidean ball centered at the origin with radius $R > 0$. In any dimension $n \geq 3$, J. M. B. do Ó, B. Ruf, and P. Ubilla [11] analyzed the following first order Sobolev type embedding for radial functions into a variable exponent Lebesgue space:

$$W^{1,2}_{0,\mathrm{rad}}(B_1) \hookrightarrow L_{2^*+|x|^\alpha}(B_1), \tag{1.1}$$

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where $2^* := \frac{2n}{n-2}$ is the critical Sobolev exponent, and $\alpha > 0$. Notice that the variable exponent $2^*+|x|^\alpha$ identifies a supercritical growth, except in the origin. In fact, if $2^*+|x|^\alpha$ is replaced by the constant 2^*+c with $c=0$ or $c>0$ then we have the classical critical or supercritical Sobolev growth. In contrast with these classical cases, the result in [11] shows the validity of (1.1) (for any $\alpha > 0$), and the existence of extremal functions for the corresponding embedding inequality (at least for suitable values of $\alpha > 0$). We mention that the fundamental work of H. Brezis and L. Nirenberg [4] shows a similar phenomenon but with a completely different growth function: if the classical critical case is perturbed by adding a suitable lower order term then the corresponding inequality is attained.

The 2-dimensional case is a *limiting* case in the framework of first order Sobolev embeddings. If $n = 2$ then the maximal growth for the integrability of functions $u \in W_0^{1,2}$ is the squared exponential $e^{\beta u^2}$, $\beta > 0$, as found by S. I. Pohozaev and N. S. Trudinger, and later J. Moser [17] discovered the *sharp* version of the embedding inequality known nowadays as Trudinger-Moser inequality. The attainability problem for this inequality is quite different from the critical Sobolev case in higher dimensions, indeed extremal functions exist as enlightned by L. Carleson and S.-Y. A. Chang in the seminal paper [6]. In [18], Q. A. Ngô and V. H. Nguyen considered the limiting case $n = 2$ of the embedding (1.1): they obtained some sharp weighted Trudinger-Moser inequalities with *supercritical* growth (except in the origin) for *radial* functions, and they proved that for such inequalities extremals exist, as in the unweighted case. We recall that some attention has been devoted to the study of weighted Trudinger-Moser inequalities, see for instance [2, 22, 5, 19, 7, 8], and we also mention the recent paper [24] about a weighted inequality in the higher order setting.

A remarkable contribution in the framework of Trudinger-Moser inequalities is due to D. R. Adams [1] who extended the sharp inequality of J. Moser to the heavily non-trivial framework of higher order Sobolev spaces with Dirichlet boundary conditions $W_0^{m,\frac{n}{m}}$, $1 < m < n$. As pointed out in [23] (see also [20, 15, 16]), the result of D. R. Adams is still valid in larger Sobolev spaces with Navier boundary conditions $W_{\mathcal{N}}^{m,\frac{n}{m}}$. Given a smooth bounded domain $\Omega \subset \mathbb{R}^n$, these spaces are defined as

$$W_{\mathcal{N}}^{m,\frac{n}{m}}(\Omega) := \Big\{ u \in W^{m,\frac{n}{m}}(\Omega) \;\Big|\; \Delta^j u\big|_{\partial\Omega} = 0 \;\text{ in the sense of traces for }\; 0 \le j < \frac{m}{2} \Big\}.$$

We focus the attention on the special second order case $m = 2 < n$. In this case

$$W_{\mathcal{N}}^{2,\frac{n}{2}}(\Omega) = W_0^{1,\frac{n}{2}}(\Omega) \cap W^{2,\frac{n}{2}}(\Omega), \qquad \text{and } W_0^{2,\frac{n}{2}}(\Omega) \subset W_{\mathcal{N}}^{2,\frac{n}{2}}(\Omega).$$

Adams' sharp inequality holds without *any* restriction to the radial case, but the sequence of functions constructed in [1] to prove the sharpness of the inequality consists of *radial* functions. Therefore the second order Adams' inequality with Navier boundary conditions is sharp also in the radial part of $W_{\mathcal{N}}^{2,\frac{n}{2}}(B_R)$, i.e. in the subspace $W_{\mathcal{N},\mathrm{rad}}^{2,\frac{n}{2}}(B_R)$ of $W_{\mathcal{N}}^{2,\frac{n}{2}}(B_R)$ consisting of spherically symmetric functions. More precisely,

$$S_n(\beta) := \sup_{u \in W_{\mathcal{N},\mathrm{rad}}^{2,\frac{n}{2}}(B_R),\, \|\Delta u\|_{\frac{n}{2}} \le 1} \int_{B_R} e^{\beta |u|^{\frac{n}{n-2}}}\, dx \begin{cases} < +\infty & \text{if } 0 < \beta \le \beta_n, \\ = +\infty & \text{if } \beta > \beta_n, \end{cases} \tag{1.2}$$

where β_n is Adams' sharp exponent defined by

$$\beta_n := \frac{n}{\omega_{n-1}} \left[\frac{\pi^{n/2} 2^m \Gamma\left(\frac{m}{2}\right)}{\Gamma\left(\frac{n-m}{2}\right)} \right]^{\frac{n}{n-m}}, \quad \omega_{n-1} := \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}.$$

In the same spirit of [11, 18], the aim of this paper is to analyze some sharp weighted versions of (1.2) with *supercritical* growth (except in the origin). Let $w(r) := r^\alpha$, $\alpha > 0$, first we consider the following weighted Adams-type problem:

$$S_n(\beta, w) := \sup_{u \in W_{\mathcal{N},\mathrm{rad}}^{2,\frac{n}{2}}(B_R),\, \|\Delta u\|_{\frac{n}{2}} \le 1} \int_{B_R} e^{\left(\beta + w(|x|)\right)|u|^{\frac{n}{n-2}}}\, dx. \tag{1.3}$$

Theorem 1.1. *For any $n \ge 3$, $R > 0$, and $\alpha > 0$, we have*

$$S_n(\beta_n, r^\alpha) < +\infty.$$

Moreover, the exponent β_n in the above inequality is sharp in the sense that for any $\beta > \beta_n$ we have $S_n(\beta, r^\alpha) = +\infty$.

Differently form the unweighted case (1.2), the restriction to the *radial* part of $W^{2,\frac{n}{2}}_{\mathcal{N}}(B_R)$ is *necessary* in the study of $S_n(\beta_n, r^\alpha)$ with $\alpha > 0$. Indeed, we have for any $\alpha > 0$

$$\sup_{u\in W^{2,\frac{n}{2}}_{\mathcal{N}}(B_R),\, \|\Delta u\|_{\frac{n}{2}}\le 1} \int_{B_R} e^{(\beta_n+|x|^\alpha)|u|^{\frac{n}{n-2}}}\, dx = +\infty.$$

This can be seen by considering any open set $\Omega \subset B_R$ with $0 \notin \overline{\Omega}$. Then $I_\alpha := \inf_{x\in\Omega} |x|^\alpha > 0$, and if $\{u_k\}_k$ is any sequence as in [1] satisfying $u_k \in W^{2,\frac{n}{2}}_0(\Omega)$, $\|\Delta u_k\|_{\frac{n}{2}} \le 1$, and

$$\lim_{k\to+\infty} \int_\Omega e^{(\beta_n+I_\alpha)|u_k|^{\frac{n}{n-2}}}\, dx = +\infty,$$

then in particular $u_k \in W^{2,\frac{n}{2}}_{\mathcal{N}}(B_R)$, and we get

$$\begin{aligned}\sup_{u\in W^{2,\frac{n}{2}}_{\mathcal{N}}(B_R),\, \|\Delta u\|_{\frac{n}{2}}\le 1} \int_{B_R} e^{(\beta_n+|x|^\alpha)|u|^{\frac{n}{n-2}}}\, dx &\ge \lim_{k\to+\infty} \int_{B_R} e^{(\beta_n+|x|^\alpha)|u_k|^{\frac{n}{n-2}}}\, dx \\ &\ge \lim_{k\to+\infty} \int_\Omega e^{(\beta_n+I_\alpha)|u_k|^{\frac{n}{n-2}}}\, dx = +\infty.\end{aligned}$$

Similarly, the restriction to radial functions is needed also in the study of

$$\tilde{S}_n(\beta, w) := \sup_{u\in W^{2,\frac{n}{2}}_{\mathcal{N},\mathrm{rad}}(B_R),\, \|\Delta u\|_{\frac{n}{2}}\le 1} \int_{B_R} e^{\left(\beta^{\frac{n-2}{n}}|u|\right)^{\frac{n}{n-2}+w(|x|)}}\, dx, \tag{1.4}$$

with $w(r) := r^\alpha$, $\alpha > 0$ (at least when $\beta = \beta_n$, as above).

