ON MINIMAX CHARACTERIZATION IN NON-LINEAR EIGENVALUE PROBLEMS
SULLA CARATTERIZZAZIONE DI MINIMAX IN PROBLEMI DI AUTOVALORI NON LINEARI

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Abstract. This is a note based on the paper [20] written in collaboration with N. Fusco and Y. Zhang. The main goal is to introduce minimax type variational characterization of non-linear eigenvalues of the p-Laplacian and other results related to shape and spectral optimization problems.

Sunto. Questa è una nota basata sul documento [20] scritto in collaborazione con N. Fusco e Y. Zhang. L'obiettivo principale è introdurre la caratterizzazione variazionale di tipo minimax di autovalori non lineari del p-Laplaciano e altri risultati relativi a problemi di forma e ottimizzazione spettrale.

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1. Introduction

Given $A \subseteq \mathbb{R}^n$ and $1 < p < \infty$, if for some $\lambda \geq 0$, the Dirichlet problem

$$
\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u \text{ in } A, \\ u \in W_0^{1,p}(A), \end{cases} \tag{1.1}
$$

admits a non-zero weak solution, then λ is called a non-linear eigenvalue (denoted $\lambda \in \sigma_p(A)$ hereafter) and the weak solution u is its corresponding eigenfunction.

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If $A \subset \mathbb{R}^n$ is an open set of finite measure, it is known that the eigenvalues are the critical values of the Rayleigh quotient

$$\mathcal{R}_A(w) = \frac{\displaystyle\int_A |\nabla w|^p \, dx}{\displaystyle\int_A |w|^p \, dx}$$

and the corresponding weak solutions of equation (1.1) are the critical points of $\mathcal{R}_A(w)$ among all non-zero functions in $w \in W_0^{1,p}(A)$. The first eigenvalue $\lambda_1(A)$ is defined as the minimum value of $\mathcal{R}_A(w)$. Also, $\lambda \in \sigma_p(A)$ of (1.1) corresponds to Lagrange multipliers of the minimization of the p-energy $E(u) = \int_A |\nabla u|^p \, dx$ constrained to the manifold

$$\mathcal{M} = \mathcal{M}_p(A) := \big\{ u \in W_0^{1,p}(A) : \|u\|_{L^p(A)} = 1 \big\}. \tag{1.2}$$

There are many ways to generate higher eigenvalues and not much is known on their behavior when $p \neq 2$.

1.1. **Minimax characterization of eigenvalues.** Critical values of constrained minimization can be defined in more general frameworks some of which we recall for the reader's convenience from classical critical point theory of eigenvalues. We refer the reader to [22] for more details.

Given a Banach space X and two functionals $E, G \in C^1(X)$, the minimizers u of the constrained problem $\min \big\{ E(w) : G(w) = 1 \big\}$ satisfy the equation

$$DE(u) = \lambda DG(u) \tag{1.3}$$

for some Lagrange multiplier $\lambda \in \mathbb{R}$, where D denotes the Fréchet derivative. Thus, (1.1) is a special case of the equation (1.3) if one takes $X = W_0^{1,p}(A)$,

$$E(w) = \int_A |\nabla w|^p \, dx \quad \text{and} \quad G(w) = \int_A |w|^p \, dx. \tag{1.4}$$

Note that for every $t \in \mathbb{R}$, $u \in W_0^{1,p}(A)$, the derivative $DE(u) - tDG(u)$ is the element of $W^{-1,p'}(A)$ such that for all $\phi \in W_0^{1,p}(A)$

$$\langle DE(u) - tDG(u), \phi \rangle = p \int_A (|\nabla u|^{p-2} \nabla u \cdot \nabla \phi - t|u|^{p-2} u \phi) \, dx.$$

From this equality it follows immediately that if u is a critical point for $\widetilde{E}$ on $\mathcal{M}_p$, then u is an eigenfunction and $E(u)$ is an eigenvalue of the p-Laplacian.

When X is a Hilbert space (corresponding to $W_0^{1,2}(A)$ in this setting), then the discreteness of the spectrum for (1.3) is well known from the classical theory of linear operators and the eigenvalues have the well known Courant-Fischer minimax theorem. But for the case of $W_0^{1,p}(A)$ with $p \neq 2$, the existence of a spectral gap is not known in general, except for the first and second eigenvalues on open connected sets (see [29, 25]) and analogous minimax theorems are much harder to prove.

If $\mathcal{M}$ of (1.2) admits a compact group of symmetries with respect to which the energy functional E is invariant, then higher eigenvalues can be generated by finding families of equivariant homeomorphism of $\mathcal{M}$ and defining infima on supremum of the energy functional over such homeomorphism-invariant families of subsets of $\mathcal{M}$. In particular, when the group is $\mathbb{Z}_2$ and $E(u)$ is even, then the invariant family of subsets are given by Σ_k for any $k \in \mathbb{N}$, which are the collection of all symmetric subsets M contained in $W_0^{1,p}(A)$ with $\gamma(M) \geq k$, where γ is the so called, *Krasnoselskii's genus*. Thus, the numbers

$$\lambda_k(A) = \inf_{\mathcal{M}\in\Sigma_k} \Big[\sup_{u\in\mathcal{M}} \mathcal{R}_A(w) \Big] \tag{1.5}$$

form an increasing sequence of eigenvalues. More details can be found in [21, 25] etc. It is not known whether all the eigenvalues of the p-Laplacian are of this form if $p \neq 2$. Just for the case of the second eigenvalue, it was proved by Anane-Tsouli [5] (see also [25]) that, given a bounded and connected open set A, if $\lambda_1(A)$ and $\lambda_2(A)$ are defined as in (1.5), then $\lambda_1(A)$ is the smallest eigenvalue and there are no other eigenvalues in the interval $(\lambda_1(A), \lambda_2(A))$.

