THE BASIC PROBLEM OF THE CALCULUS OF VARIATIONS: DU BOIS-REYMOND EQUATION, REGULARITY OF MINIMIZERS AND OF MINIMIZING SEQUENCES
IL PROBLEMA CLASSICO DEL CALCOLO DELLE VARIAZIONI: L'EQUAZIONE DI DU BOIS-REYMOND, LA REGOLARITÀ DEI MINIMI E DELLE SUCCESSIONI MINIMIZZANTI

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Abstract. We consider the basic problem of the Calculus of variations of minimizing an integral functional among the absolutely continuous functions that satisfy prescribed boundary conditions. We resume the state of the art and our recent contributions concerning the validity of the Du Bois-Reymond condition, and the Lipschitz regularity of the minimizers and of minimizing sequences (e.g., Lavrentiev phenomenon).

Sunto. In questo articolo consideriamo il problema classico in Calcolo delle Variazioni dove si minimizza un funzionale integrale sull'insieme delle funzioni assolutamente continue che soddisfano delle condizioni al bordo predeterminate. Partendo da un breve riassunto sullo stato dell'arte discutiamo alcuni nostri recenti risultati che riguardano la validità della condizione di Du Bois-Reymond, e la regolarità di tipo Lipschitz dei minimi e di successioni minimizzanti (argomento legato anche al fenomeno di Lavrentiev).

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1. Introduction

The aim of the paper is to illustrate some recent developments on necessary optimality conditions, minimizers regularity and the Lavrentiev phenomenon for the basic problem

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(P) of the calculus of variations

$$\min F(y) = \int_a^b L(t, y(t), y'(t))\, dt : \; y \in W^{1,1}([a,b]; \mathbb{R}^n),\; y(a) = A, y(b) = B.$$

When we began to work together on the subject, many results were already available in the literature, in particular for the autonomous case: we refer mainly to the Lipschitz regularity of minimizers under superlinearity and to the nonoccurrence of the Lavrentiev phenomenon. We say that problem ($\mathrm{P}_{a,b}$) exhibits the Lavrentiev phenomenon when the infimum value over the absolutely continuous admissible arcs is strictly less than the infimum over the Lipschitz continuous admissible arcs. In both cases the results were finally expressed for a wide class of autonomous Lagrangians, imposing very weak assumptions. In the autonomous case Clarke and Vinter [16] established the Lipschitz continuity of minimizers just assuming the Tonelli existence hypotheses; subsequently, this result was extended in [18] to Borel measurable (autonomous) Lagrangians which are superlinear (w.r.t. the velocity variable). The nonoccurrence of the Lavrentiev phenomenon was obtained by Alberti and Serra Cassano [1] for Borel positive Lagrangians that are bounded on bounded sets. At that time, in the nonautonomous case, no result in the literature covered the most general result achieved in the autonomous case: there were always some extra assumptions on the state or velocity variable, not needed in the autonomous case. The challenge was to remove these extra conditions, and find the 'good' additional hypotheses for the nonautonomous case. This was not due only to a taste of abstraction: many applied problems present possibly discontinuous, nonconvex Lagrangians.

We were lucky enough to establish, at the very beginning of our work, a variational inequality for minimizers that turned out to imply the classical Du Bois-Reymond equation in the smooth case. In the autonomous case it is valid once the Lagrangian is just Borel, in the nonautonomous case it requires a non smooth extension of Cesari's assumption (S) [13] (see Definition 3.4 below), concerns just the time variable. This variational inequality actually gave more than that: it implied a sort of radial convexity of the Lagrangian along the rays of the velocities of the minimizers.

The new perspective provided by these findings allowed us to derive extensions of the classical results on Lipschitz regularity of minimizers to a very general context. This includes situations in which we assume very weak growth conditions (in place of superlinearity), as the ones introduced by Clarke in the pioneering work [14] and by Cellina in [10].

Some of these ideas turned out to be useful also in the study of the Lavrentiev phenomenon: Condition (S) is the key point in [23] to extend the results of [1] to the nonautonomous case, and radial convexity is a key tool to study the Lavrentiev phenomenon in the presence of a state constraint of the form $y([a,b]) \subset \Delta$ (where $\Delta \subset \mathbb{R}^n$ is a given set), considered in a previous work by Cellina and Ferriero in [11]. A new fact that emerged in the study of the Lavrentiev phenomenon is the importance of the boundary conditions: the assumptions that ensure its nonoccurrence with just one end-point constraint are not enough, in general, when one wants to preserve both end-point conditions.

In our work we investigated extensively also the delicate case of extended valued Lagrangians (cf. [4, 6, 5, 21, 22]), that is not considered in this presentation for the sake of clarity.

Due to Covid restrictions the seminar took place online in October 2020; what, at first glance, seemed to be a drawback, actually allowed the participation of many colleagues and friends staying far away from Bologna. Two years passed before writing down these notes: we felt that many results were still in progress, many of them have just very recently been submitted. We are grateful to Giovanni Cupini and Annamaria Montanari for having organized it and twice to Annamaria for her patience.

