OPTIMIZATION OF NONLINEAR EIGENVALUES UNDER MEASURE OR PERIMETER CONSTRAINT

OTTIMIZZAZIONE DI AUTOVALORI NON LINEARI CON VINCOLO DI MISURA O DI PERIMETRO

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Abstract. In this paper we recall some recent results about variational eigenvalues of the p-Laplacian, we show new applications and point out some open problems. We focus on the continuity properties of the eigenvalues under the γ_p-convergence of capacitary measures, which have been the subject of the study of [8, 9, 10] and are needed to prove existence results for the minimization of nonlinear eigenvalues in the class of p-quasi open sets contained in a box under a measure constraint. Finally, the new contribution of this paper is to show that these continuity results can be employed to prove existence of minimizers for nonlinear eigenvalues among measurable sets contained in a box and under a *perimeter* constraint, generalizing to the case $p \neq 2$ some results of [2].

Sunto. In questo articolo richiamiamo alcuni recenti risultati riguardanti gli autovalori variazionali del p-Laplaciano, mostriamo nuove applicazioni e mettiamo in luce alcuni problemi aperti. Ci soffermiamo sulle proprietà di continuità degli autovalori rispetto alla γ_p-convergenza di misure capacitarie, che sono state l'argomento dei lavori [8, 9, 10] e che sono necessarie per dimostrare risultati di esistenza per problemi di minimizzazione di autovalori non lineari nella classe dei p-quasi aperti contenuti in un box e con vincolo di misura. Infine, il nuovo contributo di questo lavoro consiste nel dimostrare che questi risultati di continuità possono essere sfruttati per dimostrare l'esistenza di minimi per autovalori non lineari nella classe degli insiemi misurabili contenuti in un box e con vincolo di *perimetro*, estendendo al caso $p \neq 2$ alcuni risultati di [2].

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1. Introduction

In recent years, there has been a wide interest on shape optimization problems for spectral functionals, in particular concerning the eigenvalues of the Dirichlet-Laplacian, see [11] for an overview. On the other hand, the study of analogous problems for eigenvalues of nonlinear operators as the p-Laplacian (still with Dirichlet boundary conditions) has only recently been investigated.

Given an open, bounded subset Ω of $\mathbb{R}^N$ and $1 < p < \infty$, we say that $\lambda > 0$ is an eigenvalue of the p-Laplacian if there is a nonzero weak solution u, called eigenfunction, of the problem

$$\begin{cases} -\Delta_p u := -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,. \end{cases}$$

The eigenvalues can be characterized as the critical values of the functional

$$f : W_0^{1,p}(\Omega) \to \mathbb{R}\,, \qquad f(u) = \int_\Omega |\nabla u|^p \, d\mathcal{L}^N\,,$$

on the manifold $M = \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^p \, d\mathcal{L}^N = 1 \right\}$, where $W_0^{1,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$, the space of smooth functions with compact support in Ω, with respect to the $W^{1,p}$ norm. The first eigenvalue is a minimum, while higher eigenvalues (if $p \neq 2$) are less understood. More precisely, one can obtain a nondecreasing sequence of eigenvalues through the minimax procedure

$$\lambda_m^p(\Omega) = \inf_{K \in \mathcal{K}_m} \sup_{u \in K} f(u) \qquad \text{for all integer } m \geq 1\,, \tag{1}$$

where $\mathcal{K}_m$ denotes the collection of subsets K of M which are compact, symmetric (i.e. $K = -K$) and such that $i(K) \geq m$ and i denotes a suitable topological index. Unfortunately, it is still a major open problem to understand if all the eigenvalues of the p-Laplacian are of this form. Here we focus only on the "variational" eigenvalues arising from the minimax procedure described above.

We are interested in the following shape optimization problems for variational eigenvalues of the p-Laplacian: given a bounded open domain $\Omega \subseteq \mathbb{R}^N$, $c \leq \mathcal{L}^N(\Omega)$, $k \in \mathbb{N}$ and a function $F\colon \mathbb{R}^k \to \mathbb{R}$ which is nondecreasing in each variable and lower semicontinuous, we deal with

$$\min\Big\{F(\lambda_1^p(A),\dots,\lambda_k^p(A)) : A\subseteq\Omega,\ \mathcal{L}^N(A)\le c\Big\}, \tag{2}$$

and

$$\min\Big\{F(\lambda_1^p(A),\dots,\lambda_k^p(A)) : A\subseteq\Omega,\ P(A)\le c\Big\}, \tag{3}$$

where $P(\cdot)$ denotes the De Giorgi perimeter.

To investigate those problems, the key issue consists in identifying a class of sets and a topology which are suitable for proving both a compactness result and a lower semicontinuity property of the variational eigenvalues. Concerning the first and most studied problem (2), the work of Dal Maso and Murat [6] made it clear that a smart way of tackling this optimization problem is to relax it into the larger class of p-capacitary measures, endowed with the (topology induced by the) γ_p-convergence. In this setting they proved a general compactness result. On the other hand, the lower semicontinuity of the variational nonlinear eigenvalues with respect to the γ_p-convergence is a rather tricky issue, which has been understood only recently in [8] with an abstract approach, in [10] for the case $k = 1, 2$ and finally in [9] in a rather general setting, where the existence of minimizers for problem (2) is completely solved in the class of p-quasi open sets. We recall the result of [9] in Theorem 3.2.