There are other ways, different from the ones recalled above, to generate critical values using a variant of the mountain pass lemma by Ambrosetti-Rabinowitz [3]. We recall that the norm of the Fréchet derivative of the restriction $\widetilde{E}$ of E to $\mathcal{M}$ at a point $u \in \mathcal{M}$, is defined as

$$\|D\widetilde{E}(u)\|_* := \min \Big\{ \Big\| DE(u) - tDG(u) \Big\|_{X^*} : t \in \mathbb{R} \Big\},$$

where $\|\cdot\|_{X^*}$ denotes the norm of the dual space X^*. It is said that the functional E satisfies the *Palais-Smale condition on* $\mathcal{M}$, if for any sequence $u_h \in \mathcal{M}$ such that $E(u_h)$

is bounded and $\|D\widetilde{E}(u_h)\|_* \to 0$, there exists a subsequence of u_h converging strongly in X. Then, the following result holds, see [12, 22].

Theorem 1.1. *Let X be a Banach space and $E, G \in C^1(X)$. Assume that $DG \neq 0$ on $\mathcal{M}$ and that E satisfies the Palais-Smale condition on $\mathcal{M}$.*

Let $u_0, u_1 \in \mathcal{M}$ and $\rho > 0$ be such that $\|u_1 - u_0\|_X > \rho$ and

$$\inf\big\{E(u) : u \in \mathcal{M},\ \|u - u_0\|_X = \rho\big\} > \max\{E(u_0), E(u_1)\}.$$

If $\Gamma(u_0, u_1) := \big\{\gamma \in C\big([0,1], \mathcal{M}\big) : \gamma(0) = u_0,\ \gamma(1) = u_1\big\} \neq \emptyset$, then

$$\alpha = \inf_{\gamma\in\Gamma(u_0,u_1)} \Big[\max_{w\in\gamma([0,1])} E(w)\Big]$$

is a critical value for $\widetilde{E}$.

Using this, variational characterisation of $\lambda_2(A)$ was given by Cuesta-de Figueiredo-Gossez [12], who proved that for a bounded open and connected set A we have

$$\lambda_2(A) = \inf_{\gamma\in\Gamma(u_1,-u_1)} \left[\max_{w\in\gamma([0,1])} \int_A |\nabla w|^p\,dx\right], \tag{1.6}$$

where u_1 is the first nonnegative eigenfunction with $\|u_1\|_{L^p(A)} = 1$ and $\Gamma(u_1, -u_1)$ is the family of all continuous maps from $[0,1]$ to $\mathcal{M}$ with endpoints u_1 and $-u_1$. Later on, it was shown by Brasco-Franzina [9] that (1.6) still holds if A is any open set of finite measure, not necessarily connected. A different variational characterization of $\lambda_2(A)$ can be obtained by combining a result proved in [15] with the argument used by Brasco-Franzina in the proof of [10, Theorem 4.2]:

$$\lambda_2(A) = \inf_{f\in\mathcal{C}_{odd}(\mathbb{S}^1,\mathcal{M})} \left[\max_{u\in Im(f)} \int_A |\nabla w|^p\,dx\right], \tag{1.7}$$

where $\mathcal{C}_{odd}(\mathbb{S}^1, \mathcal{M})$ is the set of continuous and odd maps from $\mathbb{S}^1$ to $\mathcal{M}$. There have also been many other similar results in the literature of eigenvalue problems and critical point theory. We refer to the books [22, 32] and references therein.

1.2. **The main results.** In the paper [20], the author jointly with N. Fusco and Y. Zhang has studied the properties of the first two eigenvalues of the p-Laplacian in a p-quasi open set A (see Definition 2.1 below). For bounded open sets, it is known (see [30]) that if the first eigenvalue is simple, then it is isolated. In [20] it is proved that the same holds in the framework of p-quasi open sets, hence the notion of second eigenvalue would be well defined.

The following theorem is the main result of [20].

Theorem 1.2. *Let $A \subset \mathbb{R}^n$ be a p-quasi open set, $|A| < \infty$, let $u_1 \in W_0^{1,p}(A)$ be a normalized eigenfunction of $\lambda_1(A)$ and let*

$$\Gamma(u_1, -u_1) = \big\{\gamma \in C\big([0,1], \mathcal{M}\big) : \gamma(0) = u_1,\ \gamma(1) = -u_1\big\}.$$

Then, the second eigenvalue can be characterized as

$$\lambda_2(A) = \min_{\gamma \in \Gamma(u_1,-u_1)} \left[\max_{w \in \gamma([0,1])} \int_A |\nabla w|^p \, dx \right]. \tag{1.8}$$

The above result is also non-trivial for the case of open connected stes. The reason for consideration of quasi open sets is due to other results proved in the paper [20]. Given a bounded open $\Omega \subset \mathbb{R}^n$, the family $\mathscr{A}_p(\Omega)$ of all p-quasi open subsets of Ω is the right admissible class for shape optimization problems. The following is one of the other main results, which is the p-Laplacian counterpart of the existence theorem of Buttazzo-Dal Maso [11].

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and $F : \mathscr{A}_p(\Omega) \to \mathbb{R}$ be a decreasing function, lower semicontinuous with respect to γ_p-convergence (as defined in Definition 2.11). Then the minimization problem*

$$\min \big\{F(A) : A \in \mathscr{A}_p(\Omega),\ |A| = c\big\}, \tag{1.9}$$

where $0 < c \leq |\Omega|$, always has a solution.

In the following, there are some other results proved in [20].

