REGULARITY OF THE FREE BOUNDARY IN THE ONE-PHASE STEFAN PROBLEM: A RECENT APPROACH
REGOLARITÀ DELLA FRONTIERA LIBERA NEL PROBLEMA DI STEFAN A UNA FASE: UN APPROCCIO RECENTE

N. FORCILLO

ABSTRACT. In this note, we discuss about the regularity of the free boundary for the solutions of the one-phase Stefan problem

$$\begin{cases} u_t = \Delta u & \text{in } (\Omega \times (0,T]) \cap \{u > 0\}, \\ u_t = |\nabla u|^2 & \text{on } (\Omega \times (0,T]) \cap \partial\{u > 0\}, \end{cases}$$

with $\Omega \subset \mathbb{R}^n$, $u : \Omega \times [0,T] \to \mathbb{R}$, $u \geq 0$. We start by recalling the classical results achieved by I. Athanasopoulos, L. Caffarelli, and S. Salsa in the more general setting of the two-phase Stefan problem. Next, we focus on some recent achievements on the subject, obtained with Daniela De Silva and Ovidiu Savin starting from the techniques already known for one-phase problems governed by elliptic operators.

SUNTO. In questa nota, discutiamo della regolarità della frontiera libera per le soluzioni del problema di Stefan a una fase

$$\begin{cases} u_t = \Delta u & \text{in } (\Omega \times (0,T]) \cap \{u > 0\}, \\ u_t = |\nabla u|^2 & \text{su } (\Omega \times (0,T]) \cap \partial\{u > 0\}, \end{cases}$$

con $\Omega \subset \mathbb{R}^n$, $u : \Omega \times [0,T] \to \mathbb{R}$, $u \geq 0$. Incominciamo richiamando i risultati classici ottenuti da I. Athanasopoulos, L. Caffarelli, e S. Salsa nel setting più generale del problema di Stefan a due fasi, giungendo successivamente ad alcuni più recenti sull'argomento, trovati insieme a Daniela De Silva e Ovidiu Savin partendo dalle tecniche già note per problemi di frontiera libera a una fase governati da operatori ellittici.

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1. General facts about the Stefan problem

The Stefan problem is one of the most classical and probably motivating free boundary problems. It dates back to the 19th century, and its name precisely to around 1890, when the physicist Josef Stefan discussed it in a series of four papers relating the freezing of the ground and the formation of sea ice, see [16] for a comprehensive history of the Stefan problem. Such problem describes the phase transition processes, such as the melting of the ice (or the solidification of the water), see for example [14] and again [16].
Free boundary problems naturally arise in several fields, such as physics, industry, biology and finance. In general, in these applied problems, there is a qualitative change of a medium and thus an appearance of a phase transition, for instance ice to water, liquid to crystal, as seen before described by the Stefan problem, buying to selling (assets), active to inactive (biology), blue to red (coloring games), disorganized to organized (self -organizing criticality), see [3] for some of such examples.
We heuristically recall that a free boundary problem tipically involves a function that satisfies some partial differential equations and fulfill some conditions on unknown domains determined via the function itself. Thus, such domains are a priori unknown and depend on the problem. This is exactly the peculiarity of free boundary problems. The boundaries of these unknown domains, usually contained in the set on which the problem is stated, determine the so-called free boundary of the solution. In general, we are interested in one-phase and two-phase free boundary problems. Roughly saying, we define "positive phase" the set where the solution of the problem is positive and, in case it does not change sign, we deal with a one-phase problem. In case there also exists a set where the solution is negative, we call it "negative phase" of the solution and we have a two-phase problem. An example of a one-phase problem is the following one:

$$(1) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \\ |\nabla u| = 1 & \text{on } F(u) := \partial \Omega^+(u) \cap \Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$. An example of a two-phase problem is

$$(2) \quad \begin{cases} \Delta u = 0 & \text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \\ \Delta u = 0 & \text{in } \Omega^-(u) := \text{Int}(\{x \in \Omega : u(x) \leq 0\}), \\ |\nabla u^+|^2 - |\nabla u^-|^2 = 1 & \text{on } F(u) := \partial \Omega^+(u) \cap \Omega, \end{cases}$$

with, as usual, $u^+ := \sup\{u, 0\}$ and $u^- := \sup\{-u, 0\}$. In (1) and (2), $F(u)$ denotes the free boundary of u, while $\Omega^+(u)$ and $\Omega^-(u)$ are its positive and negative phase respectively.

In this kind of problems, we are not only interested in the regularity of the solutions, but also in the analysis of some features of the free boundary. In particular, the regularity of the free boundary is very important for possibly proving that the solutions have some further regularity properties. We will focus on this subject in a bit.

Going back to the Stefan problem, and specifically looking at its mathematical formulation, the one-phase form, on which this note mainly focuses, is

$$(3) \quad \begin{cases} u_t = \Delta u & \text{in } (\Omega \times (0, T]) \cap \{u > 0\}, \\ u_t = |\nabla u|^2 & \text{on } (\Omega \times (0, T]) \cap \partial\{u > 0\}, \end{cases}$$

where $\Omega \subset \mathbb{R}^n$, $u : \Omega \times [0, T] \to \mathbb{R}$, $u \geq 0$. The two-phase Stefan problem studied in [1] is

$$(4) \quad \begin{cases} u_t = \Delta u & \text{in } (\Omega \times (0, T]) \cap \{u > 0\}, \\ u_t = \Delta u & \text{in } (\Omega \times (0, T]) \cap \{u \leq 0\}^0, \\ \dfrac{u_t^+}{|\nabla u^+|} = |\nabla u^+| - |\nabla u^-| & \text{on } (\Omega \times (0, T]) \cap \partial\{u > 0\}. \end{cases}$$

