DEGENERATE DIFFERENTIAL PROBLEMS WITH FRACTIONAL DERIVATIVES

PROBLEMI DIFFERENZIALI DEGENERI CON DERIVATE FRAZIONARIE

MOHAMMED AL HORANI & ANGELO FAVINI

Abstract. We describe an extension of previous results on degenerate abstract equations described by Favini and Yagi in their monograph cited in the references.This allows us to study degenerate differential equations with fractional derivative in time. Both direct and relative inverse problems are studied. Finally, various applications to degenerate partial differential equations are described.

Sunto. Descriviamo un'estensione di precedenti risultati su equazioni astratte degeneri descritti da Favini e Yagi nella loro monografia citata nelle referenze. Ciò permette di studiare equazioni differenziali degeneri con derivata frazionaria nel tempo. Vengono studiati sia problemi diretti sia relativi problemi inversi. Infine, sono descritte varie applicazioni ad equazioni alle derivate parziali degeneri.

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1. Introduction

Consider the abstract equation

$$BMu - Lu = f \tag{1}$$

where B, M, L are closed linear operators on the complex Banach space E with $D(L) \subseteq D(M)$, $0 \in \rho(L)$, $f \in E$ and u is the unknown. The first approach to handle existence and uniqueness of the solution u to (1) was given by Favini-Yagi [10], see in particular the

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Dipartimento di Matematica, Università di Bologna

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monograph [11]. By using real interpolation spaces, see [14] and [15], suitable assumptions on the operators B, M, L guarantee that (1) has a unique strict solution. Such a result was improved by Favini, Lorenzi, Tanabe in [6], see also [7], [8] and [9] . In order to describe the results, we list the basic assumptions. They read

(H_1) Operator B has a resolvent $(z-B)^{-1}$ for any $z \in \mathbb{C}$, $\mathrm{Re}\, z < a$, $a > 0$ satisfying

$$\|(z-B)^{-1}\|_{\mathcal{L}(E)} \leq \frac{c}{|Re\, z|+1}, \quad Re\, z < a\,. \tag{2}$$

(H_2) Operators L, M satisfy

$$\|M(\lambda M - L)^{-1}\|_{\mathcal{L}(E)} \leq \frac{c}{(|\lambda|+1)^{\beta}} \tag{3}$$

for any $\lambda \in \Sigma_\alpha := \big\{z \in \mathbb{C}:\ Re\, z \geq -c(1+|Im\, z|)^{\alpha},\ c>0,\ 0<\beta\leq\alpha\leq 1\big\}$.

(H_3) Let A be the possibly multivalued linear operator $A = LM^{-1}$, $D(A) = M(D(L))$. Then A and B commute in the resolvent sense:

$$B^{-1}A^{-1} = A^{-1}B^{-1}\,.$$

Let $(E, D(B))_{\theta,\infty}$, $0<\theta<1$, denote the real interpolation space between E and $D(B)$. The main result holds, see [8] .

Theorem 1.1. *Suppose that $\alpha+\beta>1$, $2-\alpha-\beta<\theta<1$. Then under hypotheses (H_1)-(H_3), equation (1) admits a unique strict solution u such that $Lu,\ BMu \in (E, D(B))_{\omega,\infty}$, $\omega = \theta-2+\alpha+\beta$, provided that $f \in (E, D(B))_{\theta,\infty}$.*

It is straightforward to verify that if $-B$ generates a bounded c_0-semigroup in E exponentially decreasing to 0 as t tends to ∞, then assumption (H_1) holds for B. In a previous paper, it was also shown that Theorem 1.1 works well for solving degenerate equations on the real axis, see [1].

The first aim of this paper is to extend Theorem 1.1 to the interpolation spaces $(E, D(B))_{\theta,p}$, $1<p<\infty$. This affirmation is not immediate. Section 2 is devoted to this proof. In Section 3, we apply the abstract results to solve concrete differential equations. In Section 4, we handle corresponding inverse problems. For some related results, we refer to Guidetti [12] and Bazhlekova [2]. However, these results do not consider the degenerate problem. In Section 5, the possibly not parabolic case is considered and also some related inverse problems are studied; we remark that this treatment is really new and no previous results were devoted to last case, i.e., this treatment which solves concrete degenerate differential equations is new. Moreover, some applications illustrating our abstract results are also given.

2. Main Results

To begin with, we recall, from Favini-Yagi [11], p. 16, that if E_0, E_1 are two Banach spaces such that (E_0, E_1) is an interpolation couple, i.e., there exists a locally convex topological space X such that $E_i \subset X$, $i = 0$, 1 continuously, then the following injections hold

$$E_0 \cap E_1 \subset_d (E_0, E_1)_{\zeta, q} \subset_d (E_0, E_1)_{\eta, 1}\,, \eta < \zeta\,,$$

and

$$(E_0, E_1)_{\eta, \infty} \subset_d (E_0, E_1)_{\xi, q} \subset E_0 + E_1\,, \xi < \eta\,,$$

provided that $E_1 \subset_d E_0$, see [17], Theorem 1.3.3(e), p.26.

Taking in account the previous embedding and Theorem 1.1, we easily deduce that if ϵ, ϵ_1 are suitable small positive numbers, if $f \in (E, D(B))_{\theta+\epsilon, q} \subset (E, D(B))_{\theta, \infty}$, then equation (1) admits a unique solution u with $Lu,\ BMu \in (E, D(B))_{\theta-2+\alpha+\beta, \infty}$ and $Lu,\ BMu \in (E, D(B))_{\theta-2+\alpha+\beta-\epsilon_1, q}$, that is a weaker result than case $q = \infty$.

Our aim is to extend Theorem 1.1 to $1 < p < \infty$. In order to establish the corresponding result, we need the following lemma concerning multiplicative convolution. We recall that $L^p_*(\mathbb{R}^+) = L^p\,(R^+; t^{-1}dt)$ and that the multiplicative convolution of two (measurable) functions $f, g : \mathbb{R}^+ \to \mathbb{C}$ is defined by

$$(f * g)(x) = \int_0^\infty f(xt^{-1})g(t)t^{-1}\,dt$$

where the integral exists a.e. for $x \in \mathbb{R}_+$.