Theorem 1.4. *Let $A_m \in \mathscr{A}_p(\Omega)$ be a sequence of p-quasi open sets such that we have $A_m \xrightarrow{\gamma_p} A \in \mathscr{A}_p(\Omega)$. Then the following holds.*

(1) *If* $\lambda_m \in \sigma_p(A_m)$ *for every* $m \in \mathbb{N}$ *and* $\lambda_m \to \lambda$ *as* $m \to \infty$, *then* $\lambda \in \sigma_p(A)$ *and the eigenfunctions* u_m *of* λ_m *converge in* $W^{1,r}(\Omega)$, *up to a subsequence, to an eigenfunction* u *of* λ, *for all* $1 \leq r < p$.

(2) *For* $i \in \{1,2\}$, *we have* $\lambda_i(A) \leq \liminf_{m\to\infty} \lambda_i(A_m)$.

From Theorem 1.4, we have the following corollary of Theorem 1.3.

Corollary 1.5. *Let* $f : \mathbb{R}^2 \to \mathbb{R}$ *be a lower semicontinuous function, separately increasing in both variables, and let* $\Omega \subset \mathbb{R}^n$ *be a bounded open set. For every* $0 < c \leq |\Omega|$, *there exists a* p*-quasi open minimizer* $A_0 \in \mathscr{A}_p(\Omega)$ *satisfying*

$$f\left(\lambda_1(A_0), \lambda_2(A_0)\right) = \min\left\{f\left(\lambda_1(A), \lambda_2(A)\right) : A \in \mathscr{A}_p(\Omega),\ |A| = c\right\}.$$

All the statements in the paper [20] have been given in the context of p-quasi open sets, assuming $1 < p \leq n$. However, all the arguments and tools used in the proofs, including the characterization of the second eigenvalue, do apply without changes also for the case $p > n$ when $\mathscr{A}_p(\Omega)$ reduces to the family of open sets.

2. Overview of the structure

Here we provide some details and develop the basic structure leading to the proofs of the aforementioned results. The reader is referred to [20] for more details.

2.1. **Quasi open and finely open sets.** For any measurable set $E \subseteq \mathbb{R}^n$, the p-capacity (equivalent to the Bessel capacity $C_{1,p}$ defined via the Bessel kernel as in [26, 33] etc.) is defined by

$$\mathrm{Cap}_p(E) := \inf_{u\in\mathcal{W}} \int_{\mathbb{R}^n} \left(|u|^p + |\nabla u|^p\right) dx,$$

with $\mathcal{W} = \{u \in W^{1,p}(\mathbb{R}^n) : u \geq 1$ a.e. in an open set $U \supset E\}$.

Definition 2.1 ((Quasi open sets))**.** A set $A \subset \mathbb{R}^n$ is said to be p*-quasi open* if for every $\varepsilon > 0$ there exists an open set U_ε such that $\mathrm{Cap}_p(U_\varepsilon \triangle A) < \varepsilon$; equivalently, if there exists an open set A_ε such that $A \cup A_\varepsilon$ is open and $\mathrm{Cap}_p(A_\varepsilon) < \varepsilon$.

A function $f : A \to \mathbb{R}$ defined on a quasi open set A is said to be p*-quasi continuous* if for every $\varepsilon > 0$, there exists an open set A_ε such that $\mathrm{Cap}_p(A_\varepsilon) < \varepsilon$ and the restriction of f to $A \setminus A_\varepsilon$ is continuous. It is well known that any function $u \in W^{1,p}(\mathbb{R}^n)$ has a p-quasi continuous representative, that is equal to u almost everywhere and hence u can be assumed to be p-quasi continuous itself.

As before, for an open set $\Omega \subset \mathbb{R}^n$ we denote the collection of all p-quasi open subsets of Ω by $\mathscr{A}_p(\Omega)$. If A is p-quasi open and $\mathrm{Cap}_p(A) > 0$ the space $W_0^{1,p}(A)$ can be defined as

$$W_0^{1,p}(A) = \bigcap \big\{ W_0^{1,p}(U) : U \text{ open}, U \supset A \big\},$$

which equivalently implies that $u \in W^{1,p}(\mathbb{R}^n)$ and any p-quasi continuous representative of u vanishes p-quasi-everywhere in $\mathbb{R}^n \setminus A$. The space $W_0^{1,p}(A)$, equipped with the norm induced by $W^{1,p}(\mathbb{R}^n)$, is a Banach space as usual. Also, we denote by $W^{-1,p'}(A)$ the dual space of $W_0^{1,p}(A)$ where $p' = p/(p-1)$.

Although p-quasi open sets do not form a topology, it is very closely linked to the p-*fine topology*, which is the coarsest topology on $\mathbb{R}^n$ making all (classical) p-superharmonic functions continuous. A more robust equivalent definition can be given using the Wiener criteria, as follows.

Definition 2.2 (Finely open sets)**.** A set $U \subset \mathbb{R}^n$ is *p-finely open* if for every $x \in U$

$$\int_0^1 \left(\frac{\mathrm{Cap}_p(B_r(x) \setminus U)}{r^{n-p}} \right)^{\frac{1}{p-1}} \frac{dr}{r} < \infty.$$

The fine topology has been extensively studied in the context of nonlinear potential theory. For more details we refer the reader to [19, 18, 1, 26] and to the references therein. The link between p-quasi open and p-finely open sets is the following from Kilpeläinen-Malý [26].

Theorem 2.3. *Given a set $A \subset \mathbb{R}^n$, the following are equivalent.*

(1) A is p-quasi open.

(2) $A = U \cup E$ where U is p-finely open and $\mathrm{Cap}_p(E) = 0$.

(3) There exists a p-quasi continuous function $u \in W^{1,p}(\mathbb{R}^n)$, $u \geq 0$, such that $A = \{u > 0\}$.

