ANOTHER LOOK TO THE ORTHOTROPIC FUNCTIONAL IN THE PLANE

UN ALTRO SGUARDO AL FUNZIONALE ORTOTROPO NEL PIANO

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Abstract. We address the $\mathcal{C}^1$ regularity of the Lipschitz minimizers to the orthotropic functional in the plane.

Sunto. Studiamo la regolarità di classe $\mathcal{C}^1$ dei minimi lipschitziani del funzionale ortotropo nel piano.

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1. Introduction

1.1. **Overview.**

Given a bounded open set $\Omega \subset \mathbb{R}^2$ and two exponents $1 < p_1 \leq p_2 < \infty$, we consider the *orthotropic functional*

$$\mathcal{F}(u) = \int_\Omega \left(\frac{1}{p_1} |u_{x_1}|^{p_1} + \frac{1}{p_2} |u_{x_2}|^{p_2} \right) dx, \qquad u \in W^{1,1}(\Omega).$$

We say that $u \in W^{1,1}(\Omega)$ is a minimizer to $\mathcal{F}$ on Ω when $\mathcal{F}(u) < \infty$ and

$$\mathcal{F}(u) \leq \mathcal{F}(v), \qquad \forall v \in W_0^{1,1}(\Omega) + u.$$

In this article, we study the $\mathcal{C}^1$ regularity of minimizers to $\mathcal{F}$.

The continuity of the minimizers on a planar domain follows from a classical result in the Calculus of Variations, see Lemma 4.1 in the Appendix for details. The Lipschitz regularity is a much more challenging question: for a brief historical account on this subject, we refer to the introduction of [2] where we prove with L. Brasco that when

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$p_1 \geq 2$, any minimizer is locally Lipschitz continuous (the main result in [2] is stated for any *bounded* minimizer in *any* dimension). When $p_1 = p_2 \leq 2$, the Lipschitz continuity is a consequence of a general result due to Fonseca and Fusco [7, Theorem 2.2]. It is very plausible [4] that all minimizers to $\mathcal{F}$ are Lipschitz continuous, without any restriction on the exponents $1 < p_1 \leq p_2 < \infty$.

Let us denote by F the integrand associated to the functional $\mathcal{F}$:

$$F(\xi) = \frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2}, \qquad \forall \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

When $p_1 \geq 2$, F is at least $\mathcal{C}^2$ and its Hessian is equal to:

$$\nabla^2 F(\xi) = \begin{pmatrix} (p_1 - 1)|\xi_1|^{p_1-2} & 0 \\ 0 & (p_2 - 1)|\xi_2|^{p_2-2} \end{pmatrix}.$$

In particular, $\nabla^2 F(\xi)$ is degenerate on the *unbounded* set $\{\xi = (\xi_1, \xi_2) \in \mathbb{R}^2 : \xi_1\xi_2 = 0\}$, in the sense that for every ξ in this set, the kernel of $\nabla^2 F(\xi)$ is not trivial. When $p_2 < 2$, F is singular on the same set. Finally, when $p_1 < 2 < p_2$, F is degenerate on the ξ_1 axis and singular on the ξ_2 axis. In any case, the Lipschitz continuity of minimizers to $\mathcal{F}$ does not follow from the classical regularity theory in the Calculus of Variations.

In [3], we established the $\mathcal{C}^1$ regularity of minimizers to $\mathcal{F}$ when $p_1 = p_2 \in (1, \infty)$. This result was later extended to the case $2 \leq p_1 \leq p_2$ by Linqdvist and Ricciotti [10]. Their proof is much more simple. Moreover, they obtain an explicit modulus of continuity for the gradient: for every $a \in \Omega$ and every ball $B_r(a)$ of center a and radius $r > 0$ compactly contained in Ω, there exists $C > 0$ such that for $i = 1, 2$:

$$\mathrm{osc}_{B_s(a)} u_{x_i} \leq \frac{C}{\left(\ln \frac{r}{2s}\right)^{\frac{1}{p_i}}}, \qquad \forall s \in \left(0, \frac{r}{2}\right).$$

In the left hand side, $\mathrm{osc}_{B_s(a)} u_{x_i}$ is the *oscillation* of u_{x_i} on the ball $B_s(a)$; that is, $\mathrm{osc}_{B_s(a)} u_{x_i} = \sup_{x,y \in B_s(a)} |u_{x_i}(x) - u_{x_i}(y)|$. Here, the constant C only depends on p_1, p_2 and the following quantity:

$$\frac{1}{r^2} \int_{B_r(a)} \left(|\nabla u|^{p_1} + |\nabla u|^{p_2}\right) dx.$$