Lemma 2.1. *For any $f_1 \in L^p_*(\mathbb{R}^+)$ and $g \in L^1_*(\mathbb{R}^+)$, the multiplicative convolution $f_1 * g \in L^p_*(\mathbb{R}^+)$ and satisfies*

$$\|f_1 * g\|_{L^p_*} \leq \|f_1\|_{L^p_*} \|g\|_{L^1_*}$$

Consider now the chain of estimates

$$\begin{aligned} t^{\theta+\alpha+\beta-2}\|B(B+t)^{-1}v\| &\leq t^{\theta+\alpha+\beta-2}\int_0^\infty \frac{(1+y)^{3-\alpha-\beta-\theta}}{(1+y+t)}\,(1+y)^\theta\|B(B+1+y)^{-1}f\|\,\frac{dy}{1+y} \\ &\leq t^{\theta+\alpha+\beta-2}\int_0^\infty \frac{y^{3-\alpha-\beta-\theta}}{y+t}\,y^\theta\,\|B(B+y)^{-1}f\|\,\frac{dy}{y} \\ &= t^{\theta+\alpha+\beta-2}\int_0^\infty \frac{y^{3-\alpha-\beta-\theta}}{y(1+ty^{-1})}\,y^\theta\,\|B(B+y)^{-1}f\|\,\frac{dy}{y} \\ &= \int_0^\infty \frac{(ty^{-1})^{\theta+\alpha+\beta-2}}{1+ty^{-1}}\,y^\theta\,\|B(B+y)^{-1}f\|\,\frac{dy}{y} \end{aligned}$$

where

$$v = (2\pi i)^{-1}\int_\Gamma z^{-1}(zT-1)^{-1}B(B-z)^{-1}f\,dz, \quad T = ML^{-1}\,,$$

$\Gamma = \Gamma_\alpha$ being the oriented contour

$$\Gamma = \{z = a - c(1+|y|)^\alpha + iy, \quad -\infty < y < \infty\}\,,$$

with $a \in (c, c+a_0)$. Such an element v is the candidate solution to equation (1). Moreover, condition $2-\alpha-\beta < \theta < 1$ is assumed.

Let $f_1(y) = y^\theta\,\|B(B+y)^{-1}f\|$, $g(y) = \dfrac{y^{\theta+\alpha+\beta-2}}{1+y}$, $y \in \mathbb{R}^+$, and note that $f \in (E, D(B))_{\theta,p}$ if and only if $f_1 \in L^p_*(\mathbb{R}^+)$. Moreover, $g \in L^1_*(\mathbb{R}^+)$ since $\theta > 2-\alpha-\beta$ and obviously $\theta < 3-\alpha-\beta$. Therefore, from Lemma 1.1, we deduce that $v \in (E, D(B))_{\omega,p}$, where $\omega = \theta+\alpha+\beta-2$. Thus, we can establish the fundamental result concerning equation (1) .

Theorem 2.1. *Let B, M, L be three closed linear operators on the Banach space E satisfying (H_1)-(H_3), $0 < \beta \leq \alpha \leq 1$, $\alpha+\beta > 1$. Then for all $f \in (E, D(B))_{\theta,p}$, $2-\alpha-\beta < \theta < 1$, $1 < p < \infty$, equation (1) admits a unique solution u. Moreover, Lu, $BMu \in (E, D(B))_{\omega,p}$, $\omega = \theta+\alpha+\beta-2$.*

Remark 2.1. One can sketch a short proof of Theorem 2.1 as follows: take ϵ so small that

$$2-\alpha-\beta<\theta-\epsilon<\theta<\theta+\epsilon<1\,.$$

Then by Theorem 1.1, the operators $f \longrightarrow Lu,\ BMu$ take $\big(E, D(B)\big)_{\theta\pm\epsilon,\infty}$ into $\big(E, D(B)\big)_{\omega\pm\epsilon,\infty}$. By reiteration property, see [3],

$$\big(E, D(B)\big)_{\theta,p}=\big((E, D(B))_{\theta-\epsilon,\infty}\,,(E, D(B))_{\theta+\epsilon,\infty}\big)_{1/2,p}$$

and

$$\big(E, D(B)\big)_{\omega,p}=\big((E, D(B))_{\omega-\epsilon,\infty}\,,(E, D(B))_{\omega+\epsilon,\infty}\big)_{1/2,p}\,.$$

So, by the interpolation property, the operators $f \longrightarrow Lu,\ BMu$ take $\big(E, D(B)\big)_{\theta,p}$ into $\big(E, D(B)\big)_{\omega,p}$.

3. Fractional Derivative

Let $\tilde\alpha>0$, $m=\lceil\tilde\alpha\rceil$ is the smallest integer greater or equal to $\tilde\alpha$, $I=(0,T)$ for some $T>0$. Define

$$g_\beta(t)=\begin{cases}\frac{1}{\Gamma(\beta)}\,t^{\beta-1} & t>0,\\ 0 & t\le 0,\end{cases}\qquad \beta\ge 0\,,$$

where $\Gamma(\beta)$ is the Gamma function. Note that $g_0(t)=0$ because $\Gamma(0)^{-1}=0$. The Riemann-Liouville fractional derivative of order $\tilde\alpha$ is defined for all $f\in L^1(I)$ such that $g_{m-\tilde\alpha}*f\in W^{m,1}(I)$ by

$$D_t^{\tilde\alpha}f(t)=D_t^m(g_{m-\tilde\alpha}*f)(t)=D_t^m\,J_t^{m-\tilde\alpha}f(t)$$

where $D_t^m:=\dfrac{d^m}{dt^m}$, $m\in\mathbb{N}$. $D_t^{\tilde\alpha}$ is a left inverse of $J_t^{\tilde\alpha}$, but in general it is not a right inverse. Here, $J_t^{\tilde\alpha}$ is the Riemann-Liouville fractional integral of order $\tilde\alpha>0$ which is defined as:

$$J_t^{\tilde\alpha}f(t):=(g_{\tilde\alpha}*f)(t),\ \ f\in L^1(I),\ t>0,\quad J_t^0f(t):=f(t)\,.$$

If X is a complex Banach space, $\tilde\alpha>0$, then we define the operator $\mathcal{J}_{\tilde\alpha}$ as:

$$D\left(\mathcal{J}_{\tilde\alpha}\right):=L^p(I;X),\quad \mathcal{J}_{\tilde\alpha}u=g_{\tilde\alpha}*u,\ \ p\in[1,\infty)\,.$$

Define the spaces $R^{\tilde{\alpha},p}(I;X)$ and $R_0^{\tilde{\alpha},p}(I;X)$ as follows. If $\tilde{\alpha} \notin \mathbb{N}$, set

$$R^{\tilde{\alpha},p}(I;X) := \{u \in L^p(I;X) \,:\, g_{m-\tilde{\alpha}} * u \in W^{m,p}(I;X)\},$$

$$R_0^{\tilde{\alpha},p}(I;X) := \{u \in L^p(I;X) \,:\, g_{m-\tilde{\alpha}} * u \in W_0^{m,p}(I;X)\},$$

where

$$W_0^{m,p}(I;X) = \left\{y \in W^{m,p}(I;X),\ y^{(k)}(0) = 0, \quad k = 0,\, 1,\, ...,\, m-1\right\}.$$