Moreover, any function f that is p-quasi continuous in A, is p-finely continuous in A up to a set of zero p-capacity and the sets $\{f > c\}$ and $\{f < c\}$ are p-quasi open for all

$c \in \mathbb{R}$. It is clear that a p-quasi open set A remains quasi open if we change it by a set of zero p-capacity and hence the characterisation $A = U \cup E$ in Theorem 2.3 unique up to sets of zero p-capacity.

The notion of quasi connectedness has also been defined in [20], as follows.

Definition 2.4 (Quasi connected sets)**.** A p-quasi open set $A \subset \mathbb{R}^n$ is *p-quasi connected* when for any two p-quasi open subsets A_1, A_2, if we have $A = A_1 \cup A_2$ and $\mathrm{Cap}_p(A_1 \cap A_2) = 0$, then either $\mathrm{Cap}_p(A_1) = 0$ or $\mathrm{Cap}_p(A_2) = 0$.

The notion of p-quasi connectedness is closely related to the topological notion of p-finely connected set. In fact, the following much stronger result holds, due to Björn-Björn [7].

Theorem 2.5. *Let A be a p-quasi open set. Then the following are equivalent.*

(1) If $u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\nabla u = 0$ a.e. in A, then there exists a constant c such that $u = c$ a.e. in A.

(2) A is p-quasi connected.

(3) $A = U \cup E$, where U is finely connected and finely open and $\mathrm{Cap}_p(E) = 0$.

Some technical results have been shown in [20] dependent on properties of quasi-connected sets, which were previously known for usual open and conneted sets.

2.2. **The p-Laplacian on quasi-open sets.** Given a quasi open set $A \in \mathscr{A}_p(\Omega)$ and $f \in W^{-1,p'}(A)$, the Dirichlet problem

$$\begin{cases} -\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f \quad \text{in } A \\ \qquad\qquad\qquad\qquad u \in W^{1,p}_0(A) \end{cases} \tag{2.1}$$

is defined in the usual weak sense, i.e. $u \in W^{1,p}_0(A)$ is a weak solution of the equation (2.1) if for every $\phi \in W^{1,p}_0(A)$, we have

$$\int_A |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dx = \langle f, \phi \rangle,$$

where $\langle \cdot , \cdot \rangle$ denotes the duality pairing between $W^{-1,p'}(A)$ and $W^{1,p}_0(A)$.

Let us set $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u) \in W^{-1,p'}(A)$. Following [26], we say that a p-quasi continuous function $u \in W^{1,p}(A)$ is a *fine supersolution* of the equation $-\Delta_p u = 0$, if for every nonnegative function $\phi \in W_0^{1,p}(A)$, we have

$$\int_A |\nabla u|^{p-2}\nabla u \cdot \nabla \phi \, dx \geq 0.$$

The monotonicity of the p-Laplacian operator ensures, as in the standard case of an open set, the existence of a unique weak solution of (2.1). This enable us to define the resolvent map as usual.

Definition 2.6 (Resolvent)**.** For a quasi open set $A \in \mathscr{A}_p(\Omega)$, the *resolvent map* for the p-Laplacian operator, is defined for any $f \in W^{-1,p'}(A)$ by setting $\mathscr{R}_{p,A}(f) := u$, where $u \in W_0^{1,p}(A)$ is the unique weak solution of (2.1).

It has been shown by Kilpeläinen-Malý [26, Th. 4.3] that limits of increasing sequence of fine supersolutions are quasi-continuous. Using this a simple proof of the following minimum principle for fine supersolutions is shown in [20], that was previously present in [28, Th. 4.1].

Theorem 2.7 (Minimum principle)**.** *Let $u \in W_0^{1,p}(A)$ be a fine supersolution on a p-quasi open and quasi connected set $A \subset \mathbb{R}^n$, $u \geq 0$ quasi everywhere. Then either $u > 0$ or $u = 0$ qasi everywhere in A.*

This is used in many places throughout the paper [20] including establishing certain properties of eigenvalues and in the proof of the minimax characterization of the second eigenvalue.

2.3. **Properties of eigenvalues.** All the eigenvalues are not just non-negative but bounded away from zero since we have

$$\lambda \;\geq\; c(n,p)\,|A|^{-p/n}$$

for every $\lambda \in \sigma_p(A)$, which is easy to show using the Sobolev inequality. Moreover, the arguments used in the proof of [30, Th. 3] can be also used to prove that if we have $\lambda_k \in \sigma_p(A)$ and $\lambda_k \to \lambda$ as $k \to \infty$ then $\lambda \in \sigma_p(A)$. Hence, the following is well-defined.

Definition 2.8 (First eigenvalue)**.** The first eigenvalue of the p-Laplacian in a p-quasi open set $A \subset \mathbb{R}^n$ of finite measure, is defined as

$$\lambda_1(A) := \min\big\{\lambda > 0 : \lambda \in \sigma_p(A)\big\}.$$

Since the p-Dirichlet energy admits a minimizer u_1 on $\mathcal{M}$ from the standard methods of calculus of variations, hence we can conclude

$$\lambda_1(A) = \int_A |\nabla u_1|^p \, dx = \min_{w \in \mathcal{M}} \int_A |\nabla w|^p \, dx.$$

where u_1 is an eigenfunction for $\lambda_1(A)$. For open sets of finite measure it is well known that every eigenvalue is the first eigenvalue in its nodal domains, i.e., if λ is an eigenvalue with eigenfunction u, then $\lambda = \lambda_1(\{u > 0\})$. The proof of this result for the eigenvalues of the p-Laplacian in an open set is due to Brasco-Franzina [8, Th. 3.1]. The same proof carries on in the framework of quasi open sets as well. Moreover, if $\lambda_1(A)$ is simple and $u \in W_0^{1,p}(A)$ is an eigenfunction of $\lambda_1(A)$, then u does not change sign. As shown in [20], this together with the minimum principle Theorem 2.7, also imply that $\{u > 0\}$ is a p-quasi connected component of A and $\lambda_1(A) = \lambda_1(\{u > 0\})$. All these are used in [20] to prove the following proposition, which was previously known for open, connected sets in [29, 30].