Concerning again the physical interpretation of the problem, we point out that (3) is associated with the situation where one of the material phases may be neglected. Typically, this is achieved by assuming that a phase is everywhere at the phase change temperature and hence any variation from such temperature leads to a change of phase. As a consequence, we can just focus on the behavior of the other phase. In the particular framework of melting ice, u denotes the temperature of the water. The region $\{u = 0\}$ is the unmelted region of ice and the free boundary $\partial\{u > 0\}$ is the moving interphase separating the ice and the water. In (3), u has to satisfy the heat equation in its positive phase and the condition $u_t = |\nabla u|^2$ on $\partial\{u > 0\}$ represents the law of conservation of

energy, which defines the position of the moving interphase. Here, $\frac{u_t}{|\nabla u|}$ is the speed of the surface $\partial\{u > 0\}$, at t fixed, in the spatial direction $-\nu$, where $\nu = \frac{\nabla u}{|\nabla u|}$. The more general setting of (4), instead, describes the physical scenario in which both the two phases can not be neglected and have non-constant zero temperature. There, $\frac{u_t^+}{|\nabla u^+|}$ represents the speed of the interphase introduced before.

In this note, $\partial_x\{u(\cdot, t) > 0\}$ denotes the boundary in $\mathbb{R}^n$ of $\{u(\cdot, t) > 0\}$, with t being fixed. We say that $\partial_x\{u > 0\}$ is ε_0-flat in B_λ if, for each t, $\partial_x\{u(\cdot, t) > 0\} \cap B_\lambda$ is trapped in a strip of width $\varepsilon_0\lambda$, and $u = 0$ on one side of this strip, while $u > 0$ on the other side. Let us remark that this means that the free boundary $\partial\{u > 0\}$ is, for each t, flat in space in sense of the notion of flatness exploited by De Silva in [9], see Remark 2.2.

We are in position now to provide the main result proved in [11]. Roughly speaking, it states that a solution u to (3) in a ball of size λ in space-time, which is of size λ and has a sufficiently flat free boundary in space, must have smooth free boundary in the interior provided a necessary nondegeneracy condition holds. The nondegeneracy condition for u requires that u is bounded below by a small multiple of λ at some point in the domain at distance λ from the free boundary. The rigorous statement of the result contained in [11] is the following one.

Theorem 1.1. *Fix a constant K (large) and let u be a solution to the one-phase Stefan problem* (3) *in $B_\lambda \times [-K^{-1}\lambda, 0]$ for some $\lambda \leq 1$. Assume that*

$$|u| \leq K\lambda, \qquad u(x_0, t) \geq K^{-1}\lambda \quad for \; some \quad x_0 \in B_{\frac{3}{4}\lambda}.$$

There exists ε_0 depending only on K and n such that if, for each t, $\partial_x\{u(\cdot, t) > 0\}$ is ε_0-flat in B_λ, then the free boundary $\partial\{u > 0\}$ (and u up to the free boundary) is smooth in $B_{\frac{\lambda}{2}} \times [-(2K)^{-1}\lambda, 0]$.

This note is organized as follows. In Section 2, we recall a bit of the literature concerning the free boundary regularity in the Stefan problem and we briefly expose the approach developed in [9]. In the last Section 3, we provide a sketch of the proof of our main result Theorem 1.1.

2. A VISCOSITY APPROACH TO SOME FREE BOUNDARY PROBLEMS

In this section, we quote a bit of the literature concerning the free boundary regularity in the Stefan problem and we briefly present the approach developed in [9].

In free boundary problems, free boundary regularity is widely studied. Even in the case of the Stefan problem, of course, the behavior of the free boundary is fundamental. For this purpose, in the context of (4), I. Athanasopoulos, L. Caffarelli, and S. Salsa showed in [1] that Lipschitz free boundaries in space-time become smooth provided a nondegeneracy condition holds. In [2] they established the same conclusion for sufficiently "flat" free boundaries. Their techniques were based on the original work of Caffarelli in the elliptic case [4, 5]. At this point, we could ask to ourselves if the nondegeneracy assumption is necessary, trying to answer the more general question: if the free boundary is a Lipschitz graph in space-time, can we deduce further regularity, i.e. the free boundary is actually a $C^{1,\alpha}$ or a C^1 graph? The answer is that the nondegeneracy condition is indeed necessary. As a matter of fact, Lipschitz free boundaries in evolution problems do not enjoy, in general, instantaneous regularization. Examples in which Lipschitz free boundaries preserve corners can be found for instance in [8]. For the sake of completeness, we recall here a bidimensional example for the one-phase Stefan problem (3), see [8].

Let

$$w(\rho, \theta, t) = \rho^{g(t)}(\cos(g(t)\theta))^+,$$

where ρ, θ are polar coordinates in the plane and g is a smooth decreasing function greater than 2. Then, by direct computations, w classically solves

$$(5) \qquad \begin{cases} w_t \geq \Delta w & \text{in } C_r \cap \{w > 0\}, \\ w_t \geq |\nabla w|^2 & \text{on } C_r \cap \partial\{w > 0\}, \end{cases}$$

with $C_r := B_r \times (0, r^2)$ and $r \leq r_0$ sufficiently small. Precisely, in $\{w > 0\}$ it holds

$$w_t = g'\rho^g(\log \rho \cos(g\theta) - \sin(g\theta)\theta),$$

and

$$\Delta w = g\rho^{g-1}(((g-1)\rho^{-1}|\nabla\rho|^2 + \Delta\rho - g\rho|\nabla\theta|^2)\cos(g\theta) - (2g\langle\nabla\rho, \nabla\theta\rangle + \rho\Delta\theta)\sin(g\theta)) = 0,$$