Problem (3) has been studied, up to our knowledge, only in the case $p = 2$, starting from the paper [2], where it is highlighted that the minimization can be performed among measurable sets, as soon as a suitable definition of variational eigenvalues (and of Sobolev spaces) is given. We prove an existence result for problem (3) in Theorem 4.1 also for the case $p \neq 2$, by employing the lower semicontinuity of nonlinear variational eigenvalues with respect to γ_p-convergence of p-capacitary measures, though in this setting the situation is more delicate, since the notion of perimeter and of γ_p-convergence do not seem to interact well.

Plan of the paper. Section 2 is devoted to recalling the notions of p-capacitary measures, p-quasi open sets, generalized Sobolev spaces, γ_p-convergence, and nonlinear variational eigenvalues. In Section 3 we discuss the continuity properties of nonlinear variational eigenvalues of capacitary measures under γ_p-convergence and we prove an existence result under measure constraint in Theorem 3.2. This Section is based on results from [9]. Finally, in Section 4 we prove an existence result for nonlinear eigenvalues under perimeter constraint, Theorem 4.1.

2. Preliminaries

Throughout the paper, we fix an integer $N \geq 1$ and $1 < p < \infty$. We denote by $\mathcal{L}^N$ the $N-$dimensional Lebesgue measure. We also fix $\Omega \subseteq \mathbb{R}^N$ to be a bounded domain (with the term domain we denote a connected open set), which will be sometimes called the "box". If (X, d) is a metric space, we set $B_r(x) := \{y \in X : \ d(y,x) < r\}$ and we denote by $\mathcal{B}(X)$ the family of Borel subsets of X.

Capacity, p-quasi open sets and Sobolev spaces. We need to introduce the notion of p-capacity.

Definition 2.1. *For every subset E of $\mathbb{R}^N$, the p-capacity of E in $\mathbb{R}^N$ is defined as*

$$\mathrm{cap}_p(E) := \inf\Big\{\int (|\nabla u|^p + |u|^p)\, d\mathcal{L}^N : \ u \in W^{1,p}(\mathbb{R}^N)\,, \\ 0 \leq u \leq 1 \ \mathcal{L}^N\text{-a.e. on } \mathbb{R}^N,\ u = 1\ \mathcal{L}^N\text{-a.e. on an open set containing } E\Big\},$$

where we agree that $\inf \emptyset = +\infty$. *If $E \subseteq \mathbb{R}^N$, we say that a property $\mathcal{P}(x)$ holds* cap_p-quasi everywhere in E, *if it holds for all $x \in E$ except at most a set of zero p-capacity. We will write q.e. in E instead of cap_p-quasi everywhere in E, for the sake of simplicity.*

Definition 2.2. *A subset A of $\mathbb{R}^N$ is said to be p*-quasi open *if, for every $\varepsilon > 0$, there exists an open subset ω_ε of $\mathbb{R}^N$ such that $\mathrm{cap}_p(\omega_\varepsilon) < \varepsilon$ and $A \cup \omega_\varepsilon$ is open in $\mathbb{R}^N$.*

Definition 2.3. *A function $u : \mathbb{R}^N \to \overline{\mathbb{R}}$ is said to be p*-quasi continuous *if for every $\varepsilon > 0$ there exists an open subset ω_ε of $\mathbb{R}^N$ with $\mathrm{cap}_p(\omega_\varepsilon) < \varepsilon$ such that $u\big|_{\mathbb{R}^N\setminus\omega_\varepsilon}$ is continuous.*

For every $u \in W^{1,p}_{loc}(\mathbb{R}^N)$, there exists a Borel and p-quasi continuous representative $\tilde{u} : \mathbb{R}^N \to \mathbb{R}$ of u and, if $\tilde{u}$ and $\hat{u}$ are two p-quasi continuous representatives of the same

u, then we have $\tilde{u} = \hat{u}$ q.e. in $\mathbb{R}^N$. In the following, for every $u \in W^{1,p}_{loc}(\mathbb{R}^N)$, we will consider only its Borel and p-quasi continuous representatives.

Definition 2.4. *If A is a p-quasi open subset contained in Ω, we set*

$$W^{1,p}_0(A) := \left\{u \in W^{1,p}_0(\Omega) : \; u = 0 \; q.e. \; in \; \mathbb{R}^N \setminus A\right\} .$$

If A is an open set, then the space $W^{1,p}_0(A)$ defined above coincides with the closure of $C^\infty_c(A)$, the space of smooth functions with compact support in A, with respect to the $W^{1,p}$ norm; so there is no risk of confusion in the notation. In order to deal with the perimeter constraint, we will need to work also with measurable sets, thus a weaker notion of Sobolev space is needed.