If $\tilde{\alpha} \in \mathbb{N}$, we take

$$R^{\tilde{\alpha},p}(I;X) := W^{\tilde{\alpha},p}(I;X), \quad R_0^{\tilde{\alpha},p}(I;X) := W_0^{\tilde{\alpha},p}(I;X).$$

Denote the extensions of the operators of fractional differentiation in $L^p(I;X)$ by $\mathcal{L}_{\tilde{\alpha}}$, i.e.,

$$D(\mathcal{L}_{\tilde{\alpha}}) := R_0^{\tilde{\alpha},p}(I;X), \quad \mathcal{L}_{\tilde{\alpha}} u := D_t^{\tilde{\alpha}} u,$$

where $D_t^{\tilde{\alpha}}$ is the Riemann-Liouville fractional derivative. Notice that if $\tilde{\alpha} \in (0,1)$, $u \in D(\mathcal{L}_{\tilde{\alpha}})$, then $(g_{1-\tilde{\alpha}} * u)(0) = 0$.

Let us now list the main properties of $\mathcal{L}_{\tilde{\alpha}}$, see [2], Lemma 1.8, p. 15.

Lemma 3.1. *Let $\tilde{\alpha} > 0$, $1 < p < \infty$, X a complex Banach space, and $\mathcal{L}_{\tilde{\alpha}}$ be the operator introduced above. Then*

(a) $\mathcal{L}_{\tilde{\alpha}}$ is closed, linear and densely defined

(b) $\mathcal{L}_{\tilde{\alpha}} = \mathcal{J}_{\tilde{\alpha}}^{-1}$

(c) $\mathcal{L}_{\tilde{\alpha}} = \mathcal{L}_1^{\tilde{\alpha}}$, the $\tilde{\alpha}-$th power of the operator $\mathcal{L}_1$

(d) if $\tilde{\alpha} \in (0,2)$, operator $\mathcal{L}_{\tilde{\alpha}}$ is positive with spectral angle $\omega_{\mathcal{L}_{\tilde{\alpha}}} = \tilde{\alpha}\pi/2$

(e) if $\tilde{\alpha} \in (0,1]$, then $\mathcal{L}_{\tilde{\alpha}}$ is m-accretive

(f) $R_0^{\tilde{\alpha},p}(I;X) \hookrightarrow C^{\tilde{\alpha}-1/p}(I;X)$, $\tilde{\alpha} > 1/p$, $\tilde{\alpha} - 1/p \notin \mathbb{N}$, see [2], Theorem 1.10, p. 17

(g) if $\tilde{\alpha}\gamma - 1/p \notin \mathbb{N}_0$,

$$\Big(L^p(I;X), R_0^{\tilde{\alpha},p}(I;X)\Big)_{\gamma,p} = W_0^{\tilde{\alpha}\gamma,p}(I;X),$$

where $W_0^{\tilde{\alpha}\gamma,p}(I;X) = \left\{y \in W^{\tilde{\alpha}\gamma,p}(I;X),\ y^{(k)}(0) = 0, \quad k = 0,\, 1,\, ...,\, \lfloor\tilde{\alpha}\gamma - 1/p\rfloor\right\}$,

$\lfloor\tilde{\alpha}\gamma - 1/p\rfloor$ is the greatest integer less or equal $\tilde{\alpha}\gamma - 1/p$, see [2], Proposition 11, p. 18.

Statement (e) implies that if $\tilde{\alpha} \in (0,1]$,

$$\|\lambda(\lambda + \mathcal{L}_{\tilde{\alpha}})^{-1}\|_{L^p(I;X)} \leq C, \quad |\arg \lambda| < \pi(1 - \frac{\tilde{\alpha}}{2}).$$

But this reads equivalently $\|(\lambda - \mathcal{L}_{\tilde{\alpha}})^{-1}\|_{L^p(I;X)} \leq C/|\lambda|$ provided that λ is in a sector of the complex plane containing $Re\,\lambda \leq 0$, if $\tilde{\alpha} \in (0,1)$. If $\tilde{\alpha} = 1$, then the estimate above reads $\|(\lambda - \mathcal{L}_{\tilde{\alpha}})^{-1}\|_{L^p(I;X)} \leq C\,(|Re\,\lambda| + 1)^{-1}$, holds in the half-plane Re $\lambda \leq a$, where a is a positive number. Therefore, if $\tilde{\alpha} \leq 1$, operator $\mathcal{L}_{\tilde{\alpha}} = \dfrac{d^{\tilde{\alpha}}}{dt^{\tilde{\alpha}}} = D_t^{\tilde{\alpha}}$ satisfies assumption (H_1) in Theorem 2.1. Hence, we can handle abstract equations of the type

$$D_t^{\tilde{\alpha}}(My(t)) = Ly(t) + f(t), \quad 0 \leq t \leq T,$$

in a Banach space X with an initial condition $\big(g_{1-\tilde{\alpha}} * Mu\big)(0) = 0$. Then the results follow easily from the abstract model.

Example 3.1. Let M be the multiplication operator in $L^p(\Omega)$, Ω a bounded open set in $\mathbb{R}^n$ with a C^2 boundary $\partial\Omega$, $1 < p < \infty$, by $m(x) \geq 0$, m is continuous and bounded, and take $L = \Delta - c$, $D(L) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$, $c > 0$. Then it is seen in Favini-Yagi [11], pp. 79-80,

$$\|M(zM - L)^{-1}f\|_{L^p(\Omega)} \leq \frac{c}{(1+|z|)^{1/p}}\,\|f\|_{L^p(\Omega)}$$

for all z in a sector containing $Re\,z \geq 0$.