since

$$|\nabla\rho|^2 = 1, \qquad \Delta\rho = \rho^{-1}, \qquad \langle\nabla\rho, \nabla\theta\rangle = 0,$$

$$|\nabla\theta|^2 = \rho^{-2}, \qquad \Delta\theta = 0.$$

By these equalities the first condition in (5) immediately follows, because $g' < 0$ by definition of g and taking ρ small enough. Concerning the inequality on the free boundary $\partial\{w > 0\}$, we have

$$|\nabla w|^2 = g^2\rho^{2g-2}, \quad w_t = -\frac{\pi}{2}\frac{g'}{g}\rho^g \quad \text{on } \partial\{w > 0\},$$

which implies the second condition in (5), again by definition of g and choosing ρ sufficiently small. Hence, w is a supersolution of

(6)
$$\begin{cases} u_t = \Delta u & \text{in } C_r \cap \{u > 0\}, \\ u_t = |\nabla u|^2 & \text{on } C_r \cap \partial\{u > 0\}. \end{cases}$$

Let us consider now a solution u of (6) (in a weak or viscosity sense) with $u = w$ on $\partial_p C_r$ and the free boundary $\partial\{u > 0\}$ a Lipschitz graph in space-time. For the sake of completeness, we recall that $\partial_p C_r := \partial C_r \cap \{t = r^2\}^c$ is the parabolic boundary of C_r. Then, first, by a comparison principle, we get $u \leq w$ in C_r. Moreover, we note that at the origin $(0, 0)$, the free boundary $\partial\{w > 0\}$ has a persistent corner singularity and since $(0, 0) \in \partial\{u > 0\}$ initially with zero speed by hypothesis, $(0, 0) \in \partial\{u > 0\}$ for $0 \leq t < r_0^2$. Then, because $u \leq w$ and $\partial\{u > 0\}$ is a Lipschitz graph in space-time, the origin is a persistent corner for $\partial\{u > 0\}$ too, as we can see in Figure 1, and thus $\partial\{u > 0\}$ does not regularize instantaneously. In [8], there is a three dimensional example for the two-phase Stefan problem (4) as well, constructed via a technical lemma on spherical harmonics. Concerning previous phenomena associated with the Stefan problem, we quote the work [7] as well. There, S. Choi and I. Kim showed that solutions regularize instantaneously if the initial free boundary is locally Lipschitz with bounded Lipschitz constant and the initial data has subquadratic growth. The techniques are related to Athanasopoulos, Caffarelli, and Salsa's ones.

$\partial\{u > 0\} \cap \{t = 10^{-1}\}$

$\partial\{u > 0\} \cap \{t = 10^{-2}\}$

$\partial\{u > 0\} \cap \{t = 0\}$

$(0,0)$

$\partial\{u > 0\} \cap \{t = 0\}$

$\partial\{u > 0\} \cap \{t = 10^{-2}\}$

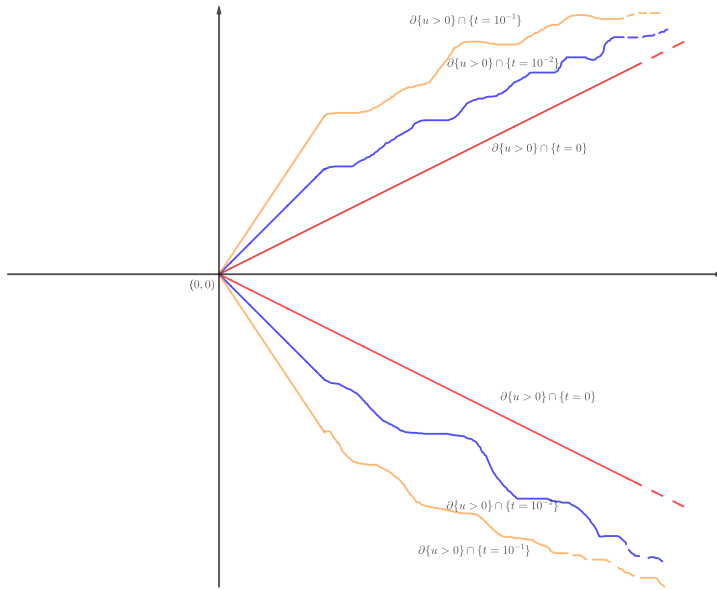$\partial\{u > 0\} \cap \{t = 10^{-1}\}$

FIGURE 1. Bidimensional example of a Lipschitz free boundary preserving corners.

Let us stress, at this point, that, as already mentioned, the contribution of Athanasopoulos, Caffarelli, and Salsa was inspired by Caffarelli's one in the elliptic case. This was fundamental to develop the study of the free boundary regularity as a research topic. The key step of his method consists in finding a family of comparison subsolutions using supconvolutions on balls of variable radii. More recently, D. De Silva in [9] improved Caffarelli's approach to obtain the $C^{1,\alpha}$ regularity of flat free boundaries for problems in the class of the one-phase nonhomogeneous Bernoulli free boundary problem

$$(7) \qquad \begin{cases} \Delta u = f & \text{in } \Omega \cap \{u > 0\}, \\ |\nabla u| = 1 & \text{on } F(u) := \Omega \cap \partial\{u > 0\}. \end{cases}$$

Specifically, she dealt with problems of the type

$$(8) \qquad \begin{cases} \text{tr}(A(x)D^2 u) = f & \text{in } \Omega \cap \{u > 0\}, \\ |\nabla u| = g & \text{on } F(u), \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $f \in C(\Omega) \cap L^\infty(\Omega)$, $g \in C^{0,\beta}(\Omega)$, $g \geq 0$, and $A(x) = (a_{ij}(x))_{1 \leq i,j \leq n}$, with $a_{ij} \in C^{0,\beta}(\Omega)$. Here, tr() denotes, as usual, the trace of a matrix. The strategy used by her was to show that the graph of a solution u to (8) satisfies an "improvement of flatness" property and then iterate it to achieve the $C^{1,\alpha}$ regularity. The main result in [9] essentially says that a sufficiently "flat" solution to (8) in a certain ball has the free boundary $F(u)$ which is a $C^{1,\alpha}$ graph in the interior. The exact statement reads as follows.