2. Notation and Basic assumptions

2.1. **Basic Assumptions.** Let $n \in \mathbb{N}, n \geq 1$. The functional F (sometimes referred to as the "energy") is defined by

$$\forall y \in W^{1,1}(I;\mathbb{R}^n) \qquad F(y) := \int_I L(s, y(s), y'(s))\, ds.$$

Basic Assumptions. *We assume the following conditions.*

- $I = [a,b]$ *is a closed, bounded interval of* $\mathbb{R}$*;*
- $L : I \times \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[$ *is Lebesgue-Borel measurable in* $(s,(y,v))$*, i.e., measurable with respect to the σ-algebra generated by the products of Lebesgue measurable*

subsets of I *(for* t*) and Borel measurable subsets of* $\mathbb{R}^n \times \mathbb{R}^n$ *(for* (y, v)*): this guarantees that if* $y, v : I \to \mathbb{R}^n$ *are measurable then* $s \mapsto L(s, y(s), v(s))$ *is measurable (see* [15, Proposition 6.34]*).*

2.2. **Notation.** We introduce the main recurring notation:

- The Euclidean norm of $x \in \mathbb{R}^n$ is denoted by $|x|$;
- The closed ball of $\mathbb{R}^n$ centered in the origin of radius $K \geq 0$ is denoted by B_K;
- If $y : I \to \mathbb{R}^n$ is a function, we denote by $y(I)$ its image, by $\|y\|_\infty$ its sup-norm and by $\|y\|_1$ its norm in $L^1(I; \mathbb{R}^n)$;
- $W^{1,1}(I; \mathbb{R}^n) = \{y : I \to \mathbb{R}^n \,:\, y, y' \in L^1(I; \mathbb{R}^n)\}$.

We fix $A, B \in \mathbb{R}^n$ and consider the problems

$$(\mathrm{P}_{a,b}) \qquad \min F(y) : y \in W^{1,1}(I; \mathbb{R}^n),\ y(a) = A, y(b) = B$$

and

$$(\mathrm{P}_{a}) \qquad \min F(y) : y \in W^{1,1}(I; \mathbb{R}^n),\ y(a) = A.$$

Absolutely continuous functions that satisfy the boundary condition(s) will be said to be *admissible.*

Definition 2.1. *L is said to be* superlinear *if there exists* $\Theta : [0, +\infty[\to [0, +\infty[$ *such that, for almost every* $t \in I$,

$$(\mathrm{G}_\Theta) \qquad L(t, y, v) \geq \Theta(|v|) \quad \forall (y, v) \in \mathbb{R}^n \times \mathbb{R}^n, \quad \lim_{r \to +\infty} \frac{\Theta(r)}{r} = +\infty.$$

3. The autonomous case

We consider here the case when the Lagrangian is *autonomous* (i.e. $L = L(y, v)$). Theorem 3.1 below is a refinement of a celebrated result by Clarke and Vinter in [16].

Theorem 3.1 (Lipschitz regularity for superlinear Lagrangians - Dal Maso & Frankowska (2003), [18])**.** *Let L be autonomous, superlinear and bounded on bounded sets. If y_* is a minimizer of* $(\mathrm{P}_{a,b})$ *then y_* is Lipschitz.*

Notice in particular that, in the autonomous case, Tonelli's existence hypotheses (L is continuous, convex in the velocity variable v and superlinear) ensure the Lipschitz regularity of the minimizers: this is no more true in the nonautonomous case. Observe also that, when L is smooth and convex in v, the proof of Theorem 3.1 is an easy matter:

Classical proof of Theorem 3.1. We assume here that L is smooth and convex in v. In this case, y_* satisfies the Du Bois-Reymond equation: there is a constant $c \in \mathbb{R}$ such that

$$L(y_*(t), y_*'(t)) - y_*'(t) \cdot \nabla_v L(y_*(t), y_*'(t)) = c \text{ a.e. } t \in I.$$

Now superlinearity and convexity in v of L imply that

$$\lim_{|v|\to+\infty} \{L(y,v) - v \cdot \nabla_v L(y,v)\} = -\infty$$

uniformly. We conclude that y_*' is bounded. □

The original proof of Theorem 3.1 in the nonsmooth general case is much more involved and based on some ad hoc arguments: indeed at that time, the validity of a suitable extended version of the Du Bois-Reymond condition was not known in the nonsmooth setting.