Theorem 1.2. *For any $n \ge 3$, $R > 0$, and $\alpha > 0$, we have*

$$\tilde{S}_n(\beta_n, r^\alpha) < +\infty.$$

Moreover, for any $\beta > \beta_n$ we have $\tilde{S}_n(\beta, r^\alpha) = +\infty$.

Up to our knowledge, few results about the existence of extremal functions for higher order Adams-type inequalities are available in the literature. In the second order framework, Y. Li and C. B. Ndiaye [13] studied the problem on 4-dimensional Riemannian manifolds, and later G. Lu and Y. Yang [14] obtained the existence of extremals in $W^{2,2}_0$ on *any* smooth bounded domain $\Omega \subset \mathbb{R}^4$. Finally, A. DelaTorre and G. Mancini [9] reached the more general case $W^{m,2}_0$ in any dimension $n = 2m > 3$. More recently, the existence of extremals was also considered in $W^{2,\frac{n}{2}}_{\mathcal{N}}$ in any dimension $n \ge 4$ on Euclidean balls, see [21]. Some of the main difficulties in the study of higher order problems are softned in these spaces with Navier boundary conditions. In particular, Talenti's comparison principle enables to perform symmetrization arguments, and if the domain is a Euclidean ball then the problem of the existence of extremals can be reduced to the radial case. In fact, the result in [21] shows that one can always find in this functional framework a *radial* extremal function, and in particular the critical supremum $S_n(\beta_n)$ in (1.2) is attained in any dimension $n \geq 4$. The attainability of $S_n(\beta_n)$ in dimension $n = 3$ is left open by [21], due to some technical difficulties.

Along the same line as in [18] and [21], we will show that both $S_n(\beta_n, r^\alpha)$ and $\tilde{S}_n(\beta_n, r^\alpha)$ are attained, i.e. there exist u, $\tilde{u} \in W^{2,\frac{n}{2}}_{\mathcal{N},\mathrm{rad}}(B_R)$ with $\|\Delta u\|_{\frac{n}{2}} = \|\Delta \tilde{u}\|_{\frac{n}{2}} = 1$ such that

$$\int_{B_R} e^{\beta(1+|x|^\alpha)|u|^{\frac{n}{n-2}}}\, dx = S_n(\beta_n, r^\alpha), \quad \text{and} \quad \int_{B_R} e^{\left(\beta^{\frac{n-2}{n}}|\tilde{u}|\right)^{\frac{n}{n-2}+|x|^\alpha}}\, dx = \tilde{S}_n(\beta_n, r^\alpha).$$

Theorem 1.3. *Let $n \geq 4$, $R > 0$, and $\alpha > 0$.*

(i) The sharp supremum $S_n(\beta_n, r^\alpha)$ is attained.

(ii) Also the sharp supremum $\tilde{S}_n(\beta_n, r^\alpha)$ is attained.

In Section 2 we analyze the sharp inequality for $S_n(\beta, r^\alpha)$, and in Section 3 we prove the attainability of $S_n(\beta_n, r^\alpha)$. Section 4 and Section 5 are devoted to the study of $\tilde{S}_n(\beta, r^\alpha)$. The proofs follows closely the arguments introduced in [18] and [21].

2. Analysis of the sharp inequality for $S_n(\beta, w)$

Let $S_n(\beta, w)$ defined by (1.3). We will analyze the problem with a general weight $w = w(r)$ satisfying the conditions:

(w_0) $w : [0, R) \to [0, +\infty)$ is continuous,

(w_1) $w(0) = 0$ and $w(r) > 0$ for any $0 < r \leq R$,

(w_2) there exists $\gamma_0 > 0$ and $r_0 \in (0, R)$ such that

$$w(r) \leq \gamma_0 \frac{1}{\log \frac{R}{r}} \quad \text{for any } r \in (0, r_0),$$

(w_3) there exists $\gamma_1 \in (0, 1)$ and $r_1 \in (0, R)$ such that

$$w(r) \leq \gamma_1 \frac{\beta_n}{n} \frac{\log \frac{R}{R-r}}{\log \frac{R}{r}} \quad \text{for any } r \in (r_1, R).$$

Clearly the weight $w(r) = r^\alpha$ with $\alpha > 0$ satisfies the above conditions (w_0), (w_1), (w_2), and (w_3), and Theorem 1.1 is a particular case of the following result.

Proposition 2.1. *For any $n \geq 3$, $R > 0$, and a weight function w satisfying (w_0), (w_1), (w_2), and (w_3), we have*

$$S(\beta_n, w) < +\infty \quad \text{if and only if} \quad \beta \leq \beta_n. \tag{2.1}$$

Proof. Let $u \in W^{2,\frac{n}{2}}_{\mathcal{N},\mathrm{rad}}(B_R)$, we set $f := -\Delta u$ in B_R, and we assume that $\|f\|_{\frac{n}{2}} \leq 1$. By assumption f is spherically symmetric, and we introduce the auxiliary functions $g : [0, |B_R|] \to \mathbb{R}$ and $G : (0, |B_R|] \to \mathbb{R}$ defined as

$$f(r) = g\left(\frac{\omega_{n-1}}{n} r^n\right), \; 0 \leq r \leq R, \quad \text{and} \quad G(t) := \frac{1}{t}\int_0^t g(s)\, ds, \; 0 < t \leq |B_R|. \tag{2.2}$$

The one-dimensional Hardy inequality [3, Chapter 3 – Lemma 3.9] yields

(2.3)

$$\begin{aligned}\left(\int_0^{|B_R|} |G(t)|^{\frac{n}{2}}\, dt\right)^{\frac{2}{n}} &\leq \frac{n}{n-2}\left(\int_0^{|B_R|} |g(t)|^{\frac{n}{2}}\, dt\right)^{\frac{2}{n}} = \frac{n}{n-2}\left(\omega_{n-1}\int_0^R |f(r)|^{\frac{n}{2}}\, dt\right)^{\frac{2}{n}} \\ &= \frac{n}{n-2}\|f\|_{\frac{n}{2}} = \frac{n}{n-2}\|\Delta u\|_{\frac{n}{2}} \leq \frac{n}{n-2}.\end{aligned}$$

Moreover

$$u(r) = \left(n^{1-\frac{1}{n}}\omega_{n-1}^{\frac{1}{n}}\right)^{-2}\int_{|B_r|}^{|B_R|} \frac{G(t)}{t^{1-\frac{2}{n}}}\, dt, \quad 0 < r \leq R, \tag{2.4}$$

and by Hölder inequality, we can estimate

$$\begin{aligned}|u(r)| &\leq \left(n^{1-\frac{1}{n}}\omega_{n-1}^{\frac{1}{n}}\right)^{-2}\left(\int_{|B_r|}^{|B_R|} |G(t)|^{\frac{n}{2}}\, dt\right)^{\frac{2}{n}}\left(n\log\frac{R}{r}\right)^{\frac{n-2}{n}} \\ &\leq \left(n^{1-\frac{1}{n}}\omega_{n-1}^{\frac{1}{n}}\right)^{-2}\left(\frac{n}{n-2}\right)\left(n\log\frac{R}{r}\right)^{\frac{n-2}{n}}.\end{aligned}$$

Hence, we have for any $0 < r \leq R$

$$\beta_n|u(r)|^{\frac{n}{n-2}} \leq \beta_n\left(n^{1-\frac{1}{n}}\omega_{n-1}^{\frac{1}{n}}\right)^{-\frac{2n}{n-2}}\left(\frac{n}{n-2}\right)^{\frac{n}{n-2}}\left(n\log\frac{R}{r}\right) = n\log\frac{R}{r}, \tag{2.5}$$

and

$$w(r)|u(r)|^{\frac{n}{n-2}} \leq \frac{n}{\beta_n}w(r)\log\frac{R}{r},$$

since $w = w(r)$ is positive on $(0, R]$.