Theorem 2.1 (Flatness implies $C^{1,\alpha}$, De Silva). *Let u be a viscosity solution to (8) with $0 \in F(u)$, $g(0) = 1$ and $A(0) = I$. There exists a universal constant $\bar\varepsilon > 0$ such that, if the graph of u is $\bar\varepsilon$-flat in B_1, that is*

$$(x_n - \bar\varepsilon)^+ \leq u(x) \leq (x_n + \bar\varepsilon)^+, \quad x \in B_1,$$

and

$$[a_{ij}]_{C^{0,\beta}(B_1)} \leq \bar\varepsilon, \quad \|f\|_{L^\infty(B_1)} \leq \bar\varepsilon, \quad [g]_{C^{0,\beta}(B_1)} \leq \bar\varepsilon$$

then $F(u)$ is $C^{1,\alpha}$ in $B_{1/2}$.

Remark 2.1. *Notice that in the theorem above viscosity solutions are considered. This notion of solution had already exploited by Caffarelli in [4, 5] and was exactly one of the most innovative elements of his work. Furthermore, the assumptions $0 \in F(u)$, $g(0) = 1$ and $A(0) = I$, and the conditions*

$$[a_{ij}]_{C^{0,\beta}(B_1)} \leq \bar\varepsilon, \quad \|f\|_{L^\infty(B_1)} \leq \bar\varepsilon, \quad [g]_{C^{0,\beta}(B_1)} \leq \bar\varepsilon$$

mean that (8) is formally close to (1).

To enter more deeply inside De Silva's work [9], in view of the main Theorem 2.1 and the strategy to prove it, it becomes crucial to clarify the notion of flatness used. In this regard, the technical definition of ε-flatness is the following.

Definition 2.1. *A function u is ε-flat in Ω if*

$$(x \cdot \nu - \varepsilon)^+ \leq u(x) \leq (x \cdot \nu + \varepsilon)^+, \quad x \in \Omega,$$
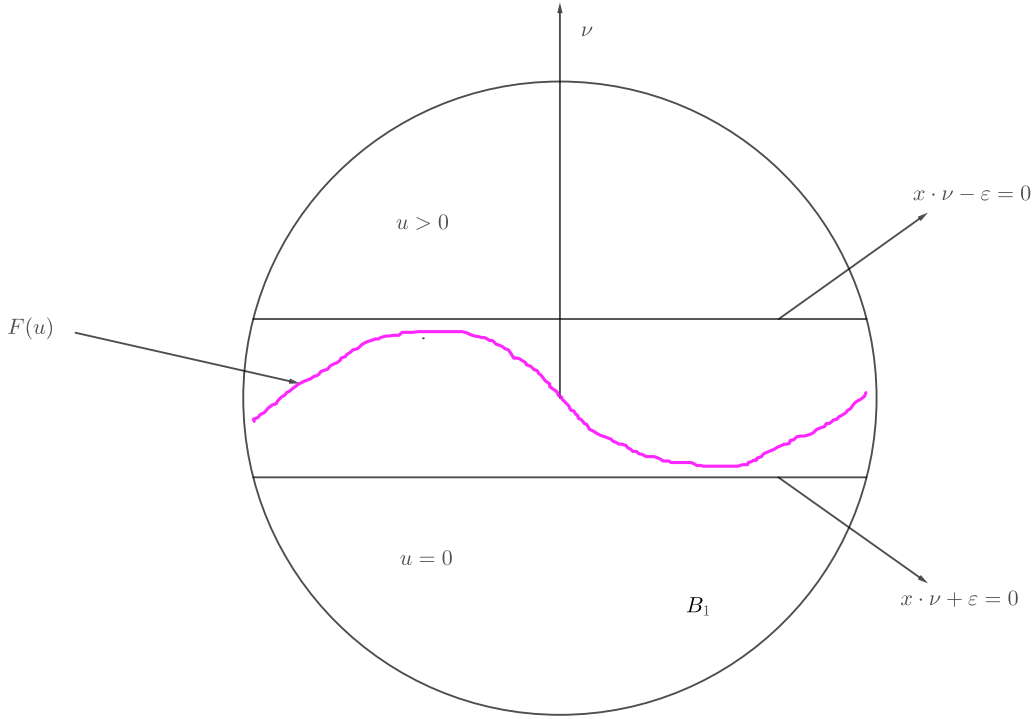
where $|\nu| = 1$.

FIGURE 2. Flatness assumption in [9].

Remark 2.2. *The previous definition, in particular, implies*

$$\begin{cases} F(u) \subseteq \{-\varepsilon \le x \cdot \nu \le \varepsilon\}, \\ u = 0 \quad in \ \{x \cdot \nu < -\varepsilon\}, \\ u > 0 \quad in \ \{x \cdot \nu > \varepsilon\}, \end{cases}$$

i.e. $F(u)$ is trapped in a strip of width 2ε (a region between two parallel hyperplanes at distance 2ε from each other), and $u = 0$ on one side of this strip, while $u > 0$ on the other side, as shown in Figure 2.