Definition 2.5. *If M is a measurable set contained in Ω, we set*

$$\widetilde{W}^{1,p}_0(M) := \left\{u \in W^{1,p}_0(\Omega) : \; u = 0 \; a.e. \; in \; \mathbb{R}^N \setminus M\right\} .$$

It is immediate to check that the spaces $W^{1,p}_0(A)$ and $\widetilde{W}^{1,p}_0(M)$ defined above are closed subspaces of $W^{1,p}_0(\Omega)$, because from any sequence converging strongly in $W^{1,p}_0(\Omega)$ it is possible to extract a subsequence converging q.e. to the same limit. For a p-quasi open set $A \subseteq \Omega$, in general $W^{1,p}_0(A) \subseteq \widetilde{W}^{1,p}_0(A)$. If $A \subseteq \Omega$ is a Lipschitz domain, then $W^{1,p}_0(A) = \widetilde{W}^{1,p}_0(A)$, while even for open sets, as for example a disk in $\mathbb{R}^2$ minus a radius, one can have a strict inclusion.

The following lemma, which clarifies the relation between the spaces $W^{1,p}_0$ and $\widetilde{W}^{1,p}_0$, is well known in the case $p = 2$, and can be proved similarly for the case $p \neq 2$. The proof that we present here is inspired by [4, Proposition 6.9].

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, $M \subseteq \Omega$ be a measurable set. There exists a quasi-open set Ω_M which is contained a.e. in M, that is, $|\Omega_M \setminus M| = 0$ and such that $\widetilde{W}^{1,p}_0(M) = \widetilde{W}^{1,p}_0(\Omega_M) = W^{1,p}_0(\Omega_M)$. Moreover, if $\Omega_M, \widetilde{\Omega}_M \subseteq M$ a.e. are such that $\widetilde{W}^{1,p}_0(M) = W^{1,p}_0(\Omega_M) = W^{1,p}_0(\widetilde{\Omega}_M)$, then $\Omega_M = \widetilde{\Omega}_M$ q.e..*

Proof. Since $\widetilde{W}_0^{1,p}(M) \subseteq W_0^{1,p}(\Omega)$ is a closed subspace of a separable Banach space, it is separable, too and we can find a dense and countable set $\{\varphi_k\}_{k=1}^\infty \subseteq \widetilde{W}_0^{1,p}(M)$. We define

$$\Omega_M := \bigcup_{k=1}^\infty \{\varphi_k \neq 0\} = \{w > 0\}, \qquad w := \sum_{k=1}^\infty \frac{|\varphi_k|}{2^k \|\varphi_k\|_{W^{1,p}}}.$$

It is clear from the definition that $\varphi_k \in W_0^{1,p}(\Omega_M)$, Ω_M is a p-quasi open set and $\Omega_M \subseteq M$ a.e., thus the inclusion $W_0^{1,p}(\Omega_M) \subseteq \widetilde{W}_0^{1,p}(\Omega_M) \subseteq \widetilde{W}_0^{1,p}(M)$ is proved.
In order to prove the reverse inclusion, let $\varphi \in \widetilde{W}_0^{1,p}(M)$ and we can find a subsequence $\{\varphi_{k_i}\}_{i=1}^\infty$ such that $\varphi_{k_i} \to \varphi$ in $W_0^{1,p}(\Omega)$ as $i \to \infty$. Then, up to pass again to a subsequence, we have that $\varphi_{k_i} \to \varphi$ q.e. and therefore $\varphi \in W_0^{1,p}(\Omega_M)$ so the first part of the statement is proved.
Concerning the second part of the statement, let $\Omega_M = \{w > 0\}$ and $\widetilde{\Omega}_M = \{w' > 0\}$. Then $w' \in \widetilde{W}_0^{1,p}(M) = W_0^{1,p}(\Omega_M)$, hence $\widetilde{\Omega}_M \subseteq \Omega_M$ q.e.. Analogously, $w \in \widetilde{W}_0^{1,p}(M) = W_0^{1,p}(\widetilde{\Omega}_M)$, hence $\widetilde{\Omega}_M \supseteq \Omega_M$ q.e. and we have concluded. □

Capacitary measures and γ_p-convergence.

Definition 2.6. *Let Ω be an open subset of $\mathbb{R}^N$. We say that a non-negative Borel measure μ in Ω is* p-capacitary *if, for every $B \in \mathcal{B}(\Omega)$ with* $\mathrm{cap}_p(B) = 0$*, we have $\mu(B) = 0$.*

Definition 2.7. *Two p-capacitary measures μ_1, μ_2 in Ω are said to be* equivalent*, if*

$$\mu_1(A) = \mu_2(A) \qquad \text{for all } A \in \mathcal{B}(\Omega) \text{ with } A \text{ } p\text{-quasi open}.$$

We denote by $\mathcal{M}_0^p(\Omega)$ the quotient of the set of all p-capacitary measures in Ω with respect to such an equivalence relation.

Definition 2.8. *If $\mu, \nu \in \mathcal{M}_0^p(\Omega)$, we write $\mu \leq \nu$ if*

$$\mu(A) \leq \nu(A) \qquad \text{for all } A \in \mathcal{B}(\Omega) \text{ with } A \text{ } p\text{-quasi open}.$$

It is easily seen that this is an order relation in $\mathcal{M}_0^p(\Omega)$.