The integral that we have to estimate can be written as

$$\begin{aligned}\int_{B_R} e^{\left(\beta_n + w(|x|)\right)|u|^{\frac{n}{n-2}}}\,dx &= \omega_{n-1}\int_0^R e^{\left(\beta_n + w(|x|)\right)|u|^{\frac{n}{n-2}}} r^{n-1}\,dr \\ &\le \omega_{n-1}\int_0^{r_0} + \int_{r_0}^{r_1} + \int_{r_1}^{R} e^{\beta_n |u|^{\frac{n}{n-2}}} e^{\frac{n}{\beta_n} w(r)\log\frac{R}{r}} r^{n-1}\,dr \qquad (2.6)\end{aligned}$$

where $r_0,\ r_1 \in (0, R)$ are given by (w_2) and (w_3) respectively. The aim is to obtain an estimate of the three terms in (2.6) which must be *independent* of u. In view of (w_2), we have

$$\omega_{n-1}\int_0^{r_0} e^{\beta_n |u|^{\frac{n}{n-2}}} e^{\frac{n}{\beta_n} w(r)\log\frac{R}{r}} r^{n-1}\,dr \le \omega_{n-1} e^{\frac{n}{\beta_n}\gamma_0}\int_0^{r_0} e^{\beta_n |u|^{\frac{n}{n-2}}} r^{n-1}\,dr \le e^{\frac{n}{\beta_n}\gamma_0} S_n(\beta_n).$$

Since $w = w(r)$ is strictly positive and continuous on $[r_0, r_1]$, we can set $M := \max_{r\in[r_0,r_1]} w(r) > 0$, and

$$\begin{aligned}\omega_{n-1}\int_{r_0}^{r_1} e^{\beta_n |u|^{\frac{n}{n-2}}} e^{\frac{n}{\beta_n} w(r)\log\frac{R}{r}} r^{n-1}\,dr &\le \omega_{n-1} e^{\frac{n}{\beta_n} M\log\frac{R}{r_0}}\int_{r_0}^{r_1} e^{\beta_n |u|^{\frac{n}{n-2}}} r^{n-1}\,dr \\ &\le e^{\frac{n}{\beta_n} M\log\frac{R}{r_0}} S_n(\beta_n).\end{aligned}$$

Finally, in view of (w_3), we get

$$\begin{aligned}\int_{r_1}^{R} e^{\beta_n |u|^{\frac{n}{n-2}}} e^{\frac{n}{\beta_n} w(r)\log\frac{R}{r}} r^{n-1}\,dr &\le R^{n+\gamma_1}\int_{r_1}^{R} \frac{1}{r(R-r)^{\gamma_1}}\,dr \\ &\le \frac{R^{n+\gamma_1}}{r_1}\int_{r_1}^{R} \frac{1}{(R-r)^{\gamma_1}}\,dr = \frac{R^{n+\gamma_1}}{r_1}\cdot\frac{(R-r_1)^{1-\gamma_1}}{1-\gamma_1},\end{aligned}$$

where we also used the assumption $\gamma_1 \in (0, 1)$.

Summarizing

$$\int_{B_R} e^{\left(\beta_n + w(|x|)\right)|u|^{\frac{n}{n-2}}}\,dx \le \left(e^{\frac{n}{\beta_n}\gamma_0} + e^{\frac{n}{\beta_n} M\log\frac{R}{r_0}}\right) S_n(\beta_n) + \omega_{n-1}\frac{R^{n+\gamma_1}}{r_1}\cdot\frac{(R-r_1)^{1-\gamma_1}}{1-\gamma_1}$$

and $S_n(\beta_n, w) < +\infty$, since the constant on the right hand side depends only on the dimension n, the radius $R > 0$, and the weight function w.

The proof of the sharpness of (2.1) is trivial, and follows directly from the sharpness of Adams' inequality (1.2). □

Remark 2.1. Let us consider a weight function w satisfying (w_0), (w_1), (w_2), and (w_3). We set

$$h(r) := \begin{cases} \frac{n}{\beta_n}\gamma_0 & \text{if } 0 < r < r_0, \\ \frac{n}{\beta_n} M \log \frac{R}{r_0} & \text{if } r_0 \le r \le r_1, \\ \gamma_1 \log \frac{R}{R-r} & \text{if } r_1 < r < R, \end{cases}$$

where the constants γ_0 and γ_1 are given by (w_2) and (w_3) respectively, and $M := \max_{r\in[r_0,r_1]} w(r) > 0$. As a by-product of the previous proof, for any $u \in W^{2,\frac{n}{2}}_{\mathcal{N},\mathrm{rad}}(B_R)$ with $\|\Delta u\|_{\frac{n}{2}} \le 1$ we have

$$\beta_n |u(r)|^{\frac{n}{n-2}} \le n \log \frac{R}{r} \quad \text{and} \quad (\beta_n + w(r))|u(r)|^{\frac{n}{n-2}} \le n \log \frac{R}{r} + h(r), \quad 0 < r \le R.$$

We emphasize that the function on the right hand side of the above pointwise estimate is *independent* of u, and satisfies

$$e^{n \log \frac{R}{r} + h(r)} r^{n-1} \in L^q(\,(a,R)\,) \quad \text{for any } a \in (0,R), \text{ and any } 1 \le q < \frac{1}{\gamma_1}.$$

This property will be useful in the study of the attainability of the critical supremum $S_n(\beta_n, w)$.

3. Attainability of $S_n(\beta_n, w)$

First, we remark the following simple relation between $S_n(\beta_n, w)$ and the critical supremum $S_n(\beta_n)$ of the *unweighted radial* Adams' inequality (1.2).

Proposition 3.1. *For any $n \ge 4$, $R > 0$, and a weight function w satisfying (w_0) and (w_1), we have*

$$S_n(\beta_n, w) > S_n(\beta_n). \tag{3.1}$$

Proof. From [21], we know that $S_n(\beta_n)$ is attained by some $U \in W^{2,\frac{n}{2}}_{\mathcal{N},\mathrm{rad}}(B_R)$ with $\|\Delta U\|_{\frac{n}{2}} = 1$. Since $w = w(r)$ is positive in $(0, R]$, then we have

$$S_n(\beta_n) = \int_{B_R} e^{\beta_n |U|^{\frac{n}{n-2}}}\, dx < \int_{B_R} e^{\left(\beta_n + w(|x|)\right)|U|^{\frac{n}{n-2}}}\, dx \le S_n(\beta_n, w)$$

and the proof is complete. □

In view of the above relation, the study of the attainability of $S_n(\beta_n, w)$ can be simplified under the following additional conditions on the weight function:

(w_4) $\lim_{r\to 0^+}\left(w(r)\log\dfrac{R}{r}\right)=0$,

(w_5) the weight $r\mapsto w(r)$ is monotone increasing in $(0,R)$, and

(w_6) there exists $\overline{r}\in(0,r_0)$ such that the function $r\mapsto w(r)\log\dfrac{R}{r}$ is monotone increasing in $(0,\overline{r})$.

For instance, the following functions satisfy (w_2) and the above conditions for $r>0$ near zero:

$$\frac{1}{\left(\log\frac{R}{r}\right)^{1+\varepsilon}},\quad \text{with } \varepsilon>0,\quad \text{and}\quad r^\alpha\left(\log\frac{R}{r}\right)^{-1+\varepsilon},\quad \text{with } \alpha>0 \text{ and } 0\le\varepsilon\le 1.$$

In particular, the weight $w(r)=r^\alpha$ with $\alpha>0$ satisfies also (w_4), (w_5) and (w_6), and hence Theorem 1.3-(i) is a particular case of the following result.

Proposition 3.2. *For any $n\ge 4$, $R>0$, and a weight function w satisfying (w_0), (w_1), (w_2), (w_3), (w_4), and (w_6), the supremum $S_n(\beta_n,w)$ is attained.*

Proof. Let $\{u_k\}_k$ be a maximizing sequence for $S_n(\beta_n,w)$, i.e. $u_k\in W^{2,\frac{n}{2}}_{\mathcal{N},\mathrm{rad}}(B_R)$, $\|\Delta u_k\|_{\frac{n}{2}}\le 1$, and

$$\lim_{k\to+\infty}\int_{B_R} e^{\left(\beta_n+w(|x|)\right)|u_k|^{\frac{n}{n-2}}}\,dx=S_n(\beta_n,w). \tag{3.2}$$

Without loss of generality, we can assume that $u_k\rightharpoonup u$ in $W^{2,\frac{n}{2}}_{\mathcal{N},\mathrm{rad}}(B_R)$, and $u_k\to u$ a.e. in B_R. First, we show that $u\neq 0$.