According to this notion of flatness, it becomes enough intuitive to think about what an improvement of flatness result means, see Figure 3. We recall here the precise statement of such property.

Lemma 2.1 (Improvement of flatness). *Let u be a solution to (8) in B_1 with*

$$\|f\|_{L^\infty(B_1)} \le \varepsilon^2, \quad \|g - 1\|_{L^\infty(B_1)} \le \varepsilon^2, \quad \|a_{ij} - \delta_{ij}\|_{L^\infty(B_1)} \le \varepsilon,$$
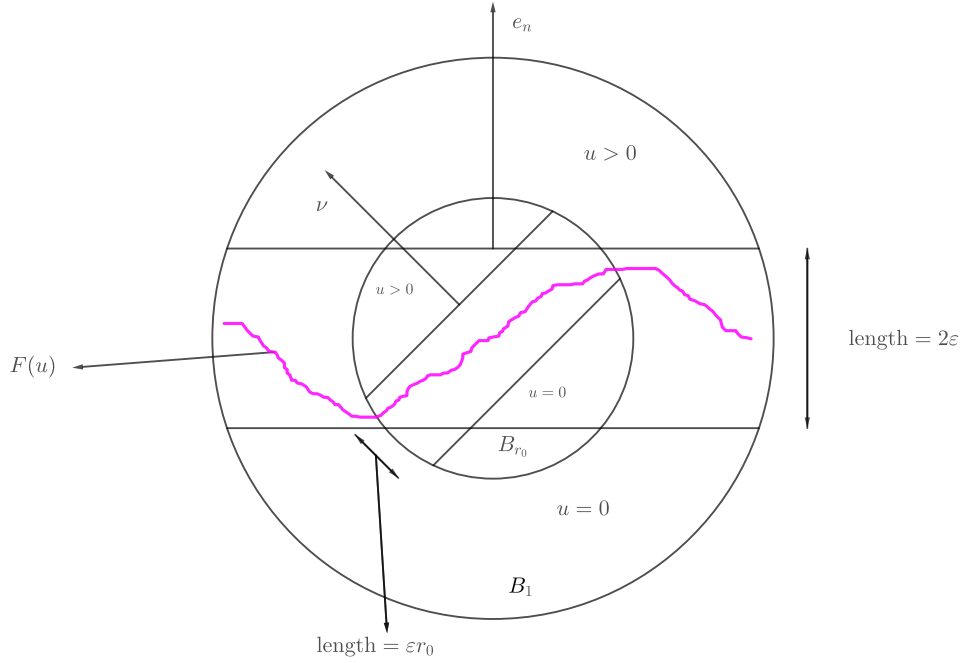
FIGURE 3. Improvement of flatness in [9]

satisfying

$$(x_n - \varepsilon)^+ \leq u(x) \leq (x_n + \varepsilon)^+ \quad \text{for } x \in B_1,$$

with $0 \in F(u)$. If $0 < r \leq r_0$ for r_0 a universal constant and $0 < \varepsilon \leq \varepsilon_0$ for some ε_0 depending on r, then

$$\left(x \cdot \nu - r\frac{\varepsilon}{2}\right)^+ \leq u(x) \leq \left(x \cdot \nu + r\frac{\varepsilon}{2}\right)^+ \quad \text{for } x \in B_r,$$

with $|\nu| = 1$ and $|\nu - e_n| \leq C\varepsilon$ for a universal constant C.

We point out that the condition $|\nu - e_n| \leq C\varepsilon$ is the crucial one to get the $C^{1,\alpha}$ regularity of $F(u)$ after the iteration of this kind of property.

The techniques developed in [9] have turned out to be very flexible and have been widely generalized to a variety of settings, including "thin" free boundary problems, two-phase nonhomogeneous problems, and constrained minimization problems, see for instance [12], [10] and [13], for which regularity results have not been proved using Caffarelli's approach yet. We remark that in [13] variational techniques, instead of viscosity ones, are employed.

It is thus important to understand if the strategy exploited in [9] might be applied in the context of time dependent problems. For this purpose, in [11], coherently with the ideas in [9], the regularity of flat free boundaries for the one-phase Stefan problem (3) is investigated by relying on perturbation arguments leading to a linearization of the problem. We end up this section pointing out that the methods developed in [11] might be suitable to further extensions, such as to the two-phase form of the Stefan problem and to the parabolic version of the "thin" one-phase problem.

3. Sketch of the proof of Theorem 1.1

In this section, we look at the proof of Theorem 1.1. Before going into the details, we remark that this theorem is basically equivalent to the previously mentioned flatness result contained in [2].

Concerning the proof, we first introduce some definitions. Coherently with [9], viscosity solutions are considered.

Definition 3.1. *We say that $u : \Omega \times [0, T] \to \mathbb{R}^+$ solves (3) in the viscosity sense if u is continuous and its graph cannot be touched by above (resp. below), at a point (x_0, t_0) in a parabolic cylinder $B_r(x_0) \times (t_0 - r^2, t_0]$, by the graph of a classical strict supersolution φ^+ (resp. subsolution).*

For the sake of completeness, we provide the notion of a function touching another one by above/below.