Remark 3.1 (Growth conditions (G) and (H)). *The proof of Theorem 3.1 shows that, at least in the smooth case, the conclusion does hold if, instead of supposing L superlinear, we impose the following hypothesis:*

(G) $$\lim_{|v|\to+\infty} L(y,v) - v \cdot \nabla_v L(y,v) = -\infty$$

uniformly when y varies on compact sets. Condition (**??**) *has a clear geometrical meaning since the tangent hyperplane to $v \mapsto z = L(y,v)$ intersects the z axis at $L(y,v) - v \cdot \nabla_v L(y,v)$. Moreover, if L is convex in v, superlinearity implies* (**??**)*. Condition* (G) *is a truly growth condition, weaker than superlinearity: it is satisfied for instance if $L(y,v) = |v| - \sqrt{|v|}$. Extensions of Condition* (**??**) *to nonsmooth Lagrangians were formulated and studied in connection with regularity by Cellina, Mariconda and Treu (see* [10, 24]*).*

The conclusion of Theorem 3.1 (in the smooth case) actually remains valid under the more subtle condition that there are an admissible $\bar{y} \in W^{1,1}(I;\mathbb{R}^n)$ and $c > 0$ such that

- $c > \operatorname{essinf} |y'|$ *whenever* $F(y) \le F(\bar{y})$,

• *for every compact subset K of $\mathbb{R}^n$ there exists $\nu_K > 0$ such that*

$$\text{(H)}\qquad \inf_{y\in K, |v|<c}\{L(y,v) - v\cdot\nabla_v L(y,v)\} > \sup_{y\in K, |v|\geq \nu_K}\{L(y,v) - v\cdot\nabla_v L(y,v)\}.$$

*Indeed, assume (**??**), set $K = y_*(I)$ (we take $\bar{y} = y_*$ in the condition above). We claim that $\|y_*'\|_\infty \leq \nu_K$. We argue by contradiction. Accordingly, suppose that $|y_*'(t)| > \nu_K$ on a non negligible set Σ_1. Denoting $\Sigma_2 = \{t \in I : |y_*'| < c\}$, the Du Bois-Reymond equation implies that, for a.e. $s_1 \in \Sigma_1, s_2 \in \Sigma_2$,*

$$\text{(1)}\quad \begin{aligned} L(y_*(s_1), y_*'(s_1)) - y_*'(s_1)\cdot\nabla_v L(y_*(s_1), y_*'(s_1)) &= c \\ &= L(y_*(s_1), y_*'(s_2)) - y_*'(s_2)\cdot\nabla_v L(y_*(s_2), y_*'(s_2)) \end{aligned}$$

so that

$$\inf_{y\in K, |v|<c}\{L(y,v) - v\cdot\nabla_v L(y,v)\} \leq \sup_{y\in K, |v|\geq \nu_K}\{L(y,v) - v\cdot\nabla_v L(y,v), \}$$

*contradicting (**??**). Condition (**??**) is strictly weaker than (**??**): $L(y,v) = \sqrt{1+|v|^2}$ satisfies (**??**) but not (**??**). Condition (**??**) was formulated and extended by Clarke [14] for nonsmooth Lagrangians (convex in v).*

A regularity (and existence) result of minimizers under the slow growth condition (**??**) was obtained in the pioneering work [14]. In addition to the assumptions of Theorem 3.1 however, it is assumed that L is lower semicontinuous and convex in v.

Theorem 3.2 (Existence and regularity with slow growth - Clarke (1993), [14])**.** *Assume that $L(y,v)$ is lower semicontinuous and convex in v. If L satisfies the growth condition (**??**) then problem ($\mathrm{P}_{a,b}$) admits a minimizer, which is Lipschitz.*

Actually, Theorem 3.2 was formulated in [14] for nonautonomous Lagrangians, assuming, in addition to the other hypotheses, the validity of Condition (S) on $t \mapsto L(t,y,v)$.

Definition 3.1 (Condition (S))**.** *$t \mapsto L(t,y,v)$ is Lipschitz for all (y,v), and there are $\kappa \geq 0$ and $\gamma \in L^1(I)$ such that, for a.e. $t \in I$ and all $(y,v) \in \mathbb{R}^n \times \mathbb{R}^n$,*

$$|L_t(t,y,v)| \leq \kappa L(t,y,v) + \gamma(t).$$

In the nonautonomous case, Condition (**??**) takes a more involved form, depending on κ, γ (see [14, §4] and below). Condition (S), in the smooth case, is known to ensure the validity of the Du Bois-Reymond equation for absolutely continuous minimizers (see [13]).

Unless the Lagrangian is continuous and convex in v, the existence of a minimizer is not guaranteed. If the Lavrentiev phenomenon does not occur, the finite element method allows to find, at least numerically, the infimum of ($P_{a,b}$). A milestone in this direction was established by Alberti and Serra Cassano in [1], without assuming any growth condition.

Theorem 3.3 (Nonoccurrence of the Lavrentiev phenomenon, Alberti & Serra Cassano (1994), [1])**.** *Suppose that L is autonomous, Borel and that*

$$\forall K > 0 \quad \exists r_K > 0 \quad L \text{ is bounded on } B_K \times B_{r_K}. \tag{B}$$

Then the Lavrentiev phenomenon does occur for the problem with one prescribed initial condition (P_a)*.*

The proof of Theorem 3.3 begins with a Lusin's type approximation, followed by a reparametrization. When (B) is not satisfied, there are cases where the Lavrentiev phenomenon occurs for (P_a) (see [12]). It was pointed out recently in [23] that Condition (B) is no more sufficient to ensure in general the nonoccurrence of the Lavrentiev phenomenon for the two end-point prescribed conditions (i.e. for problem ($P_{a,b}$)): the dependence of the phenomenon on boundary data was actually noticed in [8].