We argue by contradiction assuming that $u=0$. Let $a\in(0,R)$. In view of Remark 2.1, we can apply the Lebesgue dominated convergence Theorem and conclude that

$$\lim_{k\to+\infty}\int_{B_R\setminus B_a} e^{\left(\beta_n+w(|x|)\right)|u_k|^{\frac{n}{n-2}}}\,dx=|B_R\setminus B_a|=\lim_{k\to+\infty}\int_{B_R\setminus B_a} e^{\beta_n|u_k|^{\frac{n}{n-2}}}\,dx.$$

Moreover using again the pointwise estimate emphasized in Remark 2.1, we see that if $a\in(0,\overline{r})$ then the monotonicity condition (w_6) yields

$$\begin{aligned}\int_{B_a} e^{\left(\beta_n+w(|x|)\right)|u_k|^{\frac{n}{n-2}}}\,dx &\le e^{\frac{n}{\beta_n}w(a)\log\frac{R}{a}}\int_{B_a} e^{\beta_n|u_k|^{\frac{n}{n-2}}}\,dx\\ &\le \int_{B_a} e^{\beta_n|u_k|^{\frac{n}{n-2}}}\,dx+\left(e^{\frac{n}{\beta_n}w(a)\log\frac{R}{a}}-1\right)S_n(\beta_n).\end{aligned}\tag{3.3}$$

Combining (3.2) with (3.3), for any $a \in (0, \overline{r})$ we have

$$\begin{aligned} S_n(\beta_n, w) &= \lim_{k\to+\infty} \int_{B_R} e^{\left(\beta_n + w(|x|)\right)|u_k|^{\frac{n}{n-2}}}\, dx \\ &\leq \lim_{k\to+\infty} \int_{B_R} e^{\beta_n |u_k|^{\frac{n}{n-2}}}\, dx + \left(e^{\frac{n}{\beta_n} w(a)\log\frac{R}{a}} - 1\right) S_n(\beta_n) \\ &\leq e^{\frac{n}{\beta_n} w(a)\log\frac{R}{a}} S_n(\beta_n) \to S_n(\beta_n) \quad \text{as } a \to 0^+ \end{aligned}$$

since (w_4) holds. Therefore $S_n(\beta_n, w) \leq S_n(\beta_n)$ which contradicts (3.1), and $u \neq 0$.

We claim that there exists $q > 1$ such that

$$\sup_k \int_{B_R} e^{q\left(\beta_n + w(|x|)\right)|u_k|^{\frac{n}{n-2}}}\, dx < +\infty. \tag{3.4}$$

This yields

$$S_n(\beta_n, w) = \lim_{k\to+\infty} \int_{B_R} e^{\left(\beta_n + w(|x|)\right)|u_k|^{\frac{n}{n-2}}}\, dx = \int_{B_R} e^{\left(\beta_n + w(|x|)\right)|u|^{\frac{n}{n-2}}}\, dx,$$

which ensures that u is an extremal function for $S_n(\beta_n, w)$. Therefore the proof is complete if we show that (3.4) holds for some $q > 1$. On the one hand, from Remark 2.1, we deduce that for any $1 < q < \frac{1}{\gamma_1}$ and any $a \in (0, R)$ we have

$$\sup_k \int_{B_R \setminus B_a} e^{q\left(\beta_n + w(|x|)\right)|u_k|^{\frac{n}{n-2}}}\, dx < +\infty.$$

On the other hand, since $u \neq 0$, the concentration-compactness principle of Lions-type in [10] ensures the existence of $p > 1$ such that

$$M_p^* := \sup_k \int_{B_R} e^{p\beta_n |u_k|^{\frac{n}{n-2}}}\, dx < +\infty,$$

and we can choose $a \in (0, R)$ so small that

$$\overline{q} := p\left(1 - \frac{w(a)}{\beta_n}\right) > 1.$$

Then the monotonicity of w expressed by (w_5) enable us to estimate

$$\sup_k \int_{B_a} e^{\overline{q}\left(\beta_n + w(|x|)\right)|u_k|^{\frac{n}{n-2}}}\, dx \leq \sup_k \int_{B_a} e^{\overline{q}\beta_n\left(1 + \frac{w(a)}{\beta_n}\right)|u_k|^{\frac{n}{n-2}}}\, dx \leq M_p^* < +\infty.$$

In conclusion, it is enough to choose $1 < q < \min\{\frac{1}{\gamma_1}, \overline{q}\}$ to obtain (3.4). □

4. Analysis of the sharp inequality for $\tilde{S}_n(\beta, w)$

In this Section, we consider $\tilde{S}_n(\beta, w)$ defined by (1.4) with a weight $w = w(r)$ satisfying the two conditions (w_0) and (w_1) introduced in Section 2, and in addition

(w_2') there exists $\gamma_0 > 0$, $\mu > 2$, and $r_0 \in (0, R)$ such that

$$w(r) \leq \gamma_0 \left(\frac{1}{\log \frac{R}{r}} \right)^{\mu} \quad \text{for any } r \in (0, r_0).$$

The weight $w(r) = r^\alpha$ with $\alpha > 0$ satisfies (w_0), (w_1), and (w_2'), and Theorem 1.2 is a particular case of the next result.

Proposition 4.1. *For any $n \geq 3$, $R > 0$, and a weight function w satisfying (w_0), (w_1), and (w_2'), we have*

$$\tilde{S}_n(\beta_n, w) < +\infty \quad \textit{if and only if} \quad \beta \leq \beta_n. \tag{4.1}$$

Proof. Let $u \in W^{2,\frac{n}{2}}_{\mathcal{N},\mathrm{rad}}(B_R)$ be such that $\|\Delta u\|_{\frac{n}{2}} \leq 1$. Then arguing as in the proof of Proposition 2.1, see in particular the estimate (2.5), we have for any $0 < r \leq R$

$$\beta_n^{\frac{n-2}{n}} |u(r)| \leq \left(n \log \frac{R}{r} \right)^{\frac{n-2}{n}} \tag{4.2}$$

Set $\varrho_0 := Re^{-\frac{2}{n}}$, so that for any $\varrho_0 < r \leq R$

$$0 \leq \beta_n^{\frac{n-2}{n}} |u(r)| \leq 2^{\frac{n-2}{n}}.$$

Since $w = w(r)$ is positive in $(0, R]$ and $M_w := \max_{r \in [\varrho_0, R]} w(r) > 0$, we can estimate for any $\varrho_0 < r \leq R$

$$\left(\beta_n^{\frac{n-2}{n}} |u(r)| \right)^{\frac{n}{n-2} + w(r)} \leq 2^{1 + \frac{n-2}{n} w(r)} \leq 2^{1 + \frac{n-2}{n} M_w}.$$

Therefore, we have

$$\begin{aligned} \int_{B_R} e^{\left(\beta_n^{\frac{n-2}{n}} |u| \right)^{\frac{n}{n-2} + w(|x|)}} dx &= \omega_{n-1} \int_0^R e^{\left(\beta_n^{\frac{n-2}{n}} |u| \right)^{\frac{n}{n-2} + w(r)}} r^{n-1}\, dr \\ &\leq \omega_{n-1} \int_0^{\varrho_0} e^{\left(\beta_n^{\frac{n-2}{n}} |u| \right)^{\frac{n}{n-2} + w(r)}} r^{n-1}\, dr + e^{2^{1 + \frac{n-2}{n} M_w}} \cdot \frac{\omega_{n-1}}{n} R^n \end{aligned} \tag{4.3}$$

The proof of the inequality $\tilde{S}_n(\beta_n, w) < +\infty$ is complete if we obtain a *uniform* estimate of the integral on the right hand side of (4.3), and from now on we focus on the estimate of the integrand on $(0, \varrho_0)$.

Without loss of generality, we may assume that condition (w_2') holds up to ϱ_0, and that there exists $\tilde{\gamma}_0 > 0$ and $\mu > 2$ such that

$$w(r) \leq \tilde{\gamma}_0 \left(\frac{1}{\log \frac{R}{r}} \right)^{\mu} \quad \text{for any } r \in (0, \varrho_0).$$

Let $0 < r \leq \varrho_0$, then

$$n \log \frac{R}{r} \geq n \log \frac{R}{\varrho_0} = 2, \tag{4.4}$$

and we can estimate

$$\begin{aligned}
e^{\left(\beta_n^{\frac{n-2}{n}}|u|\right)^{\frac{n}{n-2}+w(r)}} &= e^{\beta_n|u|^{\frac{n}{n-2}}} \left[e^{\beta_n|u|^{\frac{n}{n-2}} \cdot \left(\beta_n^{\frac{n-2}{n}}|u|\right)^{w(r)} - \beta_n|u|^{\frac{n}{n-2}}} - 1 \right] + e^{\beta_n|u|^{\frac{n}{n-2}}} \\
&= e^{\beta_n|u|^{\frac{n}{n-2}}} \left[e^{\beta_n|u|^{\frac{n}{n-2}} \cdot \left[\left(\beta_n^{\frac{n-2}{n}}|u|\right)^{w(r)} - 1 \right]} - 1 \right] + e^{\beta_n|u|^{\frac{n}{n-2}}} \\
&\leq e^{\beta_n|u|^{\frac{n}{n-2}}} \left[e^{\beta_n|u|^{\frac{n}{n-2}} \cdot \left[\left(n \log \frac{R}{r}\right)^{\frac{n-2}{n} \cdot w(r)} - 1 \right]} - 1 \right] + e^{\beta_n|u|^{\frac{n}{n-2}}} \\
&\leq e^{n \log \frac{R}{r}} \left[e^{n \log \frac{R}{r} \cdot \left[\eta(r) - 1 \right]} - 1 \right] + e^{\beta_n|u|^{\frac{n}{n-2}}}
\end{aligned} \tag{4.5}$$

where

$$\eta(r) := \left(n \log \frac{R}{r} \right)^{\frac{n-2}{n} \tilde{\gamma}_0 \left(\frac{1}{\log \frac{R}{r}} \right)^{\mu}}.$$