Example 2.1. *Let us provide the two most important examples of p-capacitary measures. The first one is given by the measure ∞_E corresponding to a subset E of Ω, defined as*

$$\infty_E(B) := \begin{cases} 0 & \text{if } \operatorname{cap}_p(B \cap E) = 0\,, \\ +\infty & \text{if } \operatorname{cap}_p(B \cap E) > 0\,, \end{cases} \qquad \text{for all } B \in \mathcal{B}(\Omega)\,.$$

The other one consists in a measure absolutely continuous with respect to $\mathcal{L}^N$, that is, for a $\mathcal{L}^N$-measurable function $V : \Omega \to [0, +\infty]$, the measure $V\,\mathcal{L}^N$ is defined as

$$(V\,\mathcal{L}^N)(B) = \int_B V\,d\mathcal{L}^N \qquad \text{for all } B \in \mathcal{B}(\Omega)\,.$$

We need now to introduce a “good” notion of convergence on the space of p-capacitary measures. For every $\mu \in \mathcal{M}_0^p(\Omega)$, we denote by $w_\mu(\Omega)$ the *torsion function* in Ω associated with μ, defined as the (unique) minimizer of the functional

$$W_0^{1,p}(\Omega) \ni v \mapsto \frac{1}{p}\int_\Omega |\nabla v|^p\,d\mathcal{L}^N + \frac{1}{p}\int_\Omega |v|^p\,d\mu - \int_\Omega v\,d\mathcal{L}^N\,.$$

Thus w_μ formally solves the PDE

$$\begin{cases} -\Delta_p w_\mu + \mu |w_\mu|^{p-2} w_\mu = 1, & \text{in } \Omega, \\ w_\mu \in W_0^{1,p}(\Omega). \end{cases}$$

Definition 2.9. *The p-quasi open set $A_\mu := \{w_\mu(\Omega) > 0\}$ is called the* set of σ-finiteness *of μ.*

Remark 2.1. *The usual definition of the set A_μ is done using the p-fine topology, but it is equivalent to the one above up to sets of zero p-capacity. For more details on this (delicate) subject and more properties of A_μ, we refer to* [9, Section 5]*. The important property that we stress here is that $\infty_{\Omega \setminus A_\mu} \le \mu$, which will be crucial later.*

Definition 2.10. *If Ω is a bounded and open subset of $\mathbb{R}^N$, a sequence $(\mu^{(n)})$ in $\mathcal{M}_0^p(\Omega)$ is said to be γ_p-*convergent *to μ if $(w_{\mu^{(n)}}(\Omega))$ is weakly convergent to $w_\mu(\Omega)$ in $W_0^{1,p}(\Omega)$.*

For other equivalent definitions of the γ_p-convergence, and in particular a characterization through the Γ-convergence of suitable energy functionals, we refer to [9, Section 5]. We recall here the main result which is fundamental for our purposes and is in the line of [5], which treated the case $p = 2$. For all $\mu \in \mathcal{M}_0^p(\Omega)$ we define the functional $f_\mu \colon L^p(\Omega) \to [0, +\infty]$ as

$$f_\mu(u) = \begin{cases} \frac{1}{p}\int_\Omega |\nabla u|^p \, d\mathcal{L}^N + \frac{1}{p}\int_\Omega |u|^p \, d\mu, & \text{if } u \in W_0^{1,p}(\Omega), \\ +\infty, & \text{otherwise in } L^p(\Omega), \end{cases}$$

and note that if $\mu = \infty_{\Omega \setminus A}$ for a p-quasi open set $A \subseteq \Omega$, then

$$f_{\infty_{\Omega\setminus A}}(u) = \begin{cases} \frac{1}{p}\int_A |\nabla u|^p \, d\mathcal{L}^N, & \text{if } u \in W_0^{1,p}(A), \\ +\infty, & \text{otherwise in } L^p(\Omega). \end{cases}$$

The proof of the following result can be found in [9, Theorem 5.24].

Theorem 2.1. *A sequence* $(\mu^{(n)})$ *is* γ_p*-convergent to* μ *in* $\mathcal{M}_0^p(\Omega)$ *if and only if*

$$f_\mu(u) = \Big(\Gamma - \lim_{n\to\infty} f_{\mu^{(n)}}\Big)(u), \qquad \textit{for all } u \in L^p(\Omega).$$

The main compactness result for capacitary measures under γ_p-convergence is due to Dal Maso and Murat [6, Theorem 6.5].

Theorem 2.2. *For any sequence of capacitary measures* $(\mu^{(n)})$ *in* $\mathcal{M}_0^p(\Omega)$ *there exists a subsequence* $(\mu^{(n_j)})$ *and* $\mu \in \mathcal{M}_0^p(\Omega)$ *such that* $(\mu^{(n_j)})$ γ_p*-converges to* μ*.*

Nonlinear variational eigenvalues.

Definition 2.11. *Let A be a p-quasi open set contained in Ω. We say that a number $\lambda \in \mathbb{R}$ is an eigenvalue of the p-Laplacian if there is a nonzero $u \in W_0^{1,p}(A)$ solution to*

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u, & \textit{in } A, \\ u = 0, & \textit{on } \partial A. \end{cases} \tag{4}$$

Let us consider a topological index i, on a metrizable topological vector space $\mathcal{X}$ (which will be often $W_0^{1,p}(\Omega)$ or $L^p(\Omega)$ in our setting), satisfying the following properties.