In recent years we aimed to answer to the following questions:

(1) Concerning regularity, any known result in the literature in the nonautonomous case before 2020 required some extra assumptions on the Lagrangian with respect to the state and velocity variable, which are not imposed in Theorem 3.1 for the autonomous case. Can Theorem 3.1 be truly extended to the nonautonomous case? Could it be even extended to growth conditions weaker than superlinearity in the spirit of Theorem 3.2?

(2) For discontinuous, or nonconvex Lagrangians, the assumptions of the existence Theorem 3.2 are not fulfilled. Do growth conditions have a role in building at least equi-Lipschitz minimizing sequences?

(3) Is that possible to establish general sufficient conditions for the nonoccurrence of the Lavrentiev phenomenon for problem $(\mathrm{P}_{a,b})$ (i.e., with two end-point constraints)?

(4) Though the occurrence of the Lavrentiev phenomenon is typically associated with the nonautonomous case, are there some conditions that may prevent it?

(5) Lusin's type approximation in the proof of Theorem 3.3 is not suitable when some state constraints are present (e.g., as pointed out in [11], in the problem of Keplerian orbits where the Lagrangian is of the form $\dfrac{c}{|y|}+\dfrac{1}{2}|y'|^2$, and one needs to build minimizing sequences that avoid the origin). Is that possible to extend Theorem 3.3 to state constrained problems?

For the sake of clarity we will restrict attention to the case where the Lagrangian is real valued.

4. A Du Bois-Reymond type variational inequality

In the nonsmooth autonomous setting the validity of the Du Bois-Reymond equation was established in some particular cases (e.g., as in [18] when L is locally Lipschitz in v), or in the one-dimensional case under some further conditions as in [17]. Actually a suitable extension of the Du Bois-Reymond is valid whenever the given autonomous Lagrangian is just Borel.

Theorem 4.1 (Du Bois-Raymond variational inequality - Bettiol & Mariconda (2020), [4])**.** *Assume Condition* (S)*. Let y_* be a minimizer of* $(\mathrm{P}_{a,b})$*. Then there exists $p\in W^{1,1}(I;\mathbb{R})$ such that, for almost every $t\in I$:*

$$L\Big(t,y_*(t),\frac{y_*'(t)}{\mu}\Big)\mu-L(t,y_*(t),y_*'(t))\ge p(t)(\mu-1)\qquad\forall\mu>0. \tag{W}$$

Moreover,

$$p'(t)\in\partial_t^C L(t,y_*(t),y_*'(t))\ \text{ almost everywhere in } I, \tag{D}$$

where $\partial_t^C L(t,y,v)$ denotes the Clarke subdifferential of $t\mapsto L(t,y,v)$ (see [15] *for the definition of the Clarke subdifferential).*

The proof of Theorem 4.1 combines the classical argument of an auxiliary optimal control problem where the state variable in the auxiliary Lagrangian is the time variable.

The Lipschitz condition (S) on L with respect to the time variable then allows to apply Clarke's Maximum Principle [15, Theorem 22.26].

Remark 4.1. *(1) Theorem 4.1 is an extension of the Du Bois-Reymond equation to the nonsmooth context. Indeed, if L is smooth then the variational inequality* (**??**) *implies that, for a.e. $t \in I$,*

$$p(t) = \frac{d}{d\mu}\left[L\Big(t, y_*(t), \frac{y_*'(t)}{\mu}\Big)\mu\right]_{\mu=1}$$
$$= L(t, y_*(t), y_*'(t)) - y_*'(t) \cdot \nabla_v L(t, y_*(t), y_*'(t))$$

Moreover (**??**) *gives $p'(t) = L_t(t, y_*(t), y_*'(t))$ a.e. in I.*

(2) When L is autonomous*, the claim of Theorem 4.1 shows that the Du Bois-Reymond variational inequality* (**??**) *(with p being a constant) holds whenever L is just Borel!*

(3) The variational inequality (**??**) *shows that $p(t)$ is a subgradient, in the sense of convex analysis, of $0 < \mu \mapsto L\Big(t, y_*(t), \dfrac{y_*'(t)}{\mu}\Big)\mu$ at $\mu = 1$. In particular the map $0 < r \mapsto L(t, y_*(t), r\, y_*'(t))$ has a non-empty convex subgradient at $r = 1$ for almost every t. This property reminds the well-known relaxation result, valid under some additional growth conditions, stating that $L(t, x_*(t), x_*'(t))$ coincides for a.e. t with the bipolar at $x_*'(t)$ of $v \mapsto L(t, x_*(t), v)$.*

5. Lipschitz regularity under slow growth conditions

We now combine the Du Bois-Reymond inequality formulated in § 4 with a growth condition inspired by the ones introduced in §3. We first need to formulate the growth conditions for nonsmooth Lagrangians.