As $r \to 0^+$, we have the following Taylor expansion

$$\begin{aligned}
\eta(r) - 1 &= \exp\left\{ \frac{n-2}{n} \tilde{\gamma}_0 \left(\frac{1}{\log \frac{R}{r}} \right)^{\mu} \cdot \log\left(n \log \frac{R}{r} \right) \right\} - 1 \\
&= \frac{n-2}{n} \tilde{\gamma}_0 \left(\frac{1}{\log \frac{R}{r}} \right)^{\mu} \cdot \log\left(n \log \frac{R}{r} \right) + o\left(\left(\frac{1}{\log \frac{R}{r}} \right)^{\mu} \cdot \log\left(n \log \frac{R}{r} \right) \right)
\end{aligned}$$

and hence (recalling that $\mu > 2$)

$$e^{n \log \frac{R}{r} \cdot \left[\eta(r) - 1 \right]} - 1 = \frac{(n-2)\tilde{\gamma}_0}{(\log \frac{R}{r})^{\mu-1}} \log\left(n \log \frac{R}{r} \right) + o\left(\frac{1}{(\log \frac{R}{r})^{\mu-1}} \log\left(n \log \frac{R}{r} \right) \right).$$

In other words, when $r > 0$ is near zero, we have

$$\theta(r) = \varphi(r) + o\big(\varphi(r) \big), \tag{4.6}$$

where

$$\theta(r) := e^{n\log\frac{R}{r}\cdot\left[\,\eta(r)-1\right]} - 1, \quad \text{and} \quad \varphi(r) := \frac{(n-2)\tilde{\gamma}_0}{(\log\frac{R}{r})^{\mu-1}} \cdot \log\left(n\log\frac{R}{r}\right).$$

The two functions θ and φ are continuous and, in view of (4.4), they are strictly positive on $(0, \varrho_0)$. Combining these properties with the validity of (4.6) near zero, we deduce the existence of a constant $C_{\varrho_0} > 0$ such that

$$\theta(r) \leq C_{\varrho_0}\varphi(r) \quad \text{for any } 0 < r \leq \varrho_0.$$

This yields

$$\int_0^{\varrho_0} e^{\left(\beta_n^{\frac{n-2}{n}}|u|\right)^{\frac{n}{n-2}+w(r)}} r^{n-1}\,dr \leq \int_0^{\varrho_0} e^{n\log\frac{R}{r}}\theta(r) r^{n-1}\,dr + \int_0^{\varrho_0} e^{\beta_n|u|^{\frac{n}{n-2}}} r^{n-1}\,dr$$
$$\leq C_{\varrho_0}R^n\int_0^{\varrho_0}\frac{\varphi(r)}{r}\,dr + S_n(\beta_n) = C_{\varrho_0}R^n\int_{\frac{2}{n}}^{+\infty}\frac{(n-2)\tilde{\gamma}_0}{t^{\mu-1}}\cdot\log(nt)\,dt + S_n(\beta_n).$$

The above estimate shows that $\tilde{S}_n(\beta_n, w) < +\infty$, indeed: the above integral on $(\frac{2}{n}, +\infty)$ is *finite* (since $\mu > 2$) and it is *independent* of u.

The proof of the sharpness of (4.1) follows from the sharpenss of Adams' inequality (1.2), in fact there exists a *sufficiently small* constant $C(\beta) > 0$ such that for any $0 \leq r \leq R$ and any $s \geq 0$

$$e^{\left(\beta^{\frac{n-2}{n}}s\right)^{\frac{n}{n-2}+w(r)}} \geq C(\beta)e^{\beta s^{\frac{n}{n-2}}}.$$

□

5. Attainability of $\tilde{S}_n(\beta_n, w)$

It is *not* clear how to obtain a sharp comparison between $\tilde{S}_n(\beta_n, w)$ and $S_n(\beta_n)$ as in (3.1). In order to prove the attainability of $\tilde{S}_n(\beta_n, w)$, we perform a more careful analysis by adapting the arguments in [21] to the weighted case.

The test function constructed in [21] to study $S_n(\beta_n)$ are of the form

$$v(r) = \frac{1}{\beta_n^{\frac{n-2}{n}}}\,V\left(n\,\log\frac{R}{r}\right), \quad 0 < r \leq R,$$

where if introduce the new variable $t \geq 0$ by setting $r = Re^{-\frac{t}{n}}$ then

$$V(t) = \begin{cases} a\left(e^{\frac{n-2}{n}t} - 1\right) - \frac{A_n}{n-2}t & 0 \leq t \leq \frac{n}{2}, \\ b + \lambda e^{\frac{n-2}{n}t} - \frac{1}{n-2}(t-1)^{\frac{n-2}{n}} + H(t) & \frac{n}{2} < t \leq N_n, \\ c_1 + c_2 e^{-\alpha t} + c_3 e^{-2\alpha t} & N_n < t \leq N_n(1+\zeta), \\ d & t > N_n(1+\zeta). \end{cases}$$

We refer to [21, Section 3] for the explicit definition of the parameters b, c_1, c_2, c_3, $d \in \mathbb{R}$, the definition of the function $H = H(t)$, and the conditions on the remaining parameters a, α, λ, $\zeta > 0$. To our purposes, it is enough to recall that

$$A_n := \frac{n-2}{n}\left(\frac{n-2}{2}\right)^{-\frac{2}{n}}, \quad N_n := \frac{n-2}{2}\exp\left\{\left(\frac{n}{n-2}\right)^{\frac{n}{2}} - \frac{n}{n-2}\right\} + 1,$$

and the following result is contained in [21, (3.22)-(3.23)-(3.24)-(3.26)].

Lemma 5.1 ([21])**.** *Let $n \geq 4$. There exists $\overline{\lambda} > 0$ such that if $\lambda > \overline{\lambda}$ then all the parameters defining V, and hence v, can be chosen so that $v \in W^{2,\frac{n}{2}}_{\mathcal{N},rad}(B_R)$, and $\|\Delta v\|_{\frac{n}{2}} \leq 1$. Moreover, if $\lambda > \overline{\lambda}$ then*

$$V(t) \geq \left(\lambda\frac{n-2}{n} - \frac{A_n}{n-2}\right)t \geq 0 \quad \textit{for any } t \in \left[0, \frac{n}{2}\right], \tag{5.1}$$

and

$$V(t) \geq f(t) \geq 0 \quad \textit{for any } t \in [0, +\infty), \tag{5.2}$$

where

$$f(t) = \begin{cases} A_n t & 0 \leq t \leq \frac{n}{2}, \\ (t-1)^{\frac{n-2}{n}} & \frac{n}{2} < t \leq N_n, \\ (N_n - 1)^{\frac{n-2}{n}} & t > N_n. \end{cases}$$

The above function f corresponds to a particular case of the test functions considered by S. Hudson and M. Leckband [12, Section 2], and they obtained the following fundamental estimate.

Lemma 5.2 ([12]). *If $n \geq 4$ then*

$$\frac{N_n - 1}{e} > \exp\Big\{ \psi\Big(\frac{n}{2}\Big) + \gamma \Big\}$$

where $\psi(x) := \frac{\Gamma'(x)}{\Gamma(x)}$, Γ is the gamma Euler function, and γ is the Euler constant, i.e. $\gamma := \lim_{j\to+\infty}\Big(\sum_{i=1}^{j} \frac{1}{i} - \log j \Big)$.

Using the above results, we obtain the following estimate from below for $\tilde{S}_n(\beta_n, w)$.