(1) *$i(K)$ is an integer greater or equal than 1 and is defined whenever K is a nonempty, compact and symmetric subset of a metrizable topological vector space $\mathcal{X}$ such that $0 \notin K$;*

(2) *if* $K \subseteq \mathcal{X} \setminus \{0\}$ *is nonempty, compact and symmetric, then there exists an open subset* U *of* $\mathcal{X} \setminus \{0\}$ *such that* $K \subseteq U$ *and*

$$i(\widehat{K}) \leq i(K) \quad \text{for all nonempty, compact and symmetric } \widehat{K} \subseteq U\,;$$

(3) *if* $K_1, K_2 \subseteq \mathcal{X} \setminus \{0\}$ *are nonempty, compact and symmetric, then*

$$i(K_1 \cup K_2) \leq i(K_1) + i(K_2)\,;$$

(4) *if* $\mathcal{Y}$ *is also a metrizable topological vector space,* $K \subseteq \mathcal{X} \setminus \{0\}$ *is nonempty, compact and symmetric and* $\pi : K \to \mathcal{Y} \setminus \{0\}$ *is continuous and odd, then we have* $i(\pi(K)) \geq i(K)$*;*

(5) *if* $\mathcal{X}$ *is a real normed space with* $1 \leq \dim \mathcal{X} < \infty$*, then we have*

$$i(\{u \in \mathcal{X} : \ \|u\| = 1\}) = \dim \mathcal{X}\,.$$

Well known examples are the Krasnosel'skii genus and the $\mathbb{Z}_2$-cohomological index introduced by Fadell and Rabinowitz.

We are now able to recall the definition of variational eigenvalues of the p-Laplacian of a p-quasi open set $A \subseteq \Omega$,

$$\lambda_m^p(A) := \inf_{K \in \mathcal{K}_m} \sup_{u \in K} \int_A |\nabla u|^p \, d\mathcal{L}^N, \tag{5}$$

for all $m \geq 1$ and where $\mathcal{K}_m$ denotes the family of compact and symmetric (i.e. $K = -K$) subsets of the manifold

$$M = \left\{ u \in W_0^{1,p}(A) : \int_A |u|^p \, d\mathcal{L}^N = 1 \right\},$$

such that $i(K) \geq m$. The fact that the values $\lambda_m^p(A)$ defined above (if finite) are eigenvalues of the problem (4) follows because the functional $\int_A |\nabla u|^p \, d\mathcal{L}^N$ restricted to the manifold M satisfies the Palais Smale condition at any $c \in \mathbb{R}$ and then using classical results of critical point theory. Unfortunately, it is not known whether all the eigenvalues of (4) are of the inf-sup form (5) and this is a major open problem in this field.

We want now to extend this construction to p-capacitary measures. Let $\mu \in \mathcal{M}_0^p(\Omega)$ and we define the space

$$W_0^{1,p}(\mu) := \left\{ u \in W_0^{1,p}(\Omega) : \int_\Omega |u|^p \, d\mu < +\infty \right\}.$$

We now introduce the "variational eigenvalues" for the problem (formally written as)

$$\begin{cases} -\Delta_p u + |u|^{p-2} u \, \mu = \lambda |u|^{p-2} u & \text{in } \Omega \, , \\ u \in W_0^{1,p}(\mu) \, . \end{cases} \tag{6}$$

We define (assuming that $W_0^{1,p}(\mu) \neq \{0\}$)

$$\lambda_m^p(\mu) := \inf_{K \in \mathcal{K}_m} \sup_{u \in K} f_\mu(u), \tag{7}$$

for all $m \geq 1$ and where $\mathcal{K}_m$ denotes the family of compact and symmetric (i.e. $K = -K$) subsets of the manifold

$$M = \left\{ u \in W_0^{1,p}(\mu) : \int_\Omega |u|^p \, d\mathcal{L}^N = 1 \right\} ,$$

such that $i(K) \geq m$. For the definition of variational eigenvalues in a more general setting also with sign-changing capacitary measures and more properties, in particular the fact that the inf-sup values defined in (7) play the role of eigenvalues for problem (6), see [9, Section 4 and 8]. It is standard to check that, if $A \subseteq \Omega$ is a p-quasi open set and $\mu = \infty_{\Omega \setminus A}$, then

$$W_0^{1,p}(\mu) = W_0^{1,p}(A), \qquad \lambda_m^p(\mu) = \lambda_m^p(A), \quad \text{for all integer } m \geq 1.$$

Moreover, if $\mu_1 \leq \mu_2$, we have $\lambda_m^p(\mu_1) \leq \lambda_m^p(\mu_2)$ for all $m \geq 1$, in particular,

$$\lambda_m^p(\mu) \geq \lambda_m^p(A_\mu), \qquad \text{for all integer } m \geq 1.$$

Eventually, it is important to recall the following lower semicontinuity of the Lebesgue measure of the σ-finiteness sets with respect to the γ_p-convergence, see [9, Corollary 5.26].