In order to solve our problem, $0 < \tilde{\alpha} \leq 1$,

$$D_t^{\tilde{\alpha}}(My(t)) = Ly(t) + f(t), \quad 0 \leq t \leq T,$$

we must recall, see (g) in Lemma 3.1, that if $\tilde{\alpha}\gamma - 1/p \notin \mathbb{N}$, the interpolation space

$$\Big(L^p(I;X), R_0^{\tilde{\alpha},p}(I;X)\Big)_{\gamma,p} = W_0^{\tilde{\alpha}\gamma,p}(I;X).$$

Therefore, for any $f \in W_0^{\tilde{\alpha}\theta,p}(I;X)$, $1 - \frac{1}{p} < \theta < 1$, $1 < p < \infty$, $\tilde{\alpha}\theta - \frac{1}{p} \notin \mathbb{N}_0$, the problem above admits a unique strict solution y such that

$$\Delta y,\ D_t^{\tilde{\alpha}} m(\cdot) y \in W_0^{\tilde{\alpha}(\theta + \frac{1}{p} - 1),p}(I;X).$$

4. Inverse Problems

Given the problem

$$D_t^{\tilde{\alpha}}(My(t)) = Ly(t) + f(t)z + h(t), \quad 0 \leq t \leq T\,, \tag{4}$$

then corresponding to an initial condition and following the strategy in various previous papers, see in particular Lorenzi [16], we could study existence and regularity of solutions (y, f) to the above problem such that $\Phi[My(t)] = g(t)$, where g is a complex-valued continuous function on $[0, T]$ and $\Phi \in X^*$. This is, of course, an inverse problem. Applying $\Phi \in X^*$ to both sides of equation (4) we get

$$D_t^{\tilde{\alpha}} g(t) = \Phi[Ly(t)] + \Phi[h(t)] + f(t)\Phi[z]\,.$$

If $\Phi[z] \neq 0$, we obtain necessarily

$$f(t) = \frac{D_t^{\tilde{\alpha}} g(t) - \Phi[Ly(t)] - \Phi[h(t)]}{\Phi[z]}\,.$$

Therefore,

$$D_t^{\tilde{\alpha}}(My(t)) = Ly(t) + h(t) - \frac{\Phi[Ly(t)]}{\Phi[z]}\,z - \frac{\Phi[h(t)]}{\Phi[z]}\,z + \frac{D_t^{\tilde{\alpha}} g(t)}{\Phi[z]}\,z\,.$$

If L_1 is defined by

$$D(L_1) = D(L), \quad L_1 y = -\frac{\Phi[Ly(t)]}{\Phi[z]}\,z\,.$$

One can introduce assumptions on the given operators ensuring that the direct problem

$$D_t^{\tilde{\alpha}}(My(t)) = Ly(t) + L_1 y + h(t) - \frac{\Phi[h(t)]}{\Phi[z]}\,z + \frac{D_t^{\tilde{\alpha}} g(t)}{\Phi[z]}\,z$$

has a unique strict solution, see [4]. The main step is to verify that assumption (H_2) holds for the operators $L + L_1$ and M. Indeed, we need to introduce the multivalued linear operator $A := LM^{-1}$, $D(A) = M(D(L))$, so that A generates a weakly parabolic semigroup in X, i.e.,

$$\|(\lambda I - A)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c}{(|\lambda| + 1)^{\beta}}, \quad \lambda \in \Sigma_\alpha\,, \quad 0 < \beta \leq \alpha \leq 1\,, \text{ see } (3)\,.$$

One also introduces the space $X_A^\theta = X_A^{\theta,\infty}$, where

$$X_A^\theta = \left\{ u \in X, \ \sup_{t>0} t^\theta \|A^0(t - A)^{-1} u\|_X = \|u\|_{X_A^\theta} < \infty \right\},$$

and $A^0(t-A)^{-1}$ denotes the bounded operator $-I+t(t-A)^{-1}$. Such an operator reduces to $A(t-A)^{-1}$ as soon as A is single-valued. Note that if A is a single-valued operator with $\alpha=\beta=1$, then X_A^θ coincides with the real interpolation space $\big(X, D(A)\big)_{\theta,\infty}$; moreover, in general,

$$X_A^\theta \subseteq \big(X, D(A)\big)_{\theta,\infty}\,, \quad 0<\theta<1\,,$$

$$\big(X, D(A)\big)_{\theta,\infty} \subseteq X_A^{\theta+\beta-1}\,, \quad 1-\beta<\theta<1\,.$$

Theorem 1 in [5], pp. 148-149, affirms that if L, L_1, M are closed linear operators on X, $D(L)=D(L_1)\subseteq D(M)$, $0\in\rho(L)$, (H_2) holds and $L_1\in\mathcal{L}\big(D(L), X_A^\theta\big)$, $1-\beta<\theta<1$, then

$$\|M(\lambda M-L-L_1)^{-1}x\|_X \le c(1+|\lambda|)^{-\beta}\|x\|_X, \quad \forall\,\lambda\in\Sigma_\alpha\,,\ |\lambda| \text{ large enough}\,,\ \ x\in X\,.$$

In order to apply this theorem in our case, we must suppose that z belongs to $X_A^\theta = X^\theta_{(L+L_1)M^{-1}}$ for some $\theta\in(1-\beta,1)$. Then, see Theorem 2.1 (see also [4]), with a suitable regularity on

$$h(t)-\frac{\Phi[h(t)]}{\Phi[z]}\,z+\frac{D_t^{\tilde\alpha}g(t)}{\Phi[z]}\,z\,,$$

problem

$$\begin{aligned} &D_t^{\tilde\alpha}(My(t)) = Ly(t)+L_1y+h(t)-\frac{\Phi[h(t)]}{\Phi[z]}\,z+\frac{D_t^{\tilde\alpha}g(t)}{\Phi[z]}\,z \\ &\big(g_{1-\tilde\alpha}*My\big)(0)=0 \end{aligned}$$

will admit a unique strict solution. The general case of $p\in(1,\infty)$ could be studied taking into account new results from [4] concerning $X_A^{\theta,p}=X^{\theta,p}_{(L+L_1)M^{-1}}$.

5. The General Case: Possibly Not Parabolic

In this section, we consider the possibly not parabolic case, and again, we begin by considering the operational equation

$$TMv-Lv=f \tag{5}$$

in the Banach space X, T is a closed linear operator on X, M and L are two single valued closed linear operators on X such that $D(L)\subseteq D(M)$, L^{-1} being single valued and bounded on X, f is an element in X. Here the hypotheses on operators would allow to treat problems of hyperbolic type or highly degenerate ordinary differential equations. Of course, (5) could be written $Tu - \mathcal{A}u \ni f$ using the multivalued operator $\mathcal{A} = LM^{-1}$. Following Favini-Yagi [11], p. 128, we assume

($\mathcal{T}$) The resolvent set $\rho(T)$ contains the logarithmic region

$$\Pi = \left\{z \in \mathbb{C};\ \mathrm{Re} z \le a + b\,\log(1+|z|)\right\},$$

$a > 0$, $b \ge 0$, and the resolvent $(z-T)^{-1}$ satisfies

$$\left\|(z-T)^{-1}\right\|_{\mathcal{L}(X)} \le c(1+|z|)^p, \quad z \in \Pi \tag{6}$$

with some exponent $p \ge 0$ and constant $c > 0$.