Proposition 5.3. *For any $n \geq 4$, $R > 0$, and a weight function w satisfying (w_0) and (w_1), we have*

$$\tilde{S}_n(\beta_n, w) > |B_R| \Big(1 + \exp\Big\{ \psi\Big(\frac{n}{2}\Big) + \gamma \Big\} \Big).$$

Proof. Let v, V, and f be as in Lemma 5.1 with $\lambda > \overline{\lambda}$ to be chosen during the proof. It is enough to show that

$$\int_{B_R} e^{\big(\beta_n^{\frac{n-2}{n}}|v|\big)^{\frac{n}{n-2}+w(|x|)}}\,dx > |B_R| \Big(1 + \exp\Big\{ \psi\Big(\frac{n}{2}\Big) + \gamma \Big\} \Big). \tag{5.3}$$

If we set $\overline{w}(t) := w(\, Re^{-\frac{t}{n}}\,)$ then we can rewrite

$$\begin{aligned}\int_{B_R} e^{\big(\beta_n^{\frac{n-2}{n}}|v|\big)^{\frac{n}{n-2}+w(|x|)}}\,dx &= \omega_{n-1} \int_0^R e^{\big(\beta_n^{\frac{n-2}{n}}v\big)^{\frac{n}{n-2}+w(r)}}\,r^{n-1}\,dr \\ &= |B_R| \int_0^{+\infty} e^{[V]^{\frac{n}{n-2}+\overline{w}(t)}-t}\,dt\end{aligned}$$

and we can estimate

$$\int_0^{+\infty} e^{[V]^{\frac{n}{n-2}+\overline{w}(t)}-t}\,dt \geq \int_0^1 e^{[f]^{\frac{n}{n-2}+\overline{w}(t)}-t}\,dt + \int_1^{\frac{n}{2}} e^{[V]^{\frac{n}{n-2}+\overline{w}(t)}-t}\,dt + \int_{\frac{n}{2}}^{+\infty} e^{[f]^{\frac{n}{n-2}}-t}\,dt$$

where we used the positivity of $\overline{w} = \overline{w}(t)$ in $(0, +\infty)$, the estimate (5.2) in Lemma 5.1, and $f \geq 1$ in $[\frac{n}{2}, +\infty)$. Recalling that $f \geq 0$ in $[0, +\infty)$, in particular we have for any $t \geq 0$

$$[f]^{\frac{n}{n-2}+\overline{w}(t)} - t \geq -t,$$

and hence

$$\int_0^1 e^{[f]^{\frac{n}{n-2}+\overline{w}(t)}-t}\,dt \geq \int_0^1 e^{-t}\,dt = 1 - \frac{1}{e}. \tag{5.4}$$

Moreover, we can compute

$$\int_{\frac{n}{2}}^{+\infty} e^{[f]^{\frac{n}{n-2}}-t}\,dt = \frac{N_n-1}{e}+\frac{1}{e}\left(2-\frac{n}{2}\right). \tag{5.5}$$

Next, we point out that it is possible to choose $\lambda > \overline{\lambda}$ sufficiently large so that

$$[V(t)]^{\frac{n}{n-2}+\overline{w}(t)} \geq [f(t)]^{\frac{n}{n-2}} \quad \text{for any } t \in \left(1, \frac{n}{2}\right). \tag{5.6}$$

In fact, in view of (5.1), it is enough to check that

$$\left[\left(\lambda\frac{n-2}{n}-\frac{A_n}{n-2}\right)t\right]^{1+\frac{n-2}{n}\cdot\overline{w}(t)} \geq A_n t \quad \text{for any } t \in \left(1, \frac{n}{2}\right),$$

and this is the case if for instance we choose $\lambda > \overline{\lambda}$ satisfying $\lambda\frac{n-2}{n}-\frac{A_n}{n-2} \geq A_n\frac{n}{2} (\geq 1)$.

With this choice of λ, using (5.6), we can estimate

$$\begin{aligned}\int_1^{\frac{n}{2}} e^{[V]^{\frac{n}{n-2}+\overline{w}(t)}-t}\,dt &\geq \int_1^{\frac{n}{2}} e^{[f]^{\frac{n}{n-2}}-t}\,dt = \frac{n}{2}\int_{\frac{2}{n}}^{1} e^{\frac{n-2}{2}s^{\frac{n}{n-2}}-\frac{n}{2}s}\,ds\\ &\geq \frac{n}{2}\int_{\frac{2}{n}}^{1} e^{-1}\,ds = \frac{1}{e}\left(\frac{n}{2}-1\right),\end{aligned}$$

and if we combine this last inequality with (5.4), (5.5), and Lemma 5.2, we obtain the desired conclusion (5.3). □

We also recall another sharp estimate due to S. Hudson and M. Leckband [12] which will be crucial in our analysis.

Lemma 5.4 ([12])**.** *Let $p > 1$ and let $\{\phi_k\}_k$ be a sequence of measurable functions $\phi_k : [0,+\infty) \to [0,+\infty)$ satisfying*

$$\int_0^{+\infty} [\phi_k(\tau)]^p\,d\tau \leq 1 \quad \textit{and} \quad \lim_{k\to+\infty}\int_0^A [\phi_k(\tau)]^p\,d\tau = 0 \quad \textit{for any fixed } A > 0.$$

Then

$$\lim_{k\to+\infty}\int_0^{+\infty} \exp\Big\{\Big(\int_0^t \phi_k(\tau)\,d\tau\Big)^{p'} - t\Big\}\,dt \leq 1+\exp\Big\{\psi(p)+\gamma\Big\}.$$

We replace condition (w_2') with the following additional conditions:

(w_3') $\lim_{r\to 0^+}\left(w(r)\log\frac{R}{r}\cdot\log\left(\log\frac{R}{r}\right)\right) = 0,$

(w_4') there exists $\overline{\varrho} \in (0,R)$ such that $\frac{1}{r}w(r)\log\frac{R}{r}\cdot\log\left(\log\frac{R}{r}\right) \in L^1(0,\overline{\varrho}),$

(w_5') the weight $r \mapsto w(r)$ is monotone increasing in $(0,R)$, and

(w_6') there exists $\overline{r} \in (0, r_0)$ such that the function $r \mapsto w(r) \log\left(n \log \frac{R}{r}\right)$ is monotone increasing in $(0, \overline{r})$.

In particular, the weight $w(r) = r^\alpha$ with $\alpha > 0$ satisfies also (w_3'), (w_4'), (w_5'), and (w_6'), and hence Theorem 1.3-(ii) is a particular case of the following result.

Proposition 5.5. *For any $n \geq 4$, $R > 0$, and a weight function w satisfying (w_0), (w_1), (w_3'), (w_4'), (w_5'), and (w_6'), the supremum $\tilde{S}_n(\beta_n, w)$ is attained.*

Proof. Let $\{u_k\}_k$ be a maximizing sequence for $\tilde{S}_n(\beta_n, w)$, i.e. $u_k \in W^{2,\frac{n}{2}}_{\mathcal{N},\mathrm{rad}}(B_R)$, $\|\Delta u_k\|_{\frac{n}{2}} \leq 1$, and

$$\lim_{k \to +\infty} \int_{B_R} e^{\left(\beta_n^{\frac{n-2}{n}} |u_k|\right)^{\frac{n}{n-2} + w(|x|)}} dx = \tilde{S}_n(\beta_n, w).$$

Without loss of generality, we can assume that $u_k \rightharpoonup u$ in $W^{2,\frac{n}{2}}_{\mathcal{N},\mathrm{rad}}(B_R)$, and $u_k \to u$ a.e. in B_R. As in (4.2), we have for any $0 < r \leq R$

$$\beta_n^{\frac{n-2}{n}} |u_k(r)| \leq \left(n \log \frac{R}{r}\right)^{\frac{n-2}{n}} \tag{5.7}$$

and with this estimate, it is easy to see that for any $\varrho \in (0, R)$

$$\lim_{k \to +\infty} \int_{B_R \setminus B_\varrho} e^{\left(\beta_n^{\frac{n-2}{n}} |u_k|\right)^{\frac{n}{n-2} + w(|x|)}} dx = \int_{B_R \setminus B_\varrho} e^{\left(\beta_n^{\frac{n-2}{n}} |u|\right)^{\frac{n}{n-2} + w(|x|)}} dx. \tag{5.8}$$

In fact, if $r \geq \varrho > 0$ then

$$\left(\beta_n^{\frac{n-2}{n}} |u_k|\right)^{\frac{n}{n-2} + w(r)} \leq \left(n \log \frac{R}{r}\right) e^{\frac{n}{n-2} w(r) \log\left(n \log \frac{R}{r}\right)} \leq \left(n \log \frac{R}{\varrho}\right) e^{\frac{n}{n-2} w(\varrho) \log\left(n \log \frac{R}{\varrho}\right)}$$

where we also used the monotonicity of the weight (w_5'). Since the right hand side is a positive constant independent of k, the convergence expressed by (5.8) is a consequence of the Lebesgue dominated convergence theorem. Similarly, for any $\varrho \in (0, R)$

$$\lim_{k \to +\infty} \int_{B_R \setminus B_\varrho} e^{\beta_n |u_k|^{\frac{n}{n-2}}} dx = \int_{B_R \setminus B_\varrho} e^{\beta_n |u|^{\frac{n}{n-2}}} dx. \tag{5.9}$$