Lemma 2.2. *Let $\mu^{(n)}$ be γ_p-convergent to μ in $\mathcal{M}_0^p(\Omega)$, then we have*

$$\mathcal{L}^N(A_\mu) \leq \liminf_{n \to \infty} \mathcal{L}^N(A_{\mu^{(n)}}).$$

On the other hand, the notion of γ_p-convergence and the notion of perimeter do not seem to interact well with each other. We will deal with this topic in Section 4.

3. Continuity of nonlinear eigenvalues with respect to the γ_p-convergence

Differently from the case $p=2$, where the classical theory of linear operators provides general results asserting the continuity of eigenvalues with respect to the γ_2-convergence of capacitary measures, the case $p\neq 2$ requires a different approach and more complicated arguments when dealing with the inf-sup values (7). The results that we recall here are mostly taken from [8, 9]. Instead of presenting the proof, we give a sketch of the main ideas behind it. The first step consists in working with inf-sup values in the $L^p(\Omega)$ setting. We define

$$\hat{\lambda}_m^p(\mu) := \inf_{\hat{K}\in\hat{\mathcal{K}}_m} \sup_{u\in\hat{K}} f_\mu(u), \tag{8}$$

for all $m\geq 1$ and where $\hat{\mathcal{K}}_m$ denotes the family of compact and symmetric (i.e. $\hat{K}=-\hat{K}$) subsets of the manifold

$$\hat{M}=\left\{u\in L^p(\Omega):\int_\Omega |u|^p\,d\mathcal{L}^N=1\right\},$$

such that $i(\hat{K})\geq m$. A priori it is not clear if the inf-sup values defined in (8) coincide with the "real" variational eigenvalues (7), but one can prove for them a lower semicontinuity result (see [9, Section 7]). Then, one needs to prove the equality of the two inf-sup values, and an upper semicontinuity result. These are related to an approach proposed in [8] and expanded in [9], which consists in reducing to inf-sup over finite dimensional spaces, on which all the topologies coincide. We recall first the continuity result for the inf-sup values (8), for the proofs, we refer to [9, Section 3].

Lemma 3.1. *Let $(\mu^{(n)})$ be a sequence that γ_p-converges to μ in $\mathcal{M}_0^p(\Omega)$. Then*

$$\hat{\lambda}_m^p(\mu)=\lim_{n\to\infty}\hat{\lambda}_m^p(\mu^{(n)}),\qquad \textit{for all } m\geq 1.$$

We introduce now the subfamily $\hat{\mathcal{K}}_m^{fin}$ of K's in $\hat{\mathcal{K}}_m$ such that K is contained in a finite dimensional subspace of $L^p(\Omega)$, and it is possible to prove the following result.

Lemma 3.2. *For every integer $m\geq 1$ and every $\mu\in\mathcal{M}_0^p(\Omega)$, we have*

$$\inf_{\hat{K}\in\hat{\mathcal{K}}_m}\sup_{u\in\hat{K}} f_\mu(u)=\inf_{\hat{K}\in\hat{\mathcal{K}}_m^{fin}}\sup_{u\in\hat{K}} f_\mu(u).$$

Since in finite dimension all the topologies are equivalent, the above lemmas provide the desired continuity of nonlinear variational eigenvalues with respect to the γ_p-convergence, which is summarized here (see [9, Section 8]).

Theorem 3.1. *Let $(\mu^{(n)})$ be a sequence that γ_p-converges to μ in $\mathcal{M}_0^p(\Omega)$. Then*

$$\lambda_m^p(\mu) = \lim_{n\to\infty} \lambda_m^p(\mu^{(n)}), \qquad \textit{for all } m \geq 1.$$

3.1. **Optimization of nonlinear eigenvalues with measure constraint.** At this point we have all the tools to prove a first shape optimization result, which was solved in [9] in a more general framework and with a slightly different point of view.

Theorem 3.2. *Let $k \in \mathbb{N}$, $c \in (0, \mathcal{L}^N(\Omega)]$ and $F\colon \mathbb{R}^k \to \mathbb{R}$ be nondecreasing in each variable and lower semicontinuous. There exists an optimal p-quasi open set for the problem*

$$\min\left\{F(\lambda_1^p(A), \dots, \lambda_k^p(A)) : A \subseteq \Omega,\ p\textit{-quasi open},\ \mathcal{L}^N(A) = c\right\}. \tag{9}$$

Proof. Of course the class of admissible sets is not empty, since $c \leq \mathcal{L}^N(\Omega)$. We take a minimizing sequence $(A^{(n)})$ and consider the associated capacitary measures $\infty_{\Omega\setminus A^{(n)}}$, that (up to subsequences) γ_p-converge to a certain $\mu \in \mathcal{M}_0^p(\Omega)$, by Theorem 2.2. Moreover, we have that, by Theorem 3.1 and the monotonicity of the eigenvalues,

$$\lambda_m^p(A_\mu) \leq \lambda_m^p(\mu) = \lim_{n\to\infty} \lambda_m^p(A^{(n)}), \qquad \text{for all integer } m \geq 1.$$