($\mathtt{M}-\mathtt{L}$) The resolvent set $\rho_M(L) = \left\{z \in \mathbb{C};\ (zM-L)^{-1} \in \mathcal{L}(X)\right\}$ contains the region

$$\Lambda = \{z \in \mathbb{C};\ \mathrm{Re} z \ge \overline{a} + b\log(1+|z|)\}$$

where $0 < \overline{a} < a$ and

$$\left\|L(zM-L)^{-1}\right\|_{\mathcal{L}(X)} \le c(1+|z|)^m, \quad z \in \Lambda, \tag{7}$$

with some integer $m \ge 0$ and a positive constant c.

In addition, $B = -ML^{-1} \in \mathcal{L}(X)$ commutes with T according to

$$T^{-1}B = BT^{-1}. \tag{8}$$

The following existence result on (5) is established, see [11], Theorem 5.1, p. 129.

Theorem 5.1. *Let ($\mathcal{T}$) and ($\mathtt{M}-\mathtt{L}$) be satisfied and let $T^{-1}B = BT^{-1}$; if k is an integer greater than $m+p+1$, then (5) has at least one solution for any $f \in D(T^k)$.*

Moreover, Theorem 5.5 in [11], p. 132, established that under the assumptions of Theorem 5.1, problem (5) has at most one solution.

As an example of the operator T, we can take

$$D(T) = \left\{u \in C^1([0,\tau]);\ u(0) = 0\right\}, \quad Tu = \frac{du}{dt}.$$

Then ($\mathcal{T}$) is satisfied with $a > \tilde{a}$, $b \ge 0$ and (6) holds with $p = b\tau$.

As an important application, consider the equation

$$\frac{d}{dt}My(t) = Ly(t) + f(t)\,, \quad t \in [0,\tau] \tag{9}$$

in a Banach space X, where M, L are two closed linear operators on X with $D(L) \subseteq D(M)$, L being invertible with the property that $z=0$ is a pole of order $k+1$, $k = 0, 1, ...,$ of the bounded operator $L(zL-M)^{-1} = (z-B)^{-1}$, $B = ML^{-1} \in \mathcal{L}(X)$. Recall that the decomposition $X = N(B^m) \oplus R(B^m)$ holds for any $m \geq k+1$, $R(B^m) = \overline{R(B^m)} = R(B^{k+1})$, $N(B^m) = N(B^{k+1})$, see Yosida [18], p. 229. If $\gamma : |z| = \varepsilon$ is a circumference of sufficiently small radius with its interior not containing singularities other than $z=0$ and if

$$P = \frac{1}{2\pi i}\int_\gamma (z-B)^{-1}\,dz\,,$$

then P is a projection onto $N(B^{k+1})$ and $R(I-P) = R(B^{k+1})$. Moreover, $\|z^{k+1}(z-B)^{-1}\|_{\mathcal{L}(X)} \leq c$ for any z such that $0 < |z| \leq \varepsilon$, see [11], p. 157. Hence, if the corresponding Cauchy problem

$$\begin{cases} \dfrac{d}{dt}Bw - w = f(t)\,, \quad 0 < t \leq \tau\,, \\ \qquad Bw(0) = u_0 \end{cases} \tag{10}$$

has a solution, it is unique. The first equation of (10) is equivalent to the system

$$\frac{d}{dt}B_1(I-P)w - (I-P)w = (I-P)f(t)\,, \tag{11}$$

$$\frac{d}{dt}B_2Pw - Pw = Pf(t)\,, \tag{12}$$

where B_1 and B_2 denote the parts of B in $R(B^{k+1})$ and in $N(B^{k+1})$, respectively. It is known from Kato [13] that the spectra of B_1 and B_2 coincide with the spectrum of B minus $\{0\}$ and with $\{0\}$, respectively. Thus, $B_2 \in \mathcal{L}\left(N(B^{k+1})\right)$; moreover, B_1 is a closed operator in $R(B^{k+1})$ mapping $D(B_1) = D(B) \cap R\big(B^{k+1}\big)$ onto $R(B^{k+1})$ in a one-to-one fashion. This implies that (11) with assigned initial condition $B_1(I-P)w(0) = (I-P)u_0$ has a unique solution given by

$$(I-P)w(t) = B_1^{-1}\,e^{tB_1^{-1}}(I-P)u_0 + \int_0^t B_1^{-1}\,e^{(t-s)B_1^{-1}}(I-P)f(s)\,ds\,.$$

Regarding (12), we observe that all the powers B_2^m vanish for $m \geq k+1$.

We have the following lemma, see Favini-Yagi [11], p. 158,

Lemma 5.1. *Assume that $B \in \mathcal{L}(X)$ has $z = 0$ as the unique singularity of $(z - B)^{-1}$ with a pole of order $k + 1$. If $f \in C^k([0, \tau]; X)$, then the equation in (10) possesses the unique solution*

$$y(t) = -\sum_{j=0}^{k} B^j f^{(j)}(t), \quad 0 < t \leq \tau, \quad w(t) = -y(t)\,.$$

Proof. From $Bw(t) = \sum_{j=1}^{k} B^j f^{(j-1)}(t)$, we get

$$\frac{d}{dt} Bw(t) = \sum_{j=1}^{k} B^j f^{(j)}(t) = w(t) - f(t)\,.$$

Then $y(t) = -w(t)$ satisfies

$$\frac{d}{dt} By(t) = y(t) + f(t), \quad 0 < t \leq \tau\,. \quad \square$$

As a consequence of the lemma, we obtain the following result:

Proposition 5.1. *Assume that B has $z = 0$ as a pole of $(z - B)^{-1}$ of order $k + 1$. If $f \in C^k([0, \tau]; X)$, then the equation in (10) possesses the general solution*

$$w(t) = -\sum_{j=0}^{k} B_2^j P f^{(j)}(t) + B_1^{-1} e^{tB_1^{-1}} (I - P) u_1 + \int_0^t B_1^{-1} e^{(t-s)B_1^{-1}} (I - P) f(s)\, ds$$

where $u_1 \in X$.

Note that no initial condition can be assigned arbitrarily to $Bw(t)$ at $t = 0$.