Definition 3.2. *We say that a function φ touches a function u by above (resp. below) at (x_0, t_0) in a parabolic cylinder $B_r(x_0) \times (t_0 - r^2, t_0]$, if $\varphi(x_0, t_0) = u(x_0, t_0)$ and $u(x, t) \leq \varphi(x, t)$ (resp. $u(x, t) \geq \varphi(x, t)$) for all $(x, t) \in B_r(x_0) \times (t_0 - r^2, t_0]$.*

The initial step of the proof is to remark that if a solution u satisfies the hypotheses of Theorem 1.1, then, using an appropriate dilation, we can extend the flatness assumption to the whole function u instead of just the free boundary $\partial\{u > 0\}$. More precisely, we consider the parabolic rescaling u_λ of the function u, defined in $B_1 \times [-(K\lambda)^{-1}, 0]$,

$$
(9) \qquad u_\lambda : (x, t) \longrightarrow \frac{u(\lambda x, \lambda^2 t)}{\lambda}, \qquad (x, t) \in B_1 \times [-(K\lambda)^{-1}, 0].
$$

Then, u_λ formally solves a Stefan problem with possibly small speed coefficient λ,

$$(10) \qquad \begin{cases} (u_\lambda)_t = \Delta(u_\lambda) & \text{in } (B_1 \times [-1,0]) \cap \{u_\lambda > 0\}, \\ (u_\lambda)_t = \lambda |\nabla(u_\lambda)|^2 & \text{on } (B_1 \times [-1,0]) \cap \partial\{u_\lambda > 0\}, \end{cases}$$

and it turns out to satisfy for all small $\eta > 0$

$$a_n(t) \left(x_n - b(t) - \eta^{1+\beta}\right)^+ \leq u \leq a_n(t) \left(x_n - b(t) + \eta^{1+\beta}\right)^+ \quad \text{in } B_\eta \times [-\lambda^{-1}\eta, 0].$$

Rescaling this condition back to the original coordinates, we obtain that u is whole flat in $B_{\eta\lambda} \times [-\eta\lambda, 0]$.

At this point, Theorem 1.1 is a direct consequence of the following main result, which is the correspondent one to Theorem "Flatness implies $C^{1,\alpha}$" in [9].

Theorem 3.1. *Fix a constant K (large) and let u be a solution to the one-phase Stefan problem* (3) *in $B_{2\lambda} \times [-2\lambda, 0]$ for some $\lambda \leq 1$. Assume that $0 \in \partial\{u > 0\}$, and*

$$a_n(t) \left(x_n - b(t) - \varepsilon_1 \lambda\right)^+ \leq u \leq a_n(t) \left(x_n - b(t) + \varepsilon_1 \lambda\right)^+,$$

with

$$K^{-1} \leq a_n \leq K, \qquad |a_n'(t)| \leq \lambda^{-2}, \qquad b'(t) = -a_n(t),$$

for some small ε_1 depending only on K and n. Then in $B_\lambda \times [-\lambda, 0]$ the free boundary $\partial\{u > 0\}$ is a $C^{1,\alpha}$ graph in the x_n direction.

The fundamental step to prove Theorem 3.1 is to show that an improvement of flatness property is indeed enjoyed, according to the approach developed in [9]. The natural question could be: what type of improvement of flatness result can we expect for solutions of the one-phase Stefan problem (3)? The main difficulties with (3) lie in the answer to this question. As a matter of fact, even if at scale 1 a solution u to (3) is trapped between nearby translations of an exact traveling wave, we cannot expect that at much smaller scale the same property will still hold. This is different from what happens in the elliptic counterpart studied in [9], where the preservation of such property is exactly the improvement of flatness result, see Lemma 2.1. Specifically, this discrepancy is due to a lack of natural scaling for (3), where, in general, by natural scaling for a free boundary problem we mean a rescaling of its solutions which preserves the structure of the problem.

Indeed, we consider again the parabolic rescaling u_λ (9) of a solution u to (3), which we could think to be the natural one since (3) is a parabolic problem. Nevertheless, u_λ formally solves (10), in which if we let λ go to 0, we achieve, again formally, that the limiting solution

(11) $$\tilde{u} := \lim_{\lambda \to 0} u_\lambda$$

satisfies

$$\begin{cases} \tilde{u}_t = \Delta \tilde{u} & \text{in } (B_1 \times (-1, 0]) \cap \{\tilde{u} > 0\}, \\ \tilde{u}_t = 0 & \text{on } (B_1 \times (-1, 0]) \cap \partial\{\tilde{u} > 0\}, \end{cases}$$

where, by the sake of simplicity, we work in $B_1 \times (-1, 0]$. Then, the free boundary $\partial\{\tilde{u} > 0\}$ of $\tilde{u}$ does not move, so we can not expect to have regularization and thus to transfer regularity to the original (3), which is what, in general, the existence of a natural scaling allows to do. At the same time, if we take the hyperbolic rescaling, again, by simplicity, denoted by u_λ, of a solution u to (3)

$$u_\lambda : (x, t) \longrightarrow \frac{u(\lambda x, \lambda t)}{\lambda}, \qquad (x, t) \in B_1 \times (-1, 0],$$

then u_λ formally satisfies

$$\begin{cases} \lambda(u_\lambda)_t = \Delta(u_\lambda) & \text{in } (B_1 \times (-1, 0]) \cap \{u_\lambda > 0\}, \\ (u_\lambda)_t = |\nabla(u_\lambda)|^2 & \text{on } (B_1 \times (-1, 0]) \cap \partial\{u_\lambda > 0\}. \end{cases}$$

Letting again λ go to 0, the limiting solution $\tilde{u}$ defined in (11) formally solves

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } (B_1 \times (-1, 0]) \cap \{\tilde{u} > 0\}, \\ \tilde{u}_t = |\nabla \tilde{u}|^2 & \text{on } (B_1 \times (-1, 0]) \cap \partial\{\tilde{u} > 0\}, \end{cases}$$

which is the one-phase Hele-Shaw problem, studied in [6] by similar methods to [11]. This problem does not possess good continuity properties in time, so again we do not have regularity in the limiting problem and we can not transfer regularity to (3). Arguments of the same type can be applied to all the other possible rescalings we can consider, hence a natural scaling does not exist for (3). We stress that the existence of a natural scaling in the elliptic counterpart established in [9] allows to iterate indefinitely the improvement of flatness property.