Proposition 2.9. *Let $A \subset \mathbb{R}^n$ be a p-quasi open set of finite measure. If the first eigenvalue $\lambda_1(A)$ is simple then it is isolated.*

Furthermore, it is also shown in [20] that if A be a p-quasi open and p-quasi connected set of finite measure, then $\lambda_1(A)$ is simple. This was previously known for open sets, see [2, 6], etc. Due to Proposition 2.9, the following definition of second eigenvalue is natural and well posed.

Definition 2.10 (Second Eigenvalue)**.** Let $A \subset \mathbb{R}^n$ be a p-quasi open set of finite measure. The second eigenvalue of the p-Laplacian on A is defined as follows.

$$\lambda_2(A) := \begin{cases} \min\big\{\lambda > \lambda_1(A) : \lambda \in \sigma_p(A)\big\} & \text{if } \lambda_1(A) \text{ is simple;} \\ \lambda_1(A) & \text{otherwise.} \end{cases} \tag{2.2}$$

The advantage of the above definition is that it does not require any regularity of the set A and still ends up in the right notion of eigenvalues on standard open sets. This was used before by Brasco-Franzina [9] for posibly disconnected sets.

2.4. **The γ_p-convergence and properties.** The γ_p-convergence of p-quasi open sets used in [20] is different from the case $p = 2$ considered in [11], where the weak convergence in $W^{1,p}$ of the resolvents was required and not the strong one. Indeed, in view of the nonlinearity of the p-Laplacian, requiring the strong convergence of the resolvents would end up in a too strong topology in $\mathscr{A}_p(\Omega)$ with very few compact sets. Instead, the definition below provides plenty of compact families in $\mathscr{A}_p(\Omega)$. However, the drawback is that now the proof of the lower semicontinuity of the eigenvalues requires a more delicate argument.

Definition 2.11. Let A_m, A be p-quasi open sets in $\mathscr{A}_p(\Omega)$ for every $m \in \mathbb{N}$. We say that the sequence A_m γ_p-converges to A as $m \to \infty$ and we write $A_m \xrightarrow{\gamma_p} A$, if $\mathscr{R}_{p,A_m}(f) \rightharpoonup \mathscr{R}_{p,A}(f)$ weakly in $W_0^{1,p}(\Omega)$ for every $f \in W^{-1,p'}(\Omega)$, where $\mathscr{R}_{p,A_m}$ are the resolvents defined as in Definition 2.6.

The above definition of γ_p-convergence of p-quasi open sets is strongly related to a convergence in the space $\mathcal{M}_0^p(\Omega)$ of Borel measures with values in $[0, \infty]$ vanishing on sets of zero p-capacity introduced by Dal Maso-Murat [14], where a sequence $\mu_m \in \mathcal{M}_0^p(\Omega)$ γ-converges to a measure $\mu \in \mathcal{M}_0^p(\Omega)$ if for any $f \in W^{-1,p'}(\Omega)$ the solutions $u_m \in W_0^{1,p}(\Omega)$ of the equations

$$\int_\Omega |\nabla u_m|^{p-2}\nabla u_m \cdot \nabla\varphi \, dx + \int_\Omega |u_m|^{p-2}u_m\varphi \; d\mu_m = \langle f, \varphi\rangle \quad \text{for all } \varphi \in W_0^{1,p}(\Omega)$$

converge weakly in $W^{1,p}(\Omega)$ to the solution of the corresponding equation with μ_m replaced by μ. It is evident that the Definition 2.11 is equivalent to the γ-convergence of the measures ∞_{A_m} to ∞_A in the sense of Dal Maso-Murat [14], where by ∞_A we denote the measure in $\mathcal{M}_0^p(\Omega)$ defined by

$$\infty_A(B) := \begin{cases} 0 & \text{if } \operatorname{Cap}_p(B \cap A) = 0, \\ +\infty & \text{if } \operatorname{Cap}_p(B \setminus A) > 0 \end{cases}$$

for all Borel sets $B \subset \Omega$. With this observation in mind, the next theorem follows immediately from a general result, see [14, Th. 6.3 and Th. 6.8].

Theorem 2.12. *$A_m \xrightarrow{\gamma_p} A$ in $\mathscr{A}_p(\Omega)$ if and only if $\mathscr{R}_{p,A_m}(1) \rightharpoonup \mathscr{R}_{p,A}(1)$ weakly in $W_0^{1,p}(\Omega)$. Moreover, in this case, we have that $\mathscr{R}_{p,A_m}(f) \to \mathscr{R}_{p,A}(f)$ strongly in $W_0^{1,r}(\Omega)$ for every $f \in W^{-1,p'}(\Omega)$ and any $1 \le r < p$.*

The above theorem is used to prove if the underlying quasi open sets γ_p-converge, then the limit of the sequence of eigenvalues is still an eigenvalue and corresponding eigenfunctions converge strongly in $W^{1,r}(\Omega)$ for all $1 \le r < p$.

3. Key ideas for the proofs

Most of the proofs of the results mentioned in Section 1 are not only lengthy and technical, but also quite involved. Here we highlight some of the main ideas involved and provide some brief illustrations of the techniques used in [20].