Using the same method, Ricciotti [12] obtained a similar result when $p_1 = p_2 < 2$, namely:

$$\operatorname{osc}_{B_s(a)} u_{x_i} \leq \frac{C}{\left(\ln \frac{r}{2s}\right)^{\frac{1}{2}}},$$

where C only depends on p_i and $\frac{1}{r^2}\int_{B_r(a)} |\nabla u|^{p_i}\, dx$.

The aim of this article is twofold. First, when $p_1 \geq 2$ or $p_2 \leq 2$, we explain why the $\mathcal{C}^1$ regularity of minimizers to the orthotropic functional is an easy consequence of a very general (and earlier) result due to De Silva and Savin [6]. This covers the two situations considered in [3], [10] and [12]. More precisely, the statement in [6] is formulated in terms of *a priori* estimates for smooth and uniformly convex integrands. We shall detail how one can deduce from these estimates the $\mathcal{C}^1$ regularity result for our nonsmooth and degenerate/singular orthotropic functional. Our second objective is to extend those results to the remaining case $1 < p_1 < 2 < p_2 < \infty$.

1.2. **The main result.**

Theorem 1.1. *Given $1 < p_1 \leq p_2 < \infty$, let u be a minimizer to $\mathcal{F}$ on $\Omega \subset \mathbb{R}^2$. If u is Lipschitz continuous, then u is $\mathcal{C}^1$ on Ω.*

If $p_1 \geq 2$ or if $p_1 = p_2 \leq 2$, then any minimizer is locally Lipschitz continuous and the above statement applies on any $\Omega' \Subset \Omega$. Hence, any minimizer is $\mathcal{C}^1$ on Ω for those values of p_1 and p_2.

As a by-product of the proof of Theorem 1.1, we obtain an explicit modulus of continuity when $p_1 \geq 2$ or when $p_2 \leq 2$. More specifically, given $a \in \Omega$ and $r > 0$ such that $B_{2r}(a) \Subset \Omega$, there exists $C > 0$ such that

$$\operatorname{osc}_{B_s(a)} \nabla u \leq \frac{C}{\left(\ln \frac{r}{2s}\right)^{\frac{1}{2\max(p_2-1,1)}}}, \qquad \forall s \in \left(0, \frac{r}{2}\right).$$

Here, the constant C only depends on p_1, p_2 and $\|\nabla u\|_{L^\infty(\Omega)}$.

When $p_2 \leq 2$, the above modulus of continuity looks like the same as the one in [12], except that we rely here on the L^∞ norm of ∇u instead of its L^p norm. When $p_1 > 2$, the modulus of continuity obtained in [10] is more accurate than the above one. Indeed, Lindqvist and Ricciotti exploits the specific structure of the orthotropic functional, and in particular the fact that the functions $|u_{x_i}|^{(p_i-2)/2}u_{x_i}$ belong to $W^{1,2}_{loc}(\Omega)$, for $i = 1, 2$. We believe however that our approach can be applied to a larger class of functionals.

1.3. **Structure of the proof.** We follow the strategy introduced by De Silva and Savin in [6]. More specifically, given a smooth and strictly convex function $G : \mathbb{R}^2 \to \mathbb{R}$, one considers the functional

$$\mathcal{G} : u \in W^{1,1}(\Omega) \mapsto \int_{B_1} G(\nabla u)\, dx.$$

Here, B_1 is the unit ball of center 0 in $\mathbb{R}^2$ (for every $r > 0$, we simply denote by B_r, instead of $B_r(0)$, the ball of center 0 and radius r).

We introduce the modulus of convexity of G:

$$\nu_G(t) := \inf_{|\xi-\xi'|\geq t} |\nabla G(\xi) - \nabla G(\xi')|, \qquad \forall t \geq 0. \tag{1}$$

Given two positives numbers $\lambda, \Lambda > 0$, we also consider the sets

$$O_\lambda := \{\xi \in \mathbb{R}^2 : \nabla^2 G(\xi) \geq \lambda I\}, \qquad V_\Lambda := \{\xi \in \mathbb{R}^2 : \nabla^2 G(\xi) \leq \Lambda I\}. \tag{2}$$

Theorem 1.2. [6, Theorem 1.1] *Let u be a smooth minimizer to $\mathcal{G}$ and let $K \geq \|\nabla u\|_{L^\infty(B_1)}$. Assume that there exist $\lambda, \Lambda > 0$ such that*

$$B_K \subset (O_\lambda \cup V_\Lambda)\,. \tag{3}$$

Then in $B_{1/2}$, ∇u has a uniform modulus of continuity depending on the modulus of convexity ν_G, K, $\|\nabla G\|_{L^\infty(B_K)}$ and the sets O_λ, V_Λ.