Definition 5.1. *If $(t, y, v) \in I \times \mathbb{R}^n \times \mathbb{R}^n$ we denote by*

$$\partial_\mu \left[L\left(t, y, \frac{v}{\mu}\right)\mu\right]_{\mu=1}$$

the subgradient, in the sense of convex analysis, of the map

$$0 < \mu \mapsto L\left(t, y, \frac{v}{\mu}\right)\mu$$

at $\mu = 1$, namely the subset (possibly empty) of $\mathbb{R}$

$$\left\{p \in \mathbb{R} : \, L\left(t, y, \frac{v}{\mu}\right)\mu - L(t, y, v) \geq p(\mu - 1) \quad \forall \mu > 0\right\}.$$

Now, we show how Conditions (G) and (H) can be generalized; the major novelty with respect to previous work is that they need to be checked just where $0 < r \mapsto L(t, y, rv)$ is somehow convex at $r = 1$, namely where the map has a nonempty subgradient.

Definition 5.2 (Condition (G)). *We say that* (G) *holds if, for every selection $P(t, y, v)$ of $\partial_\mu \left[L\left(t, y, \frac{v}{\mu}\right)\mu\right]_{\mu=1}$,*

$$\text{(G)} \qquad \lim_{\substack{|v| \to +\infty \\ \partial_\mu \left[L\left(t, y, \frac{v}{\mu}\right)\mu\right]_{\mu=1} \neq \emptyset}} P(t, y, v) = -\infty$$

uniformly for y on compact sets. The above means that, for all $K > 0$, $R < 0$, there exists $M > 0$ such that $P(t, y, v) \leq R$ whenever $t \in I$, $|y| \leq K$, $|v| \geq M$ and $\partial_\mu \left[L\left(t, y, \frac{v}{\mu}\right)\mu\right]_{\mu=1} \neq \emptyset$.

Definition 5.3 (Condition (H)). *We say that L satisfies* (H) *if there are an admissible $\overline{y} \in W^{1,1}(I; \mathbb{R}^n)$ and $c > 0$ such that for every selection $P(t, y, v)$ of $\partial_\mu \left[L\left(t, y, \frac{v}{\mu}\right)\mu\right]_{\mu=1}$,*

- $c > \operatorname{essinf} |y'|$ *whenever $F(y) \leq F(\overline{y})$,*
- *for every compact subset K of $\mathbb{R}^n$ there exists $\nu_K > 0$ such that*

$$\text{(H)} \qquad \inf_{\substack{y \in K, |v| < c \\ \partial_\mu \left[L\left(t, y, \frac{v}{\mu}\right)\mu\right]_{\mu=1} \neq \emptyset}} \{P(t, y, v)\} > \sup_{\substack{y \in K, |v| \geq \nu_K \\ \partial_\mu \left[L\left(t, y, \frac{v}{\mu}\right)\mu\right]_{\mu=1} \neq \emptyset}} \{P(t, y, v)\} + \Phi,$$

where $\Phi := \kappa F(\bar{y}) + \|\gamma\|_1$ (we agree that $\Phi = 0$ when L is autonomous), κ and γ are the constant and the function that appear in Condition (S).

In [6] we show that superlinearity implies (G) and that (G) implies (H), without imposing convexity assumptions of any kind. Theorem 5.1 extends in several ways various regularity results, and covers both Theorem 3.1 when restricted to the autonomous case and the regularity part of Theorem 3.2. The proof of Theorem 5.1 is carried out similarly as to the one presented in the smooth autonomous case and easily follows from our Du Bois-Reymond variational inequality (W)–(D).

Theorem 5.1 (Lipschitz regularity - Bettiol & Mariconda (2021), [6]). *Assume Condition* (S). *Let y_* be a minimizer of* $(\mathrm{P}_{a,b})$. *Suppose in addition that L fulfills the Growth assumption* (**??**). *Then y_* is Lipschitz.*

6. Lavrentiev gap at a function and Lavrentiev phenomenon

In this section we consider Lagrangians that do not necessarily satisfy any known growth condition. Thus, one cannot expect neither existence, nor Lipschitz regularity (think about the zero functional!). We study here the possibility of avoiding the absence of any Lipschitz minimizing sequences for a variational problem, i.e. the so called non-occurrence of the Lavrentiev phenomenon.

Definition 6.1. *The Lavrentiev phenomenon does not occur for* (P_a) *(resp.* $(\mathrm{P}_{a,b})$*) if there is a minimizing sequence $(y_k)_k$ for the problem, of Lipschitz functions, satisfying the boundary condition $y_k(a) = A$ (resp. $y_k(a) = A, y_k(b) = B$).*

Unfortunately the phenomenon may happen even when the Lagrangian is a polynomial and satisfies Tonelli's existence conditions, as shown by Ball and Mizel in [2]. The following simpler example, due to Manià, illustrates the situation.