Arguing as in the proof of Proposition 2.1, see in particular the identity (2.4), we have for any $0 < r \leq R$

$$\beta_n^{\frac{n-2}{n}} u_k(r) = \frac{n-2}{n} \int_{|B_r|}^{|B_R|} \frac{G_k(t)}{t^{1-\frac{2}{n}}} dt, \quad 0 < r \leq R, \tag{5.10}$$

where G_k is defined as in (2.2), and satisfies

$$\int_0^{|B_R|} |G_k(t)|^{\frac{n}{2}}\, dt \leq \left(\frac{n}{n-2}\right)^{\frac{n}{2}},$$

as already shown in (2.3). Clearly the following alternative holds:

- either for any $0 < r \leq R$

$$\lim_{k\to+\infty} \int_{|B_r|}^{|B_R|} |G_k(t)|^{\frac{n}{2}}\, dt = 0 \tag{5.11}$$

- or there exist $\varrho_0 \in (0,R)$ and $\delta \in (0,1)$ such that

$$\int_{|B_{\varrho_0}|}^{|B_R|} |G_k(t)|^{\frac{n}{2}}\, dt \geq \left(\frac{n}{n-2}\right)^{\frac{n}{2}} \delta. \tag{5.12}$$

We will prove that the first case *cannot* happen – namely, (5.11) *cannot* hold for any $0 < r \leq R$. Then we will complete the proof by showing that the validity of (5.12) for some $\varrho_0 \in (0,R)$ and $\delta \in (0,1)$ yields the attainability of $\tilde{S}_n(\beta_n, w)$.

Let us assume that (5.11) holds for any $0 < r \leq R$. Then in particular $u = 0$, and we will show that the contribution of the weight is negligible as $k \to +\infty$ in the sense that

$$\tilde{S}_n(\beta_n, w) = \lim_{k\to+\infty} \int_{B_R} e^{\left(\beta_n^{\frac{n-2}{n}} |u_k|\right)^{\frac{n}{n-2}+w(|x|)}} dx \leq \lim_{k\to+\infty} \int_{B_R} e^{\beta_n |u_k|^{\frac{n}{n-2}}}\, dx. \tag{5.13}$$

This will enable us to reach a contradiction as a consequence of the (*unweighted*) result of S. Hudson and M. Leckband [12], see Lemma 5.4.

Since $u = 0$, the contribution of the weight is negligible on $B_R \setminus B_\varrho$ for any $\varrho \in (0,R)$. In fact, (5.8) and (5.9) yield

$$\lim_{k\to+\infty} \int_{B_R\setminus B_\varrho} e^{\left(\beta_n^{\frac{n-2}{n}} |u_k|\right)^{\frac{n}{n-2}+w(|x|)}} dx = |B_R \setminus B_\varrho| = \lim_{k\to+\infty} \int_{B_R\setminus B_\varrho} e^{\beta_n |u_k|^{\frac{n}{n-2}}}\, dx. \tag{5.14}$$

Now we assume that $0 < \varrho < Re^{-\frac{t}{n}}$. Then, arguing as in (4.5), we can estimate for any $0 < r \leq \varrho$

$$\begin{aligned} e^{\left(\beta_n^{\frac{n-2}{n}} |u_k|\right)^{\frac{n}{n-2}+w(|x|)}} &\leq e^{\beta_n |u|^{\frac{n}{n-2}}} \left[e^{\beta_n |u|^{\frac{n}{n-2}} \cdot \left[\left(n \log \frac{R}{r}\right)^{\frac{n-2}{n}\cdot w(r)} - 1\right]} - 1 \right] + e^{\beta_n |u_k|^{\frac{n}{n-2}}} \\ &\leq e^{n \log \frac{R}{r}} \left[e^{h(r)} - 1 \right] + e^{\beta_n |u_k|^{\frac{n}{n-2}}} \end{aligned}$$

where

$$h(r) := n\log\frac{R}{r}\cdot\left[\left(n\log\frac{R}{r}\right)^{\frac{n-2}{n}\cdot w(r)}-1\right]$$

and hence

$$\int_{B_\varrho} e^{\left(\beta_n^{\frac{n-2}{n}}|u_k|\right)^{\frac{n}{n-2}+w(|x|)}}\,dx \le \omega_{n-1}R^n\int_0^\varrho \frac{1}{r}\left[\,e^{h(r)}-1\,\right]\,dr+\int_{B_\varrho} e^{\beta_n|u_k|^{\frac{n}{n-2}}}\,dx. \tag{5.15}$$

Notice that in view of (w_3'), as $r\to 0^+$, we have the following Taylor expansion

$$\begin{aligned} h(r) &= n\log\frac{R}{r}\cdot\left[e^{\frac{n-2}{n}w(r)\log\left(n\log\frac{R}{r}\right)}-1\right]\\ &= n\log\frac{R}{r}\cdot\left[\frac{n-2}{n}w(r)\log\left(n\log\frac{R}{r}\right)+o\left(\,w(r)\log\left(n\log\frac{R}{r}\right)\right)\right]\\ &= (n-2)w(r)\log\frac{R}{r}\cdot\log\left(n\log\frac{R}{r}\right)+o\left(\,w(r)\log\frac{R}{r}\cdot\log\left(n\log\frac{R}{r}\right)\right) \end{aligned}$$

and hence the integrability condition (w_4') ensures that if $0<\varrho<\overline{\varrho}$ then

$$\frac{1}{r}\left[\,e^{h(r)}-1\,\right]\in L^1(\,(0,\varrho)\,)$$

which yields

$$I(\varrho) := \omega_{n-1}R^n\int_0^\varrho \frac{1}{r}\left[\,e^{h(r)}-1\,\right]\,dr\to 0 \quad \text{as } \varrho\to 0^+.$$

Combining (5.14) with (5.15), we get that for any $\varepsilon>0$ there exists $k_\varepsilon\ge 1$ such that for any $k\ge k_\varepsilon$ we have

$$\begin{aligned} \int_{B_R} e^{\left(\beta_n^{\frac{n-2}{n}}|u_k|\right)^{\frac{n}{n-2}+w(|x|)}}\,dx &= \int_{B_R\setminus B_\varrho}+\int_{B_\varrho} e^{\left(\beta_n^{\frac{n-2}{n}}|u_k|\right)^{\frac{n}{n-2}+w(|x|)}}\,dx\\ &\le \int_{B_R} e^{\beta_n|u_k|^{\frac{n}{n-2}}}\,dx+I(\varrho)+\varepsilon \end{aligned}$$

and letting $k\to+\infty$ and then $\varrho\to 0^+$, we obtain the desired estimate (5.13).

Using the change of variable $t=|B_R|e^{-\tau}$, we can equivalently rewrite

$$\beta_n^{\frac{n-2}{n}}u_k(r)=\int_0^{n\log\frac{R}{r}}\phi_k(\tau)\,d\tau,\quad 0<r\le R,$$

where

$$\phi_k(\tau) := \frac{n-2}{n}(|B_R|e^{-\tau})^{\frac{2}{n}}G_k(|B_R|e^{-\tau}),\quad \tau>0,$$

and ϕ_k satisfies

$$\int_0^{+\infty} |\phi_k(\tau)|^{\frac{n}{2}}\, d\tau = \left(\frac{n-2}{n}\right)^{\frac{n}{2}} \int_0^{|B_R|} |G_k(t)|^{\frac{n}{2}}\, dt \le 1.$$

Performing again a suitable change of variable ($|x| = r = Re^{-\frac{t}{n}}$), we have

$$\begin{aligned}\lim_{k\to+\infty} \int_{B_R} e^{\beta_n |u_k|^{\frac{n}{n-2}}}\, dx &= \lim_{k\to+\infty} \omega_{n-1} \int_0^R \exp\left\{ \left| \int_0^{n\log\frac{R}{r}} \phi_k(\tau)\, d\tau \right|^{\frac{n}{n-2}} \right\} r^{n-1}\, dr \\ &\le \lim_{k\to+\infty} |B_R| \int_0^{+\infty} \exp\left\{ \left(\int_0^t |\phi_k(\tau)|\, d\tau \right)^{\frac{n}{n-2}} - t \right\} dt.\end{aligned}$$

Moreover, for any $A > 0$

$$\begin{aligned}\int_0^A |\phi_k(\tau)|^{\frac{n}{2}}\, d\tau &= \left(\frac{n-2}{n}\right)^{\frac{n}{2}} \int_0^A |B_R| e^{-\tau} \mid G_k(|B_R| e^{-\tau}) \mid^{\frac{n}{2}} d\tau \\ &= \left(\frac{n-2}{n}\right)^{\frac{n}{2}} \int_{|B_R|e^{-A}}^{|B_R|} |G_k(t)|^{\frac{n}{2}}\, dt \to 0 \quad \text{as } k \to +\infty,\end{aligned}$$

as a consequence of (5.11) with $r = |B_R| e^{-A} \in (0, R)$, and hence Lemma 5.4 enable us to reach a contradiction. In fact, it yields

$$\tilde{S}_n(\beta_n, w) = \lim_{k\to+\infty} \int_{B_R} e^{\beta_n |u_k|^{\frac{n}{n-2}}}\, dx \le |B_R| \left(1 + \exp\left\{ \psi\left(\frac{n}{2}\right) + \gamma \right\} \right)$$

which contradicts Proposition 5.3.