Corollary 5.1. *Under the assumptions in Proposition 5.1, if $f \in C^k([0, \tau]; X)$ with $f^{(j)}(0) = 0$ for $j = 1, 2, \ldots, k - 1$, and $u_0 \in R\left(B^{k+1}\right)$, then (10) possesses a unique solution w.*

Many examples of operators B are described in Favini-Yagi [11], p. 159-161. We have already observed that operator $\mathcal{J}$ given by

$$D(\mathcal{J}) = \left\{u \in C^1([0, \tau]; X); \quad u(0) = 0\right\}$$
$$\mathcal{J} u = \frac{du}{dt}$$

satisfies $(\mathcal{T})$ with $a > \tilde{a}$, $b \geq 0$ and (6) holds with $p = b\tau$, so that $p = 0$ for $b = 0$. On the other hand, if $z = 0$ is a pole for $(z - ML^{-1})^{-1}$ of order $k+1$, i.e., $\|(z - ML^{-1})^{-1}\|_{\mathcal{L}(X)} \leq \frac{c}{|z|^{k+1}}$, then for $0 < |z| \leq \varepsilon$,

$$\|L(zL - M)^{-1}\|_{\mathcal{L}(X)} \leq \frac{c}{|z|^{k+1}}.$$

Let $z^{-1} = \mu$. Then

$$\|\mu L(\mu M - L)^{-1}\|_{\mathcal{L}(X)} \leq c\,|\mu|^{k+1}, \quad \mu \geq 1/\varepsilon\,,$$

i.e., $\|L(\mu M - L)^{-1}\|_{\mathcal{L}(X)} \leq c\,|\mu|^{k}$. Therefore Theorem 5.1 can be applied to

$$\begin{cases} \dfrac{d}{dt}My = Ly + f(t)\,, \quad 0 < t \leq \tau\,, \\ (My)(0) = 0 \end{cases} \tag{13}$$

by taking $p = 0$, $m = k$. This implies that (13) admits a unique strict solution provided that $f \in C^{k+2}([0,\tau];X)$ and its derivatives (until a certain order, determined by the resolvent estimates) vanish at $t = 0$. Thus Theorem 5.1 would give a weaker result due to its generality.

In the monograph of Favini-Yagi [11], we can find various conditions on operators L, M guaranteeing that assumption $(\mathtt{M} - \mathtt{L})$ holds.

Example 5.1. Let L be a densely defined closed linear operator acting on a Hilbert space X. Suppose that L and its adjoint L^* satisfy

$$Re(Lu,u)_X \leq -c\|u\|_X^2\,, \quad u \in D(L)$$
$$Re(L^*w,w)_X \leq -c\|w\|_X^2\,, \quad w \in D(L^*)$$

for some constant $c > 0$. Let M be a non negative bounded self adjoint operator on X. Then one sees that $(\mathtt{M} - \mathtt{L})$ is fulfilled with $b = 0$, $m = 1$.

Example 5.2. Let $-L$ be a positive self adjoint operator acting on a Hilbert space X. Let M be a densely defined closed linear operator on X with $D\left((-L)^{1/2}\right) \subseteq D(M) \cap D(M^*)$.

Moreover, suppose

$$Re(Mu,u)_X \geq 0\,, \quad u \in D(M)$$

$$Re(M^*v,v)_X \geq 0\,, \quad v \in D(M^*)\,.$$

Then it is seen in Favini-Yagi [11], p. 137, that $(\mathtt{M}-\mathtt{L})$ holds with $\tilde{a} > 0$, $b = 0$ and $m = 2$. This is typically a hyperbolic case.

Remark 5.1. In Example 5.2, if we assume that M is a non negative self adjoint operator, one sees that the exponent m can be improved to $m = 1$.

Example 5.3. Let L and M be densely defined closed linear operators on a Hilbert space X, $D(L) \subseteq D(M)$, L^{-1} being single valued and bounded in X. Assume that M (resp. M^*) is $L-$bounded (L^*-bounded) with $L-$bound (resp. L^*-bound) 0. In addition, assume

$$Re(Lu,Mu)_X \leq 0\,, \quad u \in D(L)$$

$$Re(L^*w,M^*w)_X \leq 0\,, \quad w \in D(L^*)\,.$$

Then it is shown in Favini-Yagi [11], p. 138, that $(\mathtt{M}-\mathtt{L})$ is fulfilled with $b = 0$ and $m = 1$.

Other examples could be described.

Of course, one could handle related inverse problems. Let

$$B_X : \left\{v \in C^1([0,\tau];X);\, v(0) = 0\right\} \longrightarrow C([0,\tau];X)$$

$$B_X v := D_t v \text{ (the time derivative of } v).$$

Then $\rho(B_X) = \mathbb{C}$ and B_X is a positive operator in $C([0,T];X)$ of type $\pi/2$.

For $\delta > 0$ and any $f \in C([0,T];X)$, we define the operator $B_X^{-\delta}$ as :

$$B_X^{-\delta} f(t) = \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} f(s)\, ds.$$

One sees easily that for all $\delta \in R^+$, $B_X^{-\delta}$ is injective. Then we can define

$$B_X^{\delta} = (B_X^{-\delta})^{-1} \text{ for all } \delta \in R^+\,.$$

Remark 5.2. It is shown that if $\delta \in (0,2)$, B_X^{δ} is positive of type $\dfrac{\delta\pi}{2}$.

Let $\tilde{\alpha} \in (0,1]$. Then we can consider an inverse problem as follows. To find a pair $(y,f) \in C([0,T];D(L)) \times C([0,T];\mathbb{C})$ satisfying the inverse problem

$$B_X^{\tilde{\alpha}} My(t) = Ly(t) + f(t)z + h(t), \quad 0 \le t \le T\,, \tag{14}$$

$$(My)(0) = My_0\,, \tag{15}$$

$$\Phi[My(t)] = g(t), \quad 0 \le t \le T\,, \tag{16}$$

where M, L are closed linear operators on X with $D(L) \subseteq D(M)$, $0 \in \rho(L)$, $z \in X$, $h \in C([0,T];X)$, $y_0 \in D(L)$, $\Phi \in X^*$, $g \in C([0,T];\mathbb{C})$. Of course, the compatibility relation

$$\Phi[My_0] = g(0)$$

must hold. By applying Φ to both sides of equation (14), we obtain

$$B_{\mathbb{C}}^{\tilde{\alpha}} g(t) = \Phi[Ly(t)] + \Phi[h(t)] + f(t)\Phi[z]\,.$$

If $\Phi[z] \neq 0$, then necessarily

$$f(t) = \frac{1}{\Phi[z]} \left\{ B_{\mathbb{C}}^{\tilde{\alpha}} g(t) - \Phi[Ly(t)] - \Phi[h(t)] \right\}\,.$$