3.1. **Proof of Theorem 1.2.** Here is a brief outline of the proof of the minimax characterization of the second eigenvalue, which is the main result of [20].

Towards proving (1.8), we define

$$\lambda := \inf_{\gamma \in \Gamma(u_1, -u_1)} \left[\max_{w \in \gamma([0,1])} \int_A |\nabla w|^p \, dx \right].$$

Clearly, $\lambda_1(A) \le \lambda$. Also, it is clear that that to prove (1.8), it is enough to show that there exists an admissible curve $\gamma \in \Gamma(u_1, -u_1)$ such that

$$\max_{t \in [0,1]} \int_A |\nabla \gamma(t)|^p \, dx = \lambda_2(A). \tag{3.1}$$

Indeed if $\lambda_1(A)$ is not simple, then we trivially have $\lambda = \lambda_2(A) = \lambda_1(A)$. On the other hand, if $\lambda_1(A)$ is simple, then by Theorem 1.1 (it is also shown in [20] that the p-Dirichlet energy satisfies the Palais-Smale condition) we have that $\lambda \in \sigma_p(A)$; since by Definition 2.10 there is no other eigenvalue between $\lambda_1(A)$ and $\lambda_2(A)$, from (3.1) we get $\lambda = \lambda_2(A)$, thus concluding the proof.

All that remains for the proof of Theorem 1.2 is to show the existence of such an admissible curve satisfying (3.1). Letting $u_1, u_2 \in \mathcal{M}$ as eigenfunctions of $\lambda_1(A)$ and

$\lambda_2(A)$, we can assume $u_1 \geq 0$ without loss of generality but subtle arguments are required to deal with changing signs of u_2. We set U such that $U := \{u_1 > 0\}$ if $\lambda_1(A)$ is simple so that U is a p-quasi connected component of A, otherwise U is one of the p-quasi connected components where u_1 is not identically zero. In the latter case, an admissible curve satisfying (3.1) is given by

$$\gamma(t) := \frac{a(t)u_1\chi_U \, + b(t)u_1\chi_{A\setminus U}}{\left(|a(t)|^p\|u_1\|^p_{L^p(U)} \, + \, |b(t)|^p\|u_1\|^p_{L^p(A\setminus U)}\right)^{1/p}},$$

where $a, b : [0,1] \to [-1,1]$ are continuous functions with $a(0) = b(0) = 1, a(1) = b(1) = -1$ and $|a| + |b| > 0$. If either $\lambda_1(A)$ is simple and u_2 does not change sign in U (say $u_2 \geq 0$) or $\lambda_1(A)$ is not simple and u_1 is supported in U, we define the admissible curve $\gamma \in \Gamma(u_1, -u_1)$ satisfying (3.1) by setting

$$\gamma(t) := a(t)u_1 \, + b(t)u_2 \tag{3.2}$$

where $a, b \in C([0,1])$ such that $a(0) = 1, a(1) = -1$ and $|a|^p + |b|^p = 1$. Now, the most non-trivial case remains, i.e. $\lambda_1(A)$ is simple and u_2 changes sign in U. In this case u_2^+ cannot be an eigenfunction since otherwise by the minimum principle Theorem 2.7, either $u_2^+ > 0$ or $u_2^+ = 0$ quasi everywhere in U, thereby contradicting the hypothesis. Now the construction of the curve is done in the following steps. Since $u_1 \geq 0$, the change in sign from u_1 to $-u_1$ is taken care of within the maximum energy level by the curve $w : [0,1] \to \mathcal{M}$ connecting $u_2^+/\|u_2^+\|_{L^p(A)}$ to $-u_2^-/\|u_2^-\|_{L^p(A)}$ defined as

$$w(t) := \frac{a(t)u_2^+}{\|u_2^+\|_{L^p(A)}} - \frac{b(t)u_2^-}{\|u_2^-\|_{L^p(A)}},$$

where $a, b \in C([0,1])$ are non-negative functions such that $a(0) = 1, a(1) = 0$ and b is such that $a^p + b^p = 1$. By testing the equation satisfied by u_2 with u_2^+ and u_2^-, we can find that $E(w(t)) = \lambda_2(A)$ for every $t \in [0,1]$. The required curve $\gamma \in \Gamma$ satisfying (3.1) is completed as

$$\gamma := {v_1}^{-1} * \, w \, * v_2,$$

where $*$ denotes the concatenation of curves, $E(v_i(t)) \leq \lambda_2(A)$ for all $t \in [0,1]$ and v_1 connects $u_2^+/\|u_2^+\|_{L^p(A)}$ to u_1 and v_2 connects $-u_2^-/\|u_2^-\|_{L^p(A)}$ to $-u_1$.

3.1.1. *Construction of the curves v_1, v_2.* The construction of the curves v_1 and v_2 is the non-trivial part of the proof of the Theorem 1.2. It involves construction of an energy decreasing curve of the steepest decent, given in the following lemma. This is the main lemma in [20].

Lemma 3.1. *Let $A \subset \mathbb{R}^n$ be a p-quasi open set of finite measure. Suppose that $\lambda_1(A)$ is simple and let u_1 be the first nonnegative normalized eigenfunction. If $v_0 \in \mathcal{M}$ is not an eigenfunction and $\lambda_1(A) < E(v_0) \leq \lambda_2(A)$, then there exists a curve $v \in C^{0,(p-1)/p}\big([0,\infty), \mathcal{M}\big) \cap W^{1,p}\big([0,\infty), L^p(A)\big)$ with $v(0) = v_0$, such that the following hold:*

$$(i) \quad E(v(t)) < \lambda_2(A) \quad \forall\, t > 0, \quad \text{and} \quad \int_0^\infty \|v'(t)\|^p_{L^p(A)}\, dt \leq E(v_0);$$

$$(ii) \quad \lim_{t\to\infty} E(v(t)) = \lambda_1(A);$$

$$(iii) \quad \lim_{t\to\infty} v(t) = u_1 \quad \text{or} \quad \lim_{t\to\infty} v(t) = -u_1 \quad \text{in} \quad W_0^{1,p}(A).$$

Since the limit $t \to \infty$ exists one can re-parametrize the path to $[0,1]$, preserving continuity and thus obtain the required curves v_1 and v_2.