In view of these technical difficulties, we need to find a different method to face the question of what type of improvement of flatness result can we expect for solutions of the Stefan problem (3). For this purpose, we can exploit a Hodograph transform which reduces (3) to an equivalent nonlinear problem with fixed boundary and oblique derivative boundary condition. Precisely, the graph of a solution u to (3) in $\mathbb{R}^{n+2}$

$$\Gamma := \{(x, x_{n+1}, t)| \quad x_{n+1} = u(x_1, x_2, \ldots, x_n, t)\}$$

can be viewed as the graph of a possibly multi-valued function $\bar{u}$ with respect to the x_n direction

$$\Gamma := \{(x, x_{n+1}, t)| \quad x_n = \bar{u}(x_1, x_2, \ldots, x_{n-1}, x_{n+1}, t)\}.$$

Then, abstractly, since u solves (3), $\bar{u}$ satisfies the quasilinear parabolic equation with oblique derivative boundary condition

$$
(12) \qquad
\begin{cases}
\bar{u}_t = \mathrm{tr}(\bar{A}(\nabla \bar{u})\, D^2 \bar{u}) & \text{in } \{x_{n+1} > 0\}, \\
\bar{u}_t = g(\nabla \bar{u}) & \text{on } \{x_{n+1} = 0\},
\end{cases}
$$

with $\bar{A}(p)$ symmetric, positive definite as long as $p_n \neq 0$, and $g_n(p) > 0$. We note that the free boundary $\partial\{u > 0\}$ of u is given by the graph of the trace of $\bar{u}$ on $\{x_{n+1} = 0\}$. In particular, it turns out that a solution $\bar{u}$ of (12) enjoys an improvement of flatness property.

Before providing the precise statement of this result, we introduce some notation. By the sake of simplicity, in the following we will refer to the coordinates of $\bar{u}$ as $(x_1, x_2, \ldots, x_{n-1}, x_n, t)$ instead of $(x_1, x_2, \ldots, x_{n-1}, x_{n+1}, t)$.

Notation. For $n \geq 2$, given $r > 0$ we set

$$Q_r := (-r, r)^n \qquad \mathcal{C}_r := (Q_r \cap \{x_n > 0\}) \times (-r, 0],$$
$$\mathcal{F}_r := \{(x, t)| \quad x \in Q_r \cap \{x_n = 0\}, \ t \in (-r, 0]\}.$$

Moreover, we denote by $l_{a,b}(x, t)$ functions which for each fixed t are linear in the x variable, whose coefficients in the x' variable are independent of t, and so that they

satisfy the boundary condition in (12) on $\{x_n = 0\}$. More precisely, these functions are defined as

$$l_{a,b}(x,t) := a(t) \cdot x + b(t),$$

with

$$a(t) := (a_1, \ldots, a_{n-1}, a_n(t)), \quad a_i \in \mathbb{R}, \ i = 1, \ldots, n-1,$$

and

$$b'(t) = g(a(t)).$$

Now, we are ready to state the improvement of flatness result for solutions $\bar{u}$ of (12).

Proposition 3.1 (Improvement of flatness). *Assume that u is a viscosity solution to* (12), *possibly multi-valued, which satisfies*

(13) $$|u - l_{a,b}| \leq \varepsilon\lambda \quad in \ \overline{\mathcal{C}}_\lambda, \quad |a'_n(t)| \leq \delta\varepsilon\lambda^{-2},$$

with

$$\varepsilon \leq \varepsilon_0, \quad \lambda \leq \lambda_0, \quad \lambda \leq \delta\varepsilon.$$

Then there exists $l_{\tilde{a},\tilde{b}}$ such that

$$|u - l_{\tilde{a},\tilde{b}}| \leq \frac{\varepsilon}{2}\tau\lambda \quad in \ \overline{\mathcal{C}}_{\tau\lambda},$$

with

$$|a(t) - \tilde{a}(t)| \leq C\varepsilon, \qquad |\tilde{a}'_n(t)| \leq \frac{\delta\varepsilon}{2}(\tau\lambda)^{-2}.$$

Here, the constants $\varepsilon_0, \lambda_0, \delta, \tau > 0$ small and C large depend only on n, and K.

In fact, this result holds for a more general class of problems, see [11]. For the sake of completeness, we add to Proposition 3.1 the definition of multi-valued viscosity solutions to (12).

Definition 3.3. *Assume that $u : \overline{\mathcal{C}}_\lambda \to \mathbb{R}$ is a multi-valued function with compact graph in $\mathbb{R}^{n+2}$. We say that u is a viscosity subsolution to* (12) *if u cannot be touched by above at points in $\mathcal{C}_\lambda \cup \mathcal{F}_\lambda$ (locally, in parabolic cylinders) by (single-valued) classical strict supersolutions φ of* (12).

Similarly, we can define viscosity supersolutions and viscosity solutions to (12) for multi-valued functions. Again for completeness, we give the notion of a single-valued function touching a multi-valued one by above.