Therefore, we have the direct problem

$$B_X^{\tilde{\alpha}} My(t) = Ly(t) - \frac{\Phi[Ly(t)]}{\Phi[z]}\,z + h(t) - \frac{\Phi[h(t)]}{\Phi[z]}\,z + \frac{B_{\mathbb{C}}^{\tilde{\alpha}} g(t)}{\Phi[z]}\,z\,, \quad t \in [0,T] \tag{17}$$

$$(My)(0) = My_0\,. \tag{18}$$

Let L_1 be the operator defined by

$$D(L_1) = D(L), \quad L_1 y = -\frac{\Phi[Ly]}{\Phi[z]}\,z\,.$$

Then Problem (17)-(18) reduces to

$$B_X^{\tilde{\alpha}} My(t) = (L + L_1)y(t) + h(t) - \frac{\Phi[h(t)]}{\Phi[z]}\,z + \frac{B_{\mathbb{C}}^{\tilde{\alpha}} g(t)}{\Phi[z]}\,z\,, \quad t \in [0,T] \tag{19}$$

$$(My)(0) = My_0\,. \tag{20}$$

Now we know that operator $B_X^{\tilde{\alpha}}$ satisfies $(\mathcal{T})$ and (6)

$$\left\| (z - B_X^{\tilde{\alpha}})^{-1} \right\|_{\mathcal{L}(C([0,T];X))} \le c(1 + |z|)^p\,.$$

Therefor we could apply Theorem 5.1 provided that $(zM-L-L_1)^{-1}$ satisfies

$$\begin{aligned}\left\|(L+L_1)(zM-L-L_1)^{-1}\right\| &= \left\|1-zM(zM-L-L_1)^{-1}\right\| \\ &\le 1+|z|\,\left\|M(zM-L-L_1)^{-1}\right\| \le c(1+|z|)^m\end{aligned}$$

for some integer $m \ge 0$. Now

$$\begin{aligned}M(zM-L-L_1)^{-1} &= M\left[\left(1-L_1(zM-L)^{-1}\right)(zM-L)\right]^{-1} \\ &= M(zM-L)^{-1}\left(1-L_1(zM-L)^{-1}\right)^{-1}.\end{aligned}$$

Therefore, we are compelled to assume that, in the half plane $\mathrm{Re}z \ge \tilde{a} > 0$,

$$\left\|L_1(zM-L)^{-1}\right\|_{\mathcal{L}(X)} \le \tilde{c} < 1\,.$$

Example 5.4. Consider the following system where L, L_1, M are closed linear operators on the Banach space X, $D(L) \subseteq D(M)$, $D(L_1) \subseteq D(M)$

$$D_t^{\tilde{\alpha}}\begin{bmatrix}0 & M\\ 0 & 0\end{bmatrix}\begin{bmatrix}y(t)\\ x(t)\end{bmatrix} = \begin{bmatrix}L & 0\\ 0 & L_1\end{bmatrix}\begin{bmatrix}y(t)\\ x(t)\end{bmatrix} + \begin{bmatrix}f_1(t)\\ f_2(t)\end{bmatrix}, \qquad 0 \le t \le T\,,$$

$$\begin{bmatrix}0 & M\\ 0 & 0\end{bmatrix}\begin{bmatrix}y(0)\\ x(0)\end{bmatrix} = \begin{bmatrix}0 & M\\ 0 & 0\end{bmatrix}\begin{bmatrix}y_0\\ x_0\end{bmatrix},$$

$0 < \tilde{\alpha} \le 1$. We have, provided $0 \in \rho(L) \cap \rho(L_1)$, that the resolvent system

$$\lambda\begin{bmatrix}0 & M\\ 0 & 0\end{bmatrix}\begin{bmatrix}y\\ x\end{bmatrix} - \begin{bmatrix}L & 0\\ 0 & L_1\end{bmatrix}\begin{bmatrix}y\\ x\end{bmatrix} = \begin{bmatrix}f_1\\ f_2\end{bmatrix}$$

reads

$$\begin{aligned}&\lambda Mx - Ly = f_1\,,\\ &-L_1x = f_2\,,\end{aligned}$$

so that

$x = -L_1^{-1}f_2\,, \quad Ly = -\lambda ML_1^{-1}f_2 - f_1\,,$

$x = -L_1^{-1} f_2\,,\;\; y = -\lambda L^{-1} M L_1^{-1} f_2 - L^{-1} f_1.$

Therefore,

$$\left(\lambda \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} L & 0 \\ 0 & L_1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} -L^{-1} & -\lambda L^{-1} M L_1^{-1} \\ 0 & -L_1^{-1} \end{bmatrix}$$

implying

$$\begin{bmatrix} L & 0 \\ 0 & L_1 \end{bmatrix} \left(\lambda \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} L & 0 \\ 0 & L_1 \end{bmatrix}\right)^{-1} = \begin{bmatrix} L & 0 \\ 0 & L_1 \end{bmatrix} \begin{bmatrix} -L^{-1} & -\lambda L^{-1} M L_1^{-1} \\ 0 & -L_1^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} -I & -\lambda M L_1^{-1} \\ 0 & -I \end{bmatrix} \in \mathcal{L}(X \times X)$$

for all $\lambda \in \mathbb{C}$ so that

$$\left\| \begin{bmatrix} L & 0 \\ 0 & L_1 \end{bmatrix} \left(\lambda \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} L & 0 \\ 0 & L_1 \end{bmatrix}\right)^{-1} \right\|_{\mathcal{L}(X\times X)} \leq C(1 + |\lambda|)\,.$$

We want to solve the inverse problem

$$D_t^{\tilde{\alpha}} \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} L & 0 \\ 0 & L_1 \end{bmatrix} \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} + f(t) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} h_1(t) \\ h_2(t) \end{bmatrix}, \qquad 0 \leq t \leq T\,,$$

$$\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(0) \\ x(0) \end{bmatrix} = \begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_0 \\ x_0 \end{bmatrix},$$

$$\Phi\left(\begin{bmatrix} 0 & M \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ x(t) \end{bmatrix}\right) = \Phi\left(\begin{bmatrix} Mx(t) \\ 0 \end{bmatrix}\right) = g(t) = \Phi_1\left[Mx(t)\right]$$

with $\Phi = [\Phi_1, 0]$. Since

$$D_t^{\tilde{\alpha}}[Mx(t)] = Ly(t) + f(t)\, z_1 + h_1(t)\,, \tag{21}$$

$$0 = L_1 x(t) + f(t) z_2 + h_2(t)\,, \tag{22}$$

then applying Φ_1 to (21), we get

$$D_t^{\tilde{\alpha}} g(t) = \Phi_1[Ly(t)] + f(t)\Phi_1[z_1] + \Phi_1[h_1(t)]\,.$$

On the other hand,

$$L_1x(t) = -f(t)z_2 - h_2(t)\,,$$

i.e.,

$$x(t) = -f(t)L_1^{-1}z_2 - L_1^{-1}h_2(t)\,.$$

This implies

$$Mx(t) = -f(t)ML_1^{-1}z_2 - ML_1^{-1}h_2(t)\,.$$

$$\Phi_1[Mx(t)] = g(t) = -f(t)\Phi_1[ML_1^{-1}z_2] - \Phi_1[ML_1^{-1}h_2(t)]\,.$$