To prove the lemma, the limit of a sequence of maps $v_h : A \times [0,\infty) \to \mathcal{M}$ is considered, where v_h are the doubly non-linear gradient flows of the p-energy functional restricted to the manifold $\mathcal{M}$, with respect to the $L^p(A)$-distance in $W_0^{1,p}(A)$. The maps v_h are time-discretized weak solutions of the following doubly nonlinear evolution equation

$$\begin{cases} |\partial_t v|^{p-2}\partial_t v = \operatorname{div}(|\nabla v|^{p-2}\nabla v) + \sigma(t)|v|^{p-2}v & \text{in } A \times (0,\infty) \\ v \in W_0^{1,p}(A) \cap \mathcal{M} & \text{for all } t \geq 0 \end{cases} \tag{3.3}$$

with $v(0) = v_0$, where $\lambda_1(A) < E(v_0) \leq \lambda_2(A)$ and v_0 is not an eigenfunction. It turns out that for every h, the energy functional $E(v_h(t))$ is strictly decreasing along the flow for $t > 0$ and we have $E(v_h(t)) \to \lambda_1(A)$ and $v_h(t) \to u_1$ (or $-u_1$) in $W_0^{1,p}(A)$ as $t \to +\infty$. Then, as shown in [20], the flows v_h converge to a map $v : A \times [0,\infty) \to \mathcal{M}$ weakly in $W^{1,p}((0,\infty), L^p(A))$ and strongly in $W_0^{1,p}(A)$ for almost every $t \geq 0$, as $h \to +\infty$. Although these convergences do not imply that v is also a weak solution of the equation (3.3), they are enough to conclude that $E(v(t)) < \lambda_2(A)$ for all $t > 0$ and $v(t) \to \pm u_1$ as $t \to +\infty$.

3.1.2. *Illustration for $p = 2$.* The above ideas of the proof of Lemma 3.1 can be illustrated in an easier way for the case $p = 2$ which contains the essence of the key ideas. Consider the following constrained gradient flow $v \in AC\big([0,\infty); W_0^{1,p}(A)\big)$ satisfying the equation

$$\begin{cases} \partial_t v = \Delta v + \sigma(t) v & \text{in } A \times (0,\infty); \\ v(\,\cdot\,,0) = v_0 & \text{in } A \times \{0\}, \end{cases} \tag{3.4}$$

where $\sigma(t) = \frac{E(v)}{G(v)}$, $E(v) = \frac{1}{2}\int_A |\nabla v|^2\,dx$, $G(v) = \frac{1}{2}\int_A |v|^2\,dx$. Also, given the equation is linear, we can assume as much regularity as required. Using the equation (3.4) and integration by parts, notice that

$$\frac{d}{dt}(G(v)) = \int_A v\partial_t v\,dx = -\int_A |\nabla v|^2\,dx + \sigma(t)\int_A |v|^2\,dx = 0,$$

which implies $G(v) = G(v_0)$ and hence $v(\cdot\,,t) \in \mathcal{M}$ for all $t > 0$ if we assume $v_0 \in \mathcal{M}$. Moreover, the energy E is decreasing with t; using integration by parts and the equation (3.4), we note that

$$\begin{aligned} \frac{d}{dt}(E(v)) &= \frac{1}{2}\frac{d}{dt}\Big(\int_A |\nabla v|^2\,dx\Big) = \int_A \nabla v \cdot \nabla(\partial_t v)\,dx \\ &= -\int_A \partial_t v \Delta v\,dx = -\int_A |\partial_t v|^2\,dx + \sigma(t)\int_A v\partial_t v\,dx \\ &= -\int_A |\partial_t v|^2\,dx \le 0. \end{aligned}$$

Thus we have an energy decreasing curve $v(x,t)$ on the manifold $\mathcal{M}$ with $v(x,0) = v_0(x)$. Also notice that, if $\partial_t v(\cdot\,,t) = 0$ for all $t \in (a,b)$ then $E(v)(t) = \lambda$ is a constant, in fact $\lambda \in \sigma_2(A)$ with $v(\cdot\,,t)$ as an eigenfunction for all $t \in (a,b)$. Also, it is important to assume v_0 is not an eigenfunction so that $\partial_t v(x,0) \neq 0$, in order to avoid trivial solution $v(\cdot\,,t) \equiv v_0$. Existence of $\lim_{t\to\infty} E(v(t))$ and $\lim_{t\to\infty} v(t) = \tilde{v}$ follow from compactness. Moreover, one can obtain

$$\int_0^\infty \|\partial_t v\|^2_{L^2(A)}\,dt \le E(v_0)$$

by integrating the previous equality, hence $\lim_{t\to\infty} \|\partial_t v\|^2_{L^2(A)} = 0$. So, the limit $\tilde{v}$ is an eigenfunction with eigenvalue $E(\tilde{v})$.

