A BASIS OF RESOLUTIVE SETS FOR THE HEAT EQUATION: AN ELEMENTARY CONSTRUCTION

UNA BASE DI INSIEMI RISOLUTIVI PER L'EQUAZIONE DEL CALORE: UNA COSTRUZIONE ELEMENTARE

ALESSIA E. KOGOJ AND ERMANNO LANCONELLI

Abstract. By an easy "trick" taken from the caloric polynomial theory, we prove the existence of a basis of the Euclidean topology whose elements are resolutive sets of the heat equation. This result can be used to construct, with a very elementary approach, the Perron solution of the caloric Dirichlet problem on arbitrary bounded open subsets of the Euclidean space-time.

Sunto. Con un semplice espediente preso dalla teoria dei polinomi calorici, dimostriamo l'esistenza di una base della topologia euclidea i cui elementi sono insiemi risolutivi per l'equazione del calore. Questo risultato può essere utilizzato per costruire, con un approccio elementare, la soluzione di Perron del problema di Dirichlet calorico su arbitrari insiemi aperti limitati dello spazio-tempo euclideo.

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1. Introduction and Main Theorem

The aim of this note is to draw attention to an easy trick - taken from the caloric polynomial theory - allowing to construct in a very elementary way a basis of resolutive open sets for the heat equation.

This result will make completely elementary the construction of the Perron solution of the caloric Dirichlet problem using, e.g., the Constantinescu and Cornea Potential theory

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in abstract harmonic spaces. The crucial point of our procedure is Theorem 1.1 below, the main result of this note.

To begin with, we introduce some notations. The point of $\mathbb{R}^{N+1}$ will be denoted by

$$z = (x,t), \quad x \in \mathbb{R}^N, \quad t \in \mathbb{R},$$

and the *heat operator* in $\mathbb{R}^{N+1}$ by

$$\mathcal{H} := \Delta - \partial_t.$$

Here $\Delta := \sum_{j=1}^N \partial_{x_j}^2$ is the Laplacian in $\mathbb{R}^N$. We call caloric in an open subset Ω of $\mathbb{R}^{N+1}$ a smooth solution to $\mathcal{H}u = 0$ in Ω.

If $\alpha = (\alpha_1, \dots, \alpha_N, \alpha_{N+1})$ is a multi-index with non negative integer components, we let

$$|\alpha|_c = \textit{caloric height of } \alpha := \alpha_1 + \ldots + \alpha_N + 2\alpha_{N+1}.$$

We denote by $\mathcal{P}$ the linear space of the polynomial functions in $\mathbb{R}^{N+1}$, i.e., the space of functions p of the kind

$$p(z) = \sum_{|\alpha|_c \le m} a_\alpha z^\alpha, \quad a_\alpha \in \mathbb{R} \text{ for every } \alpha, \tag{1}$$

for a suitable $m \in \mathbb{Z}$.

We say that p in (1) has caloric degree $\le m$. Moreover, if $\sum_{|\alpha|_c = m} a_\alpha z^\alpha$ is not identically zero, we will say that p has degree m. In our procedure a crucial rôle will be played by the polynomial

$$w(z) = w(x,t) = t - |x|^2,$$

where $|x|$ stands for the Euclidean norm of $x \in \mathbb{R}^N$. We explicitly point out that w has a caloric degree $m = 2$.

We denote by P the open region

$$P := \{(x,t) \in \mathbb{R}^{N+1} \ : \ t > |x|^2\}.$$

Obviously, the boundary of P is the paraboloid

$$\partial P = \{(x,t) \in \mathbb{R}^{N+1} \ : \ t = |x|^2\}.$$

We want to stress that

$$P = \{w > 0\} \quad \text{and} \quad \partial P = \{w = 0\}.$$

Here is the key theorem of our note.

Theorem 1.1. *Let p and q be polynomials in $\mathbb{R}^{N+1}$. Then, there exists a unique polynomial u in $\mathbb{R}^{N+1}$ such that*

$$\begin{cases} \mathcal{H}u = q \quad in\ \mathbb{R}^{N+1}, \\ u|_{\partial P} = p|_{\partial P}. \end{cases} \tag{2}$$

Proof. Uniqueness. If u_1 and $u_2 \in \mathcal{P}$ solve (2), then $v := u_1 - u_2 \in \mathcal{P}$ solves

$$\begin{cases} \mathcal{H}v = 0 \ \text{ in } \mathbb{R}^{N+1} \text{ (hence in } P\text{)}, \\ u|_{\partial P} = 0. \end{cases}$$

Then, by the parabolic maximum principle (see e.g., [2, Theorem 8.2]), $v \equiv 0$ in P. Since v is a polynomial function, this implies $v \equiv 0$ in $\mathbb{R}^{N+1}$, that is $u_1 \equiv u_2$.

Existence. It is enough to show the existence of a polynomial function v such that

$$\begin{cases} \mathcal{H}v = q - \mathcal{H}p \text{ in } \mathbb{R}^{N+1}, \\ v|_{\partial P} = 0. \end{cases} \tag{3}$$

Indeed, if $v \in \mathcal{P}$ solves (3), then

$$u = v + p \in \mathcal{P}$$

and u solves (2).