Suppose that $\Phi_1[ML_1^{-1}z_2] \neq 0$. Then

$$f(t) = \frac{-1}{\Phi_1[ML_1^{-1}z_2]}\left[g(t) + \Phi_1[ML_1^{-1}h_2(t)]\right]\,,$$

$$x(t) = \frac{L_1^{-1}z_2}{\Phi_1[ML_1^{-1}z_2]}\left[g(t) + \Phi_1[ML_1^{-1}h_2(t)]\right] - L_1^{-1}h_2(t)$$

$$\begin{aligned}
Ly(t) &= D_t^{\tilde{\alpha}}[Mx(t)] - f(t)\,z_1 - h_1(t) \\
&= \frac{ML_1^{-1}z_2}{\Phi_1[ML_1^{-1}z_2]}\left[g^{(\tilde{\alpha})}(t) + \Phi_1[ML_1^{-1}h_2^{(\tilde{\alpha})}(t)]\right] - ML_1^{-1}h_2^{(\tilde{\alpha})}(t) \\
&\quad + \frac{1}{\Phi_1[ML_1^{-1}z_2]}\left[g(t) + \Phi_1[ML_1^{-1}h_2(t)]\right]\,z_1 - h_1(t)\,.
\end{aligned}$$

Therefore, if $g^{(\tilde{\alpha})}$, $h_1^{(\tilde{\alpha})}$, $h_2^{(\tilde{\alpha})}$ are continuous, our problem admits a unique solution.

Observe that

$$\begin{aligned}
\Phi_1[Ly(t)] &= \frac{\Phi_1[ML_1^{-1}z_2]}{\Phi_1[ML_1^{-1}z_2]}\left[g^{(\tilde{\alpha})}(t) + \Phi_1[ML_1^{-1}h_2^{(\tilde{\alpha})}(t)]\right] - \Phi_1[ML_1^{-1}h_2^{(\tilde{\alpha})}(t)] - f(t)\Phi_1[z_1] - \Phi_1[h_1(t)] \\
&= g^{(\tilde{\alpha})}(t) - f(t)\Phi_1[z_1] - \Phi_1[h_1(t)]
\end{aligned}$$

as desired.

Example 5.5. Consider the inverse problem to find x, y and f such that

$$\frac{d}{dt}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} y(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} + f(t)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}\,, \qquad \forall\, t \in [0,T]\,,$$

using the information

$$\begin{bmatrix} a & b \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} = g(t), \quad \forall\, t \in [0,T]\,.$$

Observe that $z = 0$ is a pole for $\left(z - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)^{-1}$ of order 2. The problem reads

$$\begin{aligned} &x'(t) = y(t) + f(t)\, z_1 \\ &0 = x(t) + f(t) z_2 \\ &ax(t) = g(t)\,, \quad \forall\, t \in [0,T]\,. \end{aligned}$$

Necessarily, $a \neq 0$, so that $x(t) = g(t)/a$. Therefore, if $z_2 \neq 0$, $f(t) = -\dfrac{g(t)}{az_2}$, yielding $y(t) = g'(t)/a + g(t) z_1 (az_2)^{-1}$. If we would apply our general method, since $\Phi = \begin{bmatrix} a & b \end{bmatrix}$, we have $g'(t) = \Phi[(y,x)] + f(t)(az_1 + bz_2)$, so that, if $az_1 + bz_2 \neq 0$, $f(t) = \dfrac{g'(t) - ay(t) - bx(t)}{az_1 + bz_2}$. This implies that necessarily

$$\frac{d}{dt} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} + \frac{g'(t) - ay(t) - bx(t)}{az_1 + bz_2} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \qquad \forall\, t \in [0,T]\,,$$

with the additional information $ax(t) = g(t)$. Thus

$$\frac{d}{dt} \begin{bmatrix} x(t) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{bz_2}{az_1+bz_2} & \frac{-bz_1}{az_1+bz_2} \\ \frac{-az_2}{az_1+bz_2} & \frac{az_1}{az_1+bz_2} \end{bmatrix} \begin{bmatrix} y(t) \\ x(t) \end{bmatrix} + \frac{g'(t)}{az_1 + bz_2} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \qquad \forall\, t \in [0,T]\,,$$

or

$$\begin{aligned} \frac{g'(t)}{a} &= \frac{bz_2}{az_1 + bz_2}\, y(t) - \frac{bz_1}{az_1 + bz_2}\, x(t) + \frac{z_1}{az_1 + bz_2}\, g'(t) \\ 0 &= -\frac{az_2}{az_1 + bz_2}\, y(t) + \frac{az_1}{az_1 + bz_2}\, x(t) + \frac{z_2}{az_1 + bz_2}\, g'(t)\,. \end{aligned}$$

By using $ax(t) = g(t)$ again, we get

$$\frac{bz_2}{az_1 + bz_2}\, y(t) - \frac{bz_1}{a(az_1 + bz_2)}\, g(t) = \frac{bz_2}{a(az_1 + bz_2)}\, g'(t)\,,$$

$$\frac{az_2}{az_1 + bz_2}\, y(t) = \frac{z_1}{az_1 + bz_2}\, g(t) + \frac{z_2}{az_1 + bz_2}\, g'(t)\,,$$

or equivalently,

$$\frac{z_2}{az_1+bz_2}\,y(t)=\frac{z_1}{a(az_1+bz_2)}\,g(t)+\frac{z_2}{a(az_1+bz_2)}\,g'(t)\,,$$

$$\frac{az_2}{az_1+bz_2}\,y(t)=\frac{z_1}{az_1+bz_2}\,g(t)+\frac{z_2}{az_1+bz_2}\,g'(t)\,.$$

Observe that multiplying the first equation above by a we get the second equation. Therefore

$$y(t)=\frac{g'(t)}{a}+g(t)z_1(az_2)^{-1}$$

and

$$\begin{aligned} f(t)&=\frac{g'(t)-ay(t)-bx(t)}{az_1+bz_2}=\frac{-g(t)z_1z_2^{-1}-g(t)ba^{-1}}{az_1+bz_2}\\ &=-g(t)\,\frac{az_1+bz_2}{az_2}\,(az_1+bz_2)^{-1}=-\frac{g(t)}{az_2}\,, \end{aligned}$$

as declared.

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Mohammed AL Horani

Department of Mathematics, The University of Jordan, Amman, Jordan,

E-mail address: `horani@ju.edu.jo`

Angelo Favini

Dipartimento di Matematica, Università di Bologna, Piazza di Porta S. Donato 5, 40126 Bologna (Italia)

E-mail address: angelo.favini@unibo.it