A_μ is a p-quasi open set with $\mathcal{L}^N(A_\mu) \leq c$ thanks to the lower semicontinuity of the measure (see Lemma 2.2). If $\mathcal{L}^N(A_\mu) = c$, we have finished, otherwise it is enough to consider the set

$$\widetilde{A}_\mu := (A_\mu \cup B_R) \cap \Omega,$$

for a suitable radius $R > 0$ such that $\mathcal{L}^N(\widetilde{A}_\mu) = c$. We note that finding such a radius is possible thanks to the continuity of the Lebesgue measure and the fact that $c \leq \mathcal{L}^N(\Omega)$. Then $\widetilde{A}_\mu$ is p-quasi open and

$$\lambda_m^p(\widetilde{A}_\mu) \leq \lambda_m^p(A_\mu), \qquad \text{for all integer } m \geq 1,$$

whence the proof is concluded. $\square$

Generalizations and open problems.

(1) A major open problem is to get rid of the equiboundedness assumption in the class of the admissible sets, i.e. one wants to be able to consider the case $\Omega = \mathbb{R}^N$. This problem has been solved only for $p = 2$, see [1, 3, 13], but the extension to the nonlinear case seems highly nontrivial.

(2) Another important topic is the regularity of solutions, which has been deeply investigated in the case $p = 2$, see for example [14, 12], while for the case of nonlinear eigenvalues very little is known, up to our knowledge, and the techniques for the case $p = 2$ do not seem to be easily extendable.

4. Optimization of nonlinear eigenvalues with perimeter constraint

We want now to consider the case of a perimeter constraint. We recall the definition of De Giorgi perimeter.

Definition 4.1. *Let $M \subseteq \mathbb{R}^N$ be a measurable set, we call* (De Giorgi) perimeter *of M*

$$P(M) := \sup\left\{ \int_{\mathbb{R}^N} \chi_M \mathrm{div}\varphi \, d\mathcal{L}^N : \varphi \in C_c^\infty(\mathbb{R}^N;\mathbb{R}^N),\ \|\varphi\|_{L^\infty} \le 1 \right\},$$

where χ_M denotes the characteristic function of the set M.

First of all, it is important to note that, in principle, the γ_p-convergence (even of p-quasi open sets) does not seem to get along well with the De Giorgi perimeter.

Example 4.1. *Let $B_i \subseteq \mathbb{R}^2$ be (open) balls of radius i, for $i = 1, 2$, centered at the origin. We take $(x_k)_k \in \mathbb{N}$ an enumeration of the rational numbers in $(0,1)$ and we consider $A^{(n)} := B_2 \setminus \cup_{k=1}^n \partial B_{x_k}$. It is clear that $A^{(n)}$ γ_p-converges to $A := B_2 \setminus \overline{B}_1$, that is the capacitary measures $\infty_{\mathbb{R}^N \setminus A^{(n)}}$ γ_p-converge to $\infty_{\mathbb{R}^N \setminus A}$ in the sense of Definition 2.10. Keeping in mind the definition of De Giorgi perimeter,*

$$P(A) = P(B_2) + P(B_1), \qquad \textit{while} \qquad P(A^{(n)}) = P(B_2), \qquad \textit{for all } n.$$

This construction can be easily extended also to higher dimension.

In this section we follow the approach proposed in [2] for the case $p = 2$ and adapt it to nonlinear variational eigenvalues. First of all, we need to define eigenvalues also on measurable sets. We employ Lemma 2.1 to provide the following definition.

Definition 4.2. *Let $M \subseteq \Omega$ be a measurable set. Then for all integer $m \geq 1$, we define*

$$\widetilde{\lambda}^p_m(M) := \lambda^p_m(\Omega_M),$$

where $\Omega_M \subseteq M$ a.e. is the p-quasi open set such that $\widetilde{W}^{1,p}_0(M) = W^{1,p}_0(\Omega_M)$, see Lemma 2.1.

It is important to note that, recalling Lemma 2.1, it holds $\widetilde{\lambda}^p_m(A) \leq \lambda^p_m(A)$ for a generic p-quasi open set $A \subseteq \Omega$, while $\widetilde{\lambda}^p_m(\Omega_M) = \lambda^p_m(\Omega_M)$ for all integer $m \geq 1$. It is then possible, thanks to the lower semicontinuity of nonlinear eigenvalues with respect to the γ_p-convergence, to prove an existence result in the class of measurable sets under a perimeter constraint. We need first a technical lemma.