The above statement is formulated in terms of an priori estimate for a minimizer that is already known to be smooth, and the proof uses that G itself is smooth. However, as mentioned by the authors of [6], since the estimates do not depend on the smoothness of G, Theorem 1.2 can be proved for a nonsmooth integrand G, by the approximation technique they describe in another section of their article. Still, the sets O_λ and V_Λ in (2) can only be defined when G is at least $\mathcal{C}^2$. When G is singular, in the sense that its Hessian cannot be defined on the whole $\mathbb{R}^2$, the above approach has to be suitably modified. We have to face that difficulty for the orthotropic integrand F when $p_1 < 2$.

When $p_1 < 2 < p_2$, a more serious obstacle arises since there is no $\lambda, \Lambda > 0$ for which one could find an approximating sequence $(F_{\varepsilon_k})_{k\geq 1}$ converging to F and for which the main assumption (3) would hold true, in the sense that

$$B_K \subset \{\xi \in \mathbb{R}^2 : \nabla^2 F_{\varepsilon_k}(\xi) \geq \lambda I \text{ or } \nabla^2 F_{\varepsilon_k}(\xi) \leq \Lambda I\}, \qquad \forall k \geq 1.$$

As a matter of fact, the $\mathcal{C}^1$ regularity for the case $p_1 < 2 < p_2$ is the main novelty of the present paper. It turns out that the tools introduced by De Silva and Savin can be adapted to handle this situation as well.

The proof of Theorem 1.2 is based on two localization lemmas, that we explicitly state in the next section, see Lemma 2.1 and Lemma 2.2. In order to obtain explicit modulus of continuity when $p_1 \geq 2$ or $p_2 \leq 2$, we have established two variants, Lemma 2.3 and Lemma 2.4, which yield more precise conclusions under more restrictive assumptions. Those statements can be exploited for a large family of integrands, and not just for the orthotropic integrand F.

Regarding the approximation of F by a sequence of smooth uniformly convex integrands to which the a priori estimates apply, we rely on the same construction for the three cases $p_2 \leq 2$, $p_1 \geq 2$ and $p_1 < 2 < p_2$. To this aim, we have found convenient to exploit an approximation technique inspired from the proof of [5, Theorem 1.1, page 115].

Remark 1.1. *In view of the above introduction, one could conclude that our paper* [3] *had a sad fate: the main result (at least when $p_1 = p_2 > 2$) was essentially contained in* [6]*, up to an approximation argument (obviously, we did not know the article* [6] *at that time). In addition to this, the subsequent papers* [10] *and* [12] *improved our result by giving an explicit modulus of continuity.*

However, the approach that we followed in [3] *allows some extensions to orthotropic functionals* with a lower order term*:*

$$u \mapsto \mathcal{F}(u) + \int_\Omega \ell(x)u(x)\,dx.$$

I do not know whether such extensions are possible with the strategy introduced in [6] *or the one used in* [10]*.*

Moreover, in order to prove our main result in [3]*, we introduced a new type of Caccioppoli inequalities, which later played a crucial role in* [2] *to establish the Lipschitz regularity of bounded minimizers in any dimension.*

1.4. **Plan of the paper.** Section 2 contains the $\mathcal{C}^1$ a priori estimates for the minimizers to general functionals which are either singular or degenerate. We also establish such estimates for the *hybrid* case, namely for F when $1 < p_1 < 2 < p_2 < \infty$. In Section 3, given a minimizer u to $\mathcal{F}$, we construct an approximation sequence $(G_{\varepsilon_k})_{k\geq 1}$ for F and prove that the corresponding sequence of minimizers $(u_{\varepsilon_k})_{k\geq 1}$ converges to u. Applying the a priori estimates of Section 2 to each u_{ε_k}, we eventually deduce the desired $\mathcal{C}^1$ regularity for u.