Example 6.1 (Manià, [20]). *Consider the (nonautonomous) problem of minimizing*

$$F(y) = \int_0^1 (y^3 - s)^2 (y')^6 \, ds : \; y \in W^{1,1}(I), \, y(0) = 0, \, y(1) = 1. \tag{P}$$

Then $y_*(s) := s^{1/3}$ is a minimizer and $F(y_*) = 0$. Not only y_* is not Lipschitz; it turns out with some computations (see [8, §4.3]*) that the Lavrentiev phenomenon occurs, i.e.,*

$$0 = \min F = F(y_*) < \inf\{F(y) : \; y \in \mathrm{Lip}([0,1]), \, y(0) = 0, y(1) = 1\},$$

where by $\mathrm{Lip}([0,1])$ *we denote the Lipschitz functions defined in* $[0,1]$. *However, as it is noticed in* [8], *the situation changes drastically if one allows to vary the initial boundary condition along the minimizing sequence $(y_k)_k$. Indeed consider the sequence $(y_k)_k$, where each y_k is obtained by truncating y_* at $1/k$, $k \in \mathbb{N}_{\geq 1}$ (see Fig. 1):*

$$y_k(s) := \begin{cases} 1/k^{1/3} & \textit{if } s \in [0, 1/k], \\ s^{1/3} & \textit{otherwise.} \end{cases}$$

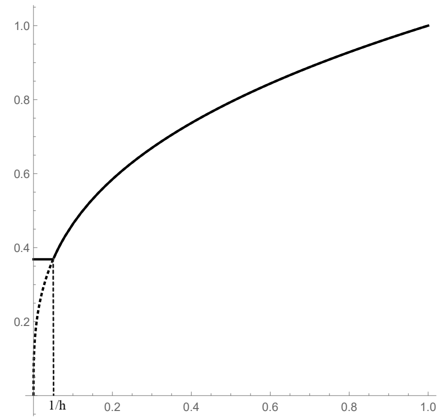

FIGURE 1. The function $y_h, h \geq 1$ in Example 6.1.

Then $(y_k)_k$ *is a sequence of Lipschitz functions satisfying*

$$y_k(1) = y(1) = 1, \quad F(y_k) \to F(y_*), \quad y_k \to y_* \text{ in } W^{1,1}([0,1]).$$

Therefore, no Lavrentiev phenomenon occurs for the variational problem with (just) the end point condition $y(1) = 1$*:*

$$\min F(y) = \int_0^1 (y^3 - s)^2 (y')^6 \, ds : \; y \in W^{1,1}(I), \; y(1) = 1.$$

We present in § 6.1 some results for both problems (P_a) and ($\mathrm{P}_{a,b}$); in § 6.2 we introduce additional state and velocity constraints.

6.1. **Unconstrained problems.** In the autonomous case the non-occurrence of the phenomenon was established by Alberti and Serra Cassano [1] for a wide range of Lagrangians. Actually it was recently realized (see [23]) that Condition (B) in Theorem 3.3 is not enough to ensure the non-occurrence when one wishes to preserve both two end-point contraints. Further conditions aimed to encompass this issue are examined in [23]. Though the occurrence of the phenomenon is often related to the nonautonomous case, there are cases of autonomous Lagrangians that exhibit it (see [12]). Similarly to the main trend in regularity, sufficient conditions to prevent the phenomenon in the nonautonomous case do usually involve regularity properties of the Lagrangian that are not present in the autonomous case (see [9]). Recently, it appeared that Condition (S) on the first (time) variable of the Lagrangian is the appropriate assumption to be added. Theorem 6.1 provides also a sufficient condition for the nonoccurrence of the gap for the two end-point condition problem $(\mathrm{P}_{a,b})$.

Theorem 6.1 (Nonoccurrence of the Lavrentiev phenomenon - Mariconda (2022), [23])**.** *Assume that L satisfies Condition* (S)*. Suppose, moreover:*

(B) *For every $K > 0$ there is $r_K > 0$ such that L is bounded on $I \times B_K \times B_{r_K}$.*

Then the Lavrentiev phenomenon does not occur for (P_a)*. Moreover, the Lavrentiev phenomenon does not occur for the two end-point constrained problem* $(\mathrm{P}_{a,b})$ *if, in addition,*