Lemma 4.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain, $(A^{(n)}) \subseteq \Omega$ be a sequence of p-quasi open sets γ_p-converging to a capacitary measure $\mu \in \mathcal{M}^p_0(\Omega)$. Let $(M^{(n)})$ be a sequence of measurable sets such that $M^{(n)} \supseteq A^{(n)}$ for all $n \in \mathbb{N}$, which is converging to a measurable set M^* in L^1 (that is, the sequence $\chi_{M^{(n)}}$ converges to χ_{M^*} in $L^1(\Omega)$ as $n \to +\infty$). Then we have*

$$|A_\mu \setminus M^*| = 0.$$

Proof. By definition of γ_p-convergence, we have that $(w_{A^{(n)}}(\Omega))$ is weakly convergent in $W^{1,p}_0(\Omega)$ to $w_\mu(\Omega)$ and we recall that $A_\mu = \{w_\mu(\Omega) > 0\}$. Up to pass to subsequences, the convergence is also pointwise a.e. in Ω. Hence, for all $x \in \Omega$ such that the pointwise convergence holds, if $w_\mu(\Omega)(x) > 0$, then for n sufficiently big it must be $w_{A^{(n)}}(\Omega)(x) > 0$, that is, $x \in A^{(n)}$. In other words, we have

$$\chi_{A_\mu}(x) \leq \liminf_{n\to+\infty} \chi_{A^{(n)}}(x).$$

We are in position to conclude, as (using also Fatou lemma)

$$|A_\mu \setminus M^*| \leq \liminf_{n\to+\infty} |A^{(n)} \setminus M^*| \leq \liminf_{n\to+\infty} |M^{(n)} \setminus M^*| = 0. \qquad \square$$

Theorem 4.1. *Let $k \in \mathbb{N}$, $L > 0$ and $F\colon \mathbb{R}^k \to \mathbb{R}$ be a function nondecreasing in each variable and lower semicontinuous. Then there exists a minimizer for the problem*

$$\min\Big\{F(\widetilde{\lambda}_1^p(M), \dots, \widetilde{\lambda}_k^p(M)) : M \subseteq \Omega,\ \textit{measurable},\ P(M) \leq L\Big\}. \tag{10}$$

Proof. Since $\Omega \subset \mathbb{R}^N$ is a bounded domain, we can find a ball $B_r \subseteq \Omega$ with $P(B_r) \leq L$, up to take $r > 0$ small enough, whence the class of admissible sets is not empty. We consider a minimizing sequence $(M^{(n)})$ with $P(M^{(n)}) \leq L$ and we can extract a subsequence (not relabeled) and a measurable set M^* such that

$$\chi_{M^{(n)}} \to \chi_{M^*}, \qquad \text{in } L^1(\Omega), \quad \text{as } n \to +\infty,$$

hence $P(M^*) \leq \liminf_n P(M^{(n)}) \leq L$. By Lemma 2.1 there exist p-quasi open sets $A^{(n)} \subseteq M^{(n)}$ with $\lambda_m^p(A^{(n)}) = \widetilde{\lambda}_m^p(M^{(n)})$ for all integer $m \geq 1$. By the compactness of the γ_p-convergence (Theorem 2.2), we find a capacitary measure $\mu \in \mathcal{M}_0^p(\Omega)$ and a subsequence (not relabeled) so that

$$\infty_{\Omega \setminus A^{(n)}} \xrightarrow{\gamma_p} \mu, \qquad \text{as } n \to +\infty,$$

and $|A_\mu \setminus M^*| = 0$ by Lemma 4.1. Then, by monotonicity of the eigenvalues and the assumptions on F, we have

$$\begin{aligned} F(\widetilde{\lambda}_1^p(M^*), \dots, \widetilde{\lambda}_k^p(M^*)) &\leq F(\widetilde{\lambda}_1^p(A_\mu), \dots, \widetilde{\lambda}_k^p(A_\mu)) \leq F(\lambda_1^p(A_\mu), \dots, \lambda_k^p(A_\mu)) \\ &\leq F(\lambda_1^p(\mu), \dots, \lambda_k^p(\mu)) = \lim_{n\to+\infty} F(\lambda_1^p(A^{(n)}), \dots, \lambda_k^p(A^{(n)})) \\ &= \lim_{n\to+\infty} F(\widetilde{\lambda}_1^p(M^{(n)}), \dots, \widetilde{\lambda}_k^p(M^{(n)})), \end{aligned}$$

therefore M^* is a solution to (10). □

Generalizations and open problems.

(1) One can consider also the double constraint of measure *and* perimeter in problem (10), as it was done in [3] for $p = 2$,

$$\min\Big\{F(\widetilde{\lambda}_1^p(M), \dots, \widetilde{\lambda}_k^p(M)) : M \subseteq \Omega,\ \text{measurable},\ P(M) \leq L,\ \mathcal{L}^N(M) \leq c\Big\}$$

and existence of a minimizer follows as in Theorem 4.1 (if L, c are such that the class of admissible sets is not empty), since the Lebesgue measure is lower semicontinuous with respect to the L^1 convergence. In this case it is important to have also the measure constraint with an inequality, as the equality constraint on the measure creates additional difficulties.

(2) It is clear that in the case of the perimeter constraint, we have proved a rather weak result, in the class of measurable sets and we are not able to say whether optimal sets are even (p-quasi)open. This is unfortunately due to the fact that the proof requires to manage both p-quasi open and measurable sets.

(3) A main open problem, as in the case of measure constraint, is to prove first existence of minimizers among measurable sets of $\mathbb{R}^N$, that is, without the a priori assumption of being contained in the box Ω. Then a second step is to prove a regularity result for optimal sets. In the case $p = 2$ these two results are contained in the seminal paper by De Philippis and Velichkov [7]. Unfortunately, their techniques rely heavily on the fact that they work with a linear operator, thus do not seem easily extendable to the case $p \neq 2$.

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