Definition 3.4. *We say that a single-valued function φ touches a multi-valued function u by above at $(x_0, t_0) \in \mathcal{C}_\lambda \cup \mathcal{F}_\lambda$ in a parabolic cylinder $B_r(x_0) \times (t_0 - r^2, t_0]$, if $\varphi(x_0, t_0) \in u(x_0, t_0)$ and $u(x, t) \leq \varphi(x, t)$ for all possible values of u at (x, t), and for all $(x, t) \in B_r(x_0) \times (t_0 - r^2, t_0]$.*

To find Proposition 3.1, we investigate the linearized problem associated to (12), namely the oblique derivative parabolic problem

$$
(14) \qquad
\begin{cases}
\lambda v_t = tr(A(t)D^2 v) & \text{in } \{x_n > 0\}, \\
v_t = \gamma(t) \cdot \nabla v & \text{on } \{x_n = 0\},
\end{cases}
$$

with $A(t)$ uniformly elliptic and $\gamma_n > 0$. The fact that we study this linearized problem to achieve the improvement of flatness property directly comes from the original idea of De Silva, which was that the proof which gives regularity in the linearized problem can be applied with appropriate differences, but following the same scheme, to the nonlinear problem as well. Nevertheless, this is not evident in [9], since the desired regularity of the Neumann problem for the Laplace operator is reached by classical arguments.

We sketch, at this point, the proof of Proposition 3.1.

Sketch of the proof of Proposition 3.1. The improvement of flatness result relies on various Hölder estimates for solutions to (14). Thus, it is crucial to prove such estimates. In particular, $C^{1,\alpha}$ estimates turn out to be the fundamental ones. In proving these, an essential role is played by estimates for the $1D$ linear problem

$$
(15) \qquad
\begin{cases}
v_t = \frac{1}{\lambda}\{a^{nn}(t)v_{nn} + h(x_n, t)\} & \text{in } \mathcal{C}_1, \\
v_t = \gamma_n(t)v_n + f(t) & \text{on } \mathcal{F}_1,
\end{cases}
$$

with

$$h(x_n, t) := \sum_{(i,j) \neq (n,n)} a^{ij}(t)\, v_{ij}((0, x_n), t),$$

$$f(t) := \sum_{i<n} \gamma_i(t)\, v_i(0, t),$$

which is obtained, roughly speaking, fixing $x' = 0$ in (14). More precisely, since problem (14) is invariant with respect to translations in the x' variable and a difference of viscosity solutions is still a viscosity solution, we have that also v_i, $i = 1, \ldots, n-1$, are viscosity solutions to (14). Therefore, by C^α estimates for solutions to (14), we get that $v \in C^\infty$ in the x' variable. Then, we are indeed left with the understanding of the $1D$ problem (15). Now, after a parabolic rescaling and a compactness argument, we further reduce ourselves to study

(16)
$$\begin{cases} \bar{v}_t = \bar{A}\, \bar{v}_{xx} & \text{in } \mathcal{P}_{1/2}, \\ \bar{v}_t = 0 & \text{on } \{x = 0\}, \end{cases}$$

with $\bar{A}$ constant and $\mathcal{P}_{1/2} := \left(0, \frac{1}{2}\right) \times \left(-\frac{1}{4}, 0\right]$. Here, we simply denote x_n by x, because we are in a one-dimensional spatial setting. We point out that the boundary condition in (16) tells us that $\bar{v}$ is constant on $\{x = 0\}$. Then C^2 estimates for the standard heat equation imply the condition

(17) $$|\bar{v} - (\bar{a}x + \bar{b})| \leq C\tau^2 \leq \frac{1}{2}\tau^{1+\alpha} \qquad \text{in} \quad \mathcal{P}_\tau, \quad \tau \text{ small},$$

where the fact that $\bar{a}x + \bar{b}$ does not depend on t is a consequence of the boundary condition in (16).

Going back, at this point, to the initial solution v of (14), (17) reads

$$|v - (a_1 x + b_1(t))| \leq \frac{3}{4}(\tau\rho)^{1+\alpha} \quad \text{in} \quad \mathcal{P}_{\tau\rho}, \quad \rho \text{ small},$$

with

$$b_1'(t) = \lambda\gamma(t)a_1,$$

which is a pointwise $C^{1,\alpha}$ estimate at the origin. This pointwise $C^{1,\alpha}$ estimate can be applied at other points on $\{x = 0\}$, and combined these with interior $C^{1,\alpha}$ estimates for parabolic equations, we have the desired $C^{1,\alpha}$ estimates for v.

We conclude the sketch of the proof with the exact statement which provides the $C^{1,\alpha}$ estimates for solutions of (14). Constants depending only on n and K are called universal.

Proposition 3.2 (Interior estimates)**.** *Let v be a viscosity solution to* (14) *such that* $\|v\|_{L^\infty} \leq 1$, *with*

$$K^{-1}I \leq A(t) \leq KI, \qquad K^{-1} \leq \gamma_n \leq K, \qquad |\gamma| \leq K,$$

$$\lambda \in (0,1], \qquad |A'(t)| \leq \lambda^{-1}, \qquad |\gamma'(t)| \leq \lambda^{-1},$$

for some large constant K. Then

$$|\nabla v|, \ |D^2 v| \ \leq C \qquad in \quad \mathcal{C}_{1/2},$$

and for each $\rho \leq 1/2$, there exists $l_{\bar{a},\bar{b}}$ such that

$$|v - l_{\bar{a},\bar{b}}| \leq C\rho^{1+\alpha} \qquad in \ \mathcal{C}_\rho,$$

where

$$\bar{b}'(t) = \gamma(t) \cdot \bar{a}, \qquad |\bar{a}'_n| \leq C\rho^{\alpha-1}\lambda^{-1}, \qquad |\bar{a}| \leq C,$$

with α, C universal.

$\square$

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DIPARTIMENTO DI MATEMATICA, ALMA MATER STUDIORUM UNIVERSITÀ DI BOLOGNA, PIAZZA DI PORTA SAN DONATO 5, 40126 BOLOGNA, ITALY

E-mail address: `nicolo.forcillo2@unibo.it`