(B^+) *L is bounded on bounded sets.*

Sketch of the proof of Theorem 6.1. The proof of the first claim in Theorem 6.1 follows narrowly that of [1, Theorem 2.4], obtained there for problem (P_a) in the case of autonomous Lagrangians.
let y be any element of a minimizing sequence. We first use a Lusin's type approximation of y, thus obtaining a Lipschitz sequence $(z_k)_k$ that converges to y in the $W^{1,1}$ norm and satisfies both boundary conditions $z_k(a) = y(a), z_k(b) = y(b)$.
We then reparametrize each z_k by setting $y_k = z_k \circ \psi_k$, where $\psi_k : I \to [a, b_k], b_k \le b$ is suitable Lipschitz, bijective function in such a way that $F(y_k) \to F(y)$ as $k \to +\infty$. It may happen, however, that $b_k < b$ whence $y_k(b) \neq y(b)$: this is avoided under assumption (B^+), since in this case one may build each ψ_k in such a way that $b_k = b$. □

6.2. **Non-occurrence of the phenomenon with state and velocity constraints.** As mentioned above, the construction in the proof of Theorem 6.1 may not preserve state or velocity constraints. We assume here that $L : I \times \Omega \times \mathbb{R}^n \to [0, +\infty[$ is Lebesgue-Borel measurable, where Ω is an open subset of $\mathbb{R}^n$. We fix a subset Δ of Ω and a cone $\mathcal{U}$ in $\mathbb{R}^n$. We consider the constrained problems

$$(\mathrm{P}_{a,b}^{\Delta,\mathcal{U}}) \qquad \begin{cases} \min F(y) : y \in W^{1,1}(I;\mathbb{R}^n),\ y(a) = A, y(b) = B \\ y(t) \in \Delta \ \text{ for all } t \in I, \quad y'(t) \in \mathcal{U} \ \text{ a.e. } t \in I \end{cases}$$

and

$$(\mathrm{P}_a^{\Delta,\mathcal{U}}) \qquad \begin{cases} \min F(y) : y \in W^{1,1}(I;\mathbb{R}^n),\ y(a) = A \\ y(t) \in \Delta \ \text{ for all } t \in I, \quad y'(t) \in \mathcal{U} \ \text{ a.e. } t \in I. \end{cases}$$

In this context the non-occurrence of the phenomenon means, referring to Definition 6.1, that there exists a minimizing sequence of Lipschitz functions $(y_k)_k$ that satisfy the given constraints. In addition to the assumptions of Theorem 6.1, we require here the radial convexity of L with respect to the velocity variable, a quite natural condition in view of Theorem 4.1.

Theorem 6.2 (Nonoccurrence of the Lavrentiev phenomenon for problems with constraints - Mariconda (2022), [21, 22])**.** *Assume that L satisfies Condition* (S) *and that $0 < r \mapsto L(t,y,rv)$ is convex for all $(t,y,v) \in I \times \Omega \times \mathbb{R}^n$.*

(1) Suppose

($\mathrm{B}_{\Delta,\mathcal{U}}$) *For every compact subset $\mathcal{K}$ of Δ there is $r_{\mathcal{K}} > 0$ such that L is bounded on $(I \times \mathcal{K} \times (B_{r_{\mathcal{K}}} \cap \mathcal{U}))$.*

Then the Lavrentiev phenomenon does not occur for ($P_a^{\Delta,\mathcal{U}}$).

(2) Suppose

($\mathrm{B}^+_{\Delta,\mathcal{U}}$) *For every compact subset K of Δ, L is bounded on the bounded subsets of $I \times \mathcal{K} \times \mathcal{U}$.*

Then the Lavrentiev phenomenon does not occur for the two end-point constrained problem ($\mathrm{P}_{a,b}^{\Delta,\mathcal{U}}$).

(3) Assume, in addition to ($\mathrm{B}_{\Delta,\mathcal{U}}$) *(resp.* ($\mathrm{B}^+_{\Delta,\mathcal{U}}$)*), the growth condition* (H)*, that $\Omega = \mathbb{R}^n$ and that $L(t,y,v) \geq \alpha|v| - d$ for some $\alpha > 0, d \geq 0$. Then, in Claim 1 (resp. Claim 2), the infimum of* (P_a) *(resp.* ($\mathrm{P}_{a,b}$)*) may be reached via a sequence of equi-Lipschitz functions.*

Remark 6.1. *Whereas Theorem 6.1 requires that some suitable rectangles of $I \times \mathbb{R}^n \times \mathbb{R}^n$ are contained in the effective domain of L, in Theorem 6.2 it is enough to check the local boundedness of L on the subsets of $I \times \Delta \times \mathcal{U}$. Notice that, as in Theorem 6.1, no growth conditions are involved in Claims 1 and 2 of Theorem 6.2. In the autonomous case, Claim*

3 of the theorem was obtained by Cellina and Ferriero [11] *under the additional hypotheses that L is continuous and convex and the more restrictive growth assumption* (G)*.*