Since we ruled out the validity of (5.11) for any $0 \le r \le R$, condition (5.12) must hold for some $\varrho_0 \in (0, R)$ and $\delta \in (0, 1)$. In particular, for any $k \ge k_0$

$$\int_0^{|B_{\varrho_0}|} |G_k(t)|^{\frac{n}{2}}\, dt \le \left(\frac{n}{n-2}\right)^{\frac{n}{2}} (1 - \delta), \tag{5.16}$$

and the idea is to exploit this property to show that for any $\varepsilon > 0$ there exists $\overline{\varrho}_\varepsilon \in (0, \varrho_0)$ such that

$$\int_{B_{\overline{\varrho}_\varepsilon}} e^{\left(\beta_n^{\frac{n-2}{n}} |u_k|\right)^{\frac{n}{n-2} + w(|x|)}}\, dx \le \varepsilon \quad \text{for any } k \ge k_0. \tag{5.17}$$

This ensures that the weak limit u is an extremal function for $\tilde{S}_n(\beta_n, w)$. In fact, combining (5.8) with (5.17), we get

$$\int_{B_R} e^{\left(\beta_n^{\frac{n-2}{n}}|u|\right)^{\frac{n}{n-2}+w(|x|)}}\,dx \geq \lim_{k\to+\infty}\int_{B_R\setminus B_{\overline{\varrho}_\varepsilon}} e^{\left(\beta_n^{\frac{n-2}{n}}|u_k|\right)^{\frac{n}{n-2}+w(|x|)}}\,dx$$
$$\geq \lim_{k\to+\infty}\int_{B_R} e^{\left(\beta_n^{\frac{n-2}{n}}|u_k|\right)^{\frac{n}{n-2}+w(|x|)}}\,dx - \varepsilon = \tilde{S}_n(\beta_n, w) - \varepsilon.$$

Passing to the limit as $\varepsilon \to 0^+$, we obtain the desired conclusion, i.e.

$$\int_{B_R} e^{\left(\beta_n^{\frac{n-2}{n}}|u|\right)^{\frac{n}{n-2}+w(|x|)}}\,dx \geq \tilde{S}_n(\beta_n, w).$$

Therefore, the proof is complete if we show that (5.17) holds. Using (5.10), (5.16), and Hölder inequality, we can estimate for any $k \geq k_0$ and any $0 < r \leq \varrho_0$

$$\beta_n^{\frac{n-2}{n}}\left|u_k(r) - u_k(\varrho_0)\right| = \frac{n-2}{n}\left|\int_{|B_r|}^{|B_{\varrho_0}|}\frac{G_k(t)}{t^{1-\frac{2}{n}}}\,dt\right|$$
$$\leq \frac{n-2}{n}\left(\int_{|B_r|}^{|B_{\varrho_0}|}|G_k(t)|^{\frac{n}{2}}\,dt\right)^{\frac{2}{n}}\left(\int_{|B_r|}^{|B_{\varrho_0}|}\frac{1}{t}\,dt\right)^{\frac{n-2}{n}}$$
$$\leq (1-\delta)^{\frac{2}{n}}\left(n\,\log\frac{\varrho_0}{r}\right)^{\frac{n-2}{n}}$$

and hence

$$\beta_n\,|u_k(r) - u_k(\varrho_0)|^{\frac{n}{n-2}} \leq (1-\delta)^{\frac{2}{n-2}}\,n\,\log\frac{\varrho_0}{r}.$$

Since

$$(1-\delta)^{\frac{2}{n-2}} \leq 1 - c_n\delta \quad \text{with } c_n := \begin{cases} 1 & \text{if } n = 3, 4, \\ \frac{2}{n-2} & \text{otherwise,}\end{cases}$$

we conclude that

$$\beta_n\,|u_k(r) - u_k(\varrho_0)|^{\frac{n}{n-2}} \leq (1 - c_n\delta)\,n\,\log\frac{\varrho_0}{r}. \tag{5.18}$$

In view of the following elementary inequality

$$(a+b)^p \leq (1+\varepsilon)a^p + \left(1 - \frac{1}{(1+\varepsilon)^{\frac{1}{p-1}}}\right)^{1-p} b^p\,,$$

which holds for any a, $b \geq 0$, $p > 1$ and $\varepsilon > 0$, we can use together (5.7) with $r = \varrho_0$ and the above estimate (5.18) to get

$$\begin{aligned}\beta_n |u_k(r)|^{\frac{n}{n-2}} &\leq (1+\varepsilon)\, \beta_n\, |u_k(r) - u_k(\varrho_0)|^{\frac{n}{n-2}} + C(\varepsilon, n)\, \beta_n\, |u_k(\varrho_0)|^{\frac{n}{n-2}} \\ &\leq (1+\varepsilon)\,(1 - c_n\delta)\, n\, \log\frac{\varrho_0}{r} + C(\varepsilon, n)\, n\, \log\frac{R}{\varrho_0}.\end{aligned}$$

If we choose $\varepsilon = c_n\delta$ and we denote by $\varphi = \varphi(r)$ the corresponding right hand side of the above estimate, that is

$$\varphi(r) := (1 - c_n^2\delta^2)\, n\, \log\frac{\varrho_0}{r} + C(\delta, n)\, n\, \log\frac{R}{\varrho_0},$$

then we obtain for any $k \geq k_0$ and any $0 < r \leq \varrho_0$

$$\beta_n^{\frac{n-2}{n}} |u_k(r)| \leq [\varphi(r)]^{\frac{n-2}{n}}$$

and

$$\left(\beta_n^{\frac{n-2}{n}} |u_k(r)|\right)^{\frac{n}{n-2} + w(r)} \leq [\varphi(r)]^{1 + \frac{n-2}{n} w(r)} = \varphi(r) e^{\frac{n-2}{n} w(r) \log\varphi(r)}.$$

Let $0 < \overline{\varrho} < \min\{\varrho_0, \overline{r}\}$ with $\overline{r} > 0$ given by (w_6'). Then (w_6') ensures the monotonicity of the function $r \mapsto w(r)\log\varphi(r)$ in $(0, \overline{\varrho})$ and

$$\varphi(r) e^{\frac{n-2}{n} w(r) \log\varphi(r)} \leq \varphi(r) e^{\frac{n-2}{n} w(\overline{\varrho}) \log\varphi(\overline{\varrho})}, \quad 0 < r < \overline{\varrho}.$$

Moreover, we can choose $\overline{\varrho}$ even smaller so that (w_3') guarantees

$$e^{\frac{n-2}{n} w(\overline{\varrho}) \log\varphi(\overline{\varrho})} \leq 1 + c_n^2\delta^2.$$

This choice of $\overline{\varrho} > 0$ small is *independent* of k, and hence for any $k \geq k_0$ we have

$$\begin{aligned}\int_{B_{\overline{\varrho}}} e^{\left(\beta_n^{\frac{n-2}{n}} |u_k|\right)^{\frac{n}{n-2} + w(|x|)}} dx &= \omega_{n-1} \int_0^{\overline{\varrho}} e^{\left(\beta_n^{\frac{n-2}{n}} |u_k(r)|\right)^{\frac{n}{n-2} + w(r)}} r^{n-1}\, dr \\ &\leq \omega_{n-1} \int_0^{\overline{\varrho}} e^{(1 + c_n^2\delta^2)\varphi(r)} r^{n-1}\, dr = \omega_{n-1} \varrho_0^{n(1 - c_n^4\delta^4)} \left(\frac{R}{\varrho_0}\right)^{\tilde{C}(\delta, n)} \int_0^{\overline{\varrho}} r^{nc_n^4\delta^4 - 1}\, dr \\ &= \omega_{n-1} \varrho_0^{n(1 - c_n^4\delta^4)} \left(\frac{R}{\varrho_0}\right)^{\tilde{C}(\delta, n)} \frac{\overline{\varrho}^{nc_n^4\delta^4}}{nc_n^4\delta^4}\end{aligned}$$

Since $nc_n^4\delta^4 > 0$, the last term tends to zero as $\overline{\varrho} \to 0^+$. This yields (5.17), and the proof is complete.

□

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