To solve (3) we argue as follows.

Let m be the caloric degree of $q - \mathcal{H}p$ ($\mathcal{H}p \in \mathcal{P}$ if $p \in \mathcal{P}$!) and denote by $\mathcal{P}_m$ the linear space of the polynomials having the degree less or equal to m. Since w has caloric degree two, then $\mathcal{H}(wf) \in \mathcal{P}_m$ if $f \in \mathcal{P}_m$. Therefore,

$$f \longmapsto T(f) := \mathcal{H}(wf)$$

maps $\mathcal{P}_m$ into $\mathcal{P}_m$. To prove that (3) has a polynomial solution we only have to prove that T is surjective. Indeed, if $f \in \mathcal{P}_m$ and $T(f) = q - \mathcal{H}p$, then $v := wf$ is a polynomial function solving (3).

To prove that T is surjective, since $\mathcal{P}_m$ is a linear space of finite dimension and T linearly maps $\mathcal{P}_m$ into $\mathcal{P}_m$, it is enough to show that T is injective. Let f in $\mathcal{P}_m$ be such that $T(f) = 0$. Then $g := wf$ is caloric in $\mathbb{R}^{N+1}$, hence, in particular, in the open set P.

Moreover, since $w = 0$ on ∂P,

$$g|_{\partial P} = wf|_{\partial P} = 0.$$

As a consequence, by the caloric maximum principle ([2, Theorem 8.2]), $g = 0$ in P. Since $w > 0$ in P, this implies $f = 0$ in P, hence in $\mathbb{R}^{N+1}$, as f is a polynomial function.

We have hence proved that $f = 0$ is $T(f) = 0$, that is the injectivity of T. This completes the proof. □

A straightforward consequence of our main theorem is the following corollary.

Corollary 1.1. *For every p in $\mathcal{P}$, there exists a unique u_p in $\mathcal{P}$ such that*

$$\begin{cases} \mathcal{H}u_p = 0 & \text{in } P, \\ u_p = p & \text{on } \partial P. \end{cases}$$

Proof. The existence follows from Theorem 1.1 by taking in it $q = 0$. The uniqueness is a consequence of the caloric maximum principle on the parabolic region P. □

2. Existence of a basis of $\mathcal{H}$-risolutive open sets.

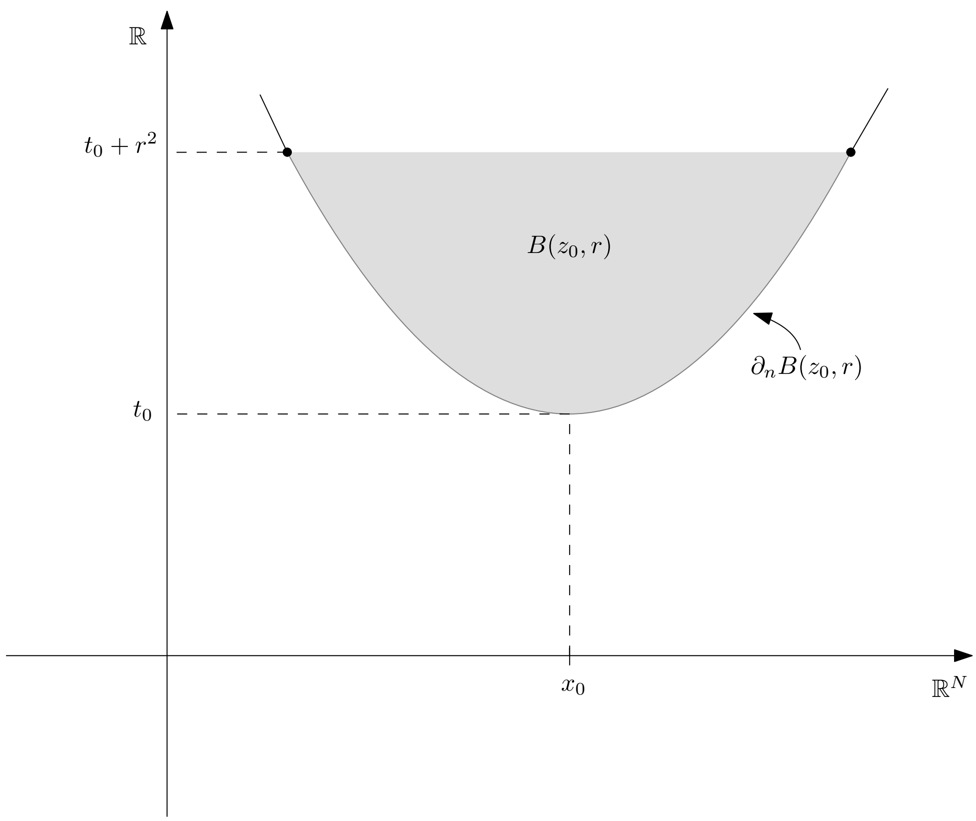
Let $z_0 = (x_0, t_0) \in \mathbb{R}^{N+1}$ and let $r > 0$. We call:

(*i*) *caloric bowl* of *bottom* z_0 and *opening* r the open set

$$B(z_0, r) := \{(x, t) \in \mathbb{R}^{N+1} \ : \ |x - x_0|^2 < t - t_0 < r^2\};$$

(*ii*) *normal* or *caloric boundary* of $B(z_0, r)$ the subset of $\partial B(z_0, r)$

$$\partial_n B(z_0, r) := \{(x, t) \in \mathbb{R}^{N+1} \ : \ |x - x_0|^2 = t - t_0, \ 0 \le t - t_0 \le r^2\}.$$



Obviously,

$$\mathscr{B} := \{B(z_0, r) \ : \ z_0 \in \mathbb{R}^{N+1}, \ r > 0\}$$

is a basis of the Euclidean topology.