Remark 6.2. *Theorem 6.2 can be appreciated with the following example, quoted and solved in* [11]*. Let $L(t,y,v)=\dfrac{1}{|y|}+\dfrac{1}{2}|v|^2$ for $t\in[0,1], y\in\mathbb{R}^3\setminus\{0\}, v\in\mathbb{R}^3$. The problem* (P) *of minimizing $F(y)=\displaystyle\int_0^1 L(t,y(t),y'(t))$ among the absolutely continuous function that satisfy given end-point conditions is related to the existence of Keplerian orbits. It is shown in* [19] *that there exists a solution to* (P) *in $W^{1,1}([0,1];\mathbb{R}^3\setminus\{0\})$. A natural question is whether the Lavrentiev phenomenon occurs here. Now, Theorem 6.1 is of no help, since $L(t,y,v)$ is unbounded on* every *strip $I\times B_K\times B_r$ for any $K,r>0$. Instead, the assumptions of Claim 2 of Theorem 6.2 are satisfied. Indeed, here $\Omega=\Delta=\mathbb{R}^3\setminus\{0\}$, $\mathcal{U}=\mathbb{R}^3$ and*

- *$L(t,y,v)$ is convex in v and thus radially convex;*
- *$L(t,y,v)\geq\dfrac{1}{2}|v|^2$ is superlinear;*
- *If $K\subset\Delta=\mathbb{R}^3\setminus\{0\}$ is compact and $r>0$ then L is continuous and thus bounded on $I\times K\times B_r$, thus L fulfills condition* ($\mathrm{B}^+_{\Delta,\mathbb{R}^3}$) *of Theorem 6.2.*

Il follows from Theorem 6.2 that (P) *has a Lipschitz minimizing sequence.*

Sketch of the proof of Theorem 6.2. The argument here differs from that of Theorem 6.1 and is inspired by the one used by Cellina an Ferriero in [11]; it can be summarized as follows. Let φ be a smooth, increasing change of variable on I, y be an admissible trajectory for ($\mathrm{P}_{a,b}$), and set $\overline{y}(s):=y(\varphi^{-1}(s))$. Notice that, by taking high values of $\varphi'(\tau)$, one lowers the norm of the derivative of $\overline{y}(\varphi(\tau))$. The change of variable $t=\varphi(\tau)$ yields

$$F(\overline{y})=\int_a^b L(t,\overline{y}(t),\overline{y}'(t))\,dt=\int_a^b L\Big(\varphi(\tau),y(\tau),\frac{y'(\tau)}{\varphi'(\tau)}\Big)\,\varphi'(\tau)\,d\tau.$$

Supposing that L smooth, the derivative of $v\mapsto L\Big(\varphi,y,\dfrac{v}{\mu}\Big)\,\mu$ at $\mu=1$ is

$$L(\varphi,y,v)-v\cdot\nabla_v L(\varphi,y,v).$$

The proof of Theorem 6.2 begins by taking an arbitrary absolutely continuous function y of a given minimizing sequence. Claims 1 and 2 consist, correspondingly to y, in finding a suitable sequence of increasing and one-to-one change of variable $\varphi_\nu : I \to I$ in such a way that:

- $y_\nu := y \circ \varphi_\nu$ is Lipschitz;
- $y_\nu \to y$ in the $W^{1,1}$ norm;
- $F(y_\nu)$ tends to $F(y)$ as $\nu \to +\infty$.

Actually in Claim 1 it may happen that $\varphi_\nu(b) > b$, whereas in Claim 2 Assumption ($\mathrm{B}^+_{\Delta,\mathcal{U}}$) allows to take higher values of φ_ν' and thus to build φ_ν in such a way that $\varphi_\nu(b) = b$.
In Claim 3, the growth from below implies that the union of the ranges of the elements of a given minimizing sequence lie in a compact subset K. Assuming that (H) holds, choose $\nu = \nu_K, c > 0$ as in (**??**). One then compensate the values of the integral of $L(t, y \circ \varphi_\nu(t), (y \circ \varphi_\nu)'(t))$ on the sets where $|y'| > \nu_K$ with the ones where $|y'| < c$, up to obtain $F(y \circ \varphi_\nu) \le F(y)$: one thus obtains a minimizing sequence of Lipschitz functions, all with rank less than ν_K. □

6.3. **Application to the regularity of the value function.** In [3] we apply the methods of §6.2 to deduce the Lipschitz regularity of the value function

$$V(t,x) = \inf\left\{\int_t^b L(s, y(s), y'(s))\, ds : \, y(t) = x, y(b) = B\right\}$$

without assuming, as it is often common in this kind of results, neither the existence minimizers nor superlinearity.

Corollary 6.1 (Regularity of the value function - Bernis, Bettiol & Mariconda [3, 7])**.** *Assume that* $L : I \times \mathbb{R}^n \times \mathbb{R}^n \to [0, +\infty[$ *satisfies Condition* (S)*, and that* $0 < r \mapsto L(t, y, rv)$ *is convex for all* $(t, y, v) \in I \times \mathbb{R}^n \times \mathbb{R}^n$*. Assume the growth condition* (H) *and that* $L(t, y, v) \ge \alpha|v| - d$ *for some* $\alpha > 0, d \ge 0$*. Then the value function* $V(t, x)$ *is locally Lipschitz on* $[a, b[\times\mathbb{R}^n$*.*

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