3.1.3. *Minimizing movements.* In the non-linear case for $p \neq 2$, it is not well known if the weak solutions of the gradient flow has enough regularity to carry out the steps of the above. To deal with the difficulties, time-discretization is done and the construction relies on the theory of minimizing movements, a technique that was introduced by De Giorgi and developed in the book by Ambrosio-Gigli-Savaré [4]. For the functional Φ defined as

$$\Phi(v) = \begin{cases} \frac{1}{p} E(v) & \text{if } v \in \mathcal{M} \\ +\infty & \text{if } v \in L^p(A) \setminus \mathcal{M}, \end{cases} \tag{3.5}$$

with E as the p-Dirichlet energy, the local slope $\partial\Phi(v)$ of Φ defined as in [4] at every points v in the effective domain, more general energy inequalities of the form

$$\frac{1}{p}\int_0^{\tau_i} \|v_i'(s)\|^p_{L^p(A)}\, ds + \frac{1}{p'}\int_0^{\tau_i} (|\partial\Phi|(v_i(s)))^{p'}\, ds \leq \Phi(v_0) - \Phi(v_i),$$

can be obtained using the inequalities in [4], where the v_i's are particular time-discretizations. Similar inequalities have been obtained in [20]. The convergence of the discrete schemes to a curve of maximal slope follows from a refined version of Arzelà-Ascoli theorem, see [4, Prop. 3.3.1].

3.2. **Proof of Theorem 1.4.** The main non-trivial part of the theorem is to prove the lower semicontinuity of the second eigenvalue when $A_m \xrightarrow{\gamma_p} A \in \mathscr{A}_p(\Omega)$. The proof is trivial for the first eigenvalue since it is the absolute minimum in $\sigma_p(A)$ and it also implies that for the case of the second eigenvalue, we can assume only the case when $\lambda_1(A)$ is simple. The proof of lower semicontinuity, as written in [20], has similarities to the structure of the proof of Theorem 1.2.

Without loss of generality, we can assume

$$\lambda_2(A_m) \to \lambda \quad \text{as } m \to \infty \tag{3.6}$$

for some $\lambda \in \sigma_p(A)$. For every $m \in \mathbb{N}$, let $u_{1,m}, u_{2,m} \in \mathcal{M}$ be eigenfunctions for $\lambda_1(A_m)$ and $\lambda_2(A_m)$ and let $\lambda_1(A_m)$ supported in a p-quasi connected component U_m of A_m. A suitable sequence of curves $\gamma_m \in W^{1,p}\big([0,1], \mathcal{M}_p(A_m)\big)$ with endpoints $\pm u_{1,m}$ is constructed similarly as in the proof of Theorem 1.2, such that for all $m \in \mathbb{N}$, we have

$$\int_0^1 \|\gamma_m'(t)\|^p_{L^p(\Omega)}\,dt \le C \quad \text{and} \quad E(\gamma_m(t)) \le \lambda_2(A_m), \qquad \text{for all } t > 0.$$

However, the construction of γ_m is much easier than before. This is used to conclude the proof of lower semicontinuity by proving that $\lambda \ge \lambda_2(A)$, where λ is the limit in (3.6); it is argued by contradiction, assuming $\lambda < \lambda_2(A)$. Then we have that $\lambda = \lambda_1(A)$ and that $u_{1,m}$ converges to u_1. Using Arzelà-Ascoli theorem of [4] as before, we conclude that there exists $\gamma \in W^{1,p}\big([0,1], L^p(\Omega)\big)$ such that, up to a subsequence, $\gamma_m(t) \to \gamma(t)$ in $L^p(\Omega)$ and weakly in $W^{1,p}_0(\Omega)$ for all $t \in [0,1]$. Since the endpoints of γ are $\pm u_1$, from Theorem 1.2 we conclude that

$$\begin{aligned}\lambda_2(A) \le \max_{t\in[0,1]} \int_A |\nabla\gamma(t)|^p\,dx &\le \liminf_{m\to\infty} \left[\max_{t\in[0,1]} \int_A |\nabla\gamma_m(t)|^p\,dx\right] \\ &\le \lim_{m\to\infty} \lambda_2(A_m) = \lambda_1(A),\end{aligned}$$

which is impossible since $\lambda_1(A)$ is simple and the contradiction concludes $\lambda \ge \lambda_2(A)$.

3.3. **Proof of Theorem 1.3.** The proof of Theorem 1.3 in [20], follows closely the proof for the case of $p = 2$ by Buttazzo-Dal Maso [11]. A closed convex subset $K \subset W^{1,p}_0(\Omega)$ defined by imposing an obstacle condition, i.e.

$$K := \big\{w \in W^{1,p}_0(\Omega) : w \ge 0,\ -\Delta_p w - 1 \le 0\big\},$$

being fixed, the proof of the existence of a minimizer of the problem (1.9) is reduced to showing the existence of a minimizer of the problem

$$\min\big\{G(w) : w \in K,\ |\{w > 0\}| \le c\big\}.$$

Similarly as in [11], letting $J : K \to \mathbb{R}$ as $J(w) = \inf\big\{F(A) : A \in \mathscr{A}_p(\Omega),\ w_A \le w\big\}$ with $w_A = \mathscr{R}_{p,A}(1)$, the function G is defined as the $L^p(\Omega)$-lower semicontinuous envelope of J, i.e.

$$G(w) = \inf\big\{\liminf_{h\to\infty} J(w_h) : w_h \in K,\ w_h \to w \text{ in } L^p(\Omega)\big\}.$$

The following properties of G is used to prove the theorem.

(1) For every u, $v \in K$ with $u \le v$ q.e. in Ω, then $G(u) \ge G(v)$;

(2) G is lower semicontinuous on K with respect to $L^p(\Omega)$;

(3) $G(w_A) = F(A)$ for every $A \in \mathscr{A}_p(\Omega)$, where $w_A = \mathscr{R}_{p,A}(1)$.

Verification of the properties is done similarly as in [11], however some difficulties are encountered arising from the non-linearity of the p-Laplacian and has been handled differently in [20].

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