Every caloric bowl is *resolutive* in the sense of the following theorem.

Theorem 2.1. *Let $z_0 \in \mathbb{R}^{N+1}$ and $r > 0$ be arbitrarily fixed and let $B = B(z_0, r)$ be the caloric bowl of bottom z_0 and opening r. Then for every $\varphi \in C(\partial_n B, \mathbb{R})$ there exists a unique solution to the boundary value problem*

$$\begin{cases} \mathcal{H}u = 0 \quad \text{in } B, \\ u|_{\partial_n B} = \varphi. \end{cases}$$

Precisely: there exists a unique function u_φ^B caloric in B and continuous up to $B \cup \partial_n B$ such that

$$u_\varphi^B(z) = \varphi(z) \text{ for every } z \in \partial_n B.$$

Moreover $u_\varphi^B \geq 0$ if $\varphi \geq 0$.

Proof. Since $\mathcal{H}$ is left translation invariant and

$$B(z_0, r) = z_0 + B(0, r), \quad 0 \in \mathbb{R}^{N+1},$$

we may assume $z_0 = (0, 0)$. In this case,

$$B = P \cap \{(x, t) \mid t < r^2\}$$

and the existence of u_φ^B follows from Theorem 1.1 if φ is the restriction to $\partial_n B$ of a polynomial function. If φ is merely continuous we get the existence of u_φ^B by approximating φ with a sequence of polynomial functions uniformly convergent to φ on $\partial_n B$. For the details of this argument, as well as for the proof of the remaining part of the theorem we directly refer to our paper [4]. □

3. Some comments

3.1. The sheaf of the caloric functions endows $\mathbb{R}^{N+1}$ with a structure of Harmonic Space in the sense of Constantinescu and Cornea (see [3]). By "simply"applying the Potential Theory in such a space one gets the Perron solution to the caloric Dirichlet problem on every bounded open subset of $\mathbb{R}^{N+1}$.

One of the biggest difficulties of this approach consists in proving that the caloric sheaf actually satisfies the Resolutive Axiom of the abstract Potential Theory, i.e., the existence of a basis of *resolutive* open sets for the heat equation. In their monograph [3], Constantinescu and Cornea proved that the caloric sheaf satisfies the Resolutive Axiom by showing that the *rectangular cylinders* are resolutive for the heat equation. This proof is quite difficult and lengthy (see [3, Chapter 3, Section 3.3]). Our Theorem 2.1, which shows the resolutivity of the caloric bowls in an elementary way, provides a simple proof of that Axiom.

3.2. The axiomatic Potential Theory of Heinz Bauer assumes the Regularity Axiom, a stronger version of the Resolutivity one. Precisely, Bauer requires the existence of a basis of the topology formed by sets which are *regular* for the Dirichlet problem. More explicitly, to apply Bauer theory to the heat equation, one has to prove that the boundary value problem

$$\begin{cases} Hu = 0 \ \text{ in } V, \\ u|_{\partial V} = \varphi, \end{cases} \tag{4}$$

has a solution u_φ^V caloric in V, continuous in $\overline{V}$ and assuming the boundary data φ on the entire boundary, for every $\varphi \in C(\partial V, \mathbb{R})$ and for every V of a basis of the Euclidean topology. Bauer showed that the caloric sheaf satisfies this axiom by proving that the circular cones with vertex above their bases are *regular* for the heat equation. To do this, Bauer used Volterra integral equation theory, a procedure that is not at all elementary (see [1]).

3.3. Neil Watson directly constructs a generalized solution to the caloric Dirichlet problem on general open subsets of $\mathbb{R}^{N+1}$. His construction is based on the solvability of the first boundary value problem for the heat equation on a circular cylinder. Watson proved this solvability result by using a not elementary double layer potential method.

3.4. In our paper [4] we sketch how to construct the Perron solution of the caloric Dirichlet problem on an arbitrary open subset of $\mathbb{R}^{N+1}$, starting from Theorem 2.1. Thanks to this theorem, our procedure turns out to be completely elementary.

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Dipartimento di Scienze Pure e Applicate, Piazza della Repubblica 13, 61029, Urbino (PU), Italy.

E-mail address: alessia.kogoj@uniurb.it

Dipartimento di Matematica, Piazza di Porta San Donato 5, 40126, Bologna (BO), Italy.

E-mail address: ermanno.lanconelli@unibo.it