A technical appendix concludes the paper: it contains the continuity result for minimizers on a planar domain and also a uniform estimate on the modulus of convexity of the approximating sequence $(G_{\varepsilon_k})_{k\geq 1}$.

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2. A PRIORI ESTIMATES

As explained in the Introduction, the proof of our main result Theorem 1.1 relies on several tools introduced in [6], and more specifically on two localization lemmas that we proceed to quote explicitly.

Let $G : \mathbb{R}^2 \to \mathbb{R}^+$ be a smooth function such that $\nabla^2 G > 0$ on $\mathbb{R}^2$. We then consider the functional:

$$\mathcal{G}(v) := \int_{B_1} G(\nabla v(x))\, dx, \qquad v \in W^{1,1}(B_1).$$

For every $\lambda, \Lambda > 0$, one defines the *non degenerate set* O_λ and the *non singular set* V_Λ as in (2).

Given a unit vector e in $\mathbb{R}^2$ and constants $c, c_0, c_1 \in \mathbb{R}$ with $c_0 < c_1$, let us define the sets

$$H_e^+(c) := \{p \in \mathbb{R}^2 : \langle p, e\rangle \geq c\}, \quad H_e^-(c) := \{p \in \mathbb{R}^2 : \langle p, e\rangle \leq c\},$$
$$S_e(c_0, c_1) := \{p \in \mathbb{R}^2 : c_0 \leq \langle p, e\rangle \leq c_1\},$$

where we have denoted by $\langle p, e\rangle$ the standard inner product of the two vectors p and e in $\mathbb{R}^2$.

Let u be a minimizer to $\mathcal{G}$ on the unit ball B_1: $\mathcal{G}(u) < \infty$ and

$$\mathcal{G}(u) \leq \mathcal{G}(v), \qquad \forall v \in W_0^{1,1}(B_1) + u.$$

In this section, one assumes that u is globally Lipschitz on $\overline{B_1}$ and smooth on B_1.

In the first localization lemma, one considers the region of B_1 where ∇u takes its values in the non degenerate set:

Lemma 2.1. [6, Lemma 2.1] *Let $K \geq \|\nabla u\|_{L^\infty(B_1)}$ and $L \geq \|\nabla G\|_{L^\infty(B_K)}$. Assume that there exist a direction e and constants $c_0 < c_1$ such that*

$$S_e(c_0, c_1) \cap \nabla u(B_1) \subset O_\lambda. \tag{4}$$

Then, there exists $\delta > 0$ only depending on $c_1 - c_0$, λ, K, L such that either $\nabla u(B_\delta) \subset H_e^+(c_0)$ or $\nabla u(B_\delta) \subset H_e^-(c_1)$.

The second localization lemma is related to the non singular set:

Lemma 2.2. [6, Lemma 2.2] *Let $K \geq \|\nabla u\|_{L^\infty(B_1)}$ and $\nu : (0, +\infty) \to (0, +\infty)$ such that $\nu \leq \nu_G$ on $(0, +\infty)$, where ν_G is the modulus of convexity of G, see (1). Assume that there exist a unit vector e and constants $\widetilde{c} \in \mathbb{R}$, $\varepsilon > 0$ such that*

$$H_e^+(\widetilde{c} - \varepsilon) \cap \nabla u(B_1) \subset V_\Lambda.$$

Then, there exists $\delta > 0$ only depending on $\widetilde{c}, \varepsilon, \Lambda, K, \nu$ such that either $\nabla u(B_\delta) \subset H_e^+(\widetilde{c} - \varepsilon)$ or $\nabla u(B_\delta) \subset H_e^-(\widetilde{c} + \varepsilon)$.

A close inspection of the proofs of the above lemmas leads to the two next statements, where we strengthen the assumptions in order to get more specific conclusions. More precisely, in the first statement below, we replace the assumption (4) by the requirement that ∇u maps the whole ball B_1 into O_λ. As a consequence, we obtain an *explicit estimate* on the modulus of continuity of ∇u.

Lemma 2.3. *Let* $K \geq \|\nabla u\|_{L^\infty(B_1)}$. *Assume that there exists* $\lambda > 0$ *such that* $\nabla u(B_1) \subset O_\lambda$. *Then*

$$\operatorname{osc}_{B_r} \nabla u \leq C \left(1 + \frac{\|\nabla G\|_{L^\infty(B_K)}}{\lambda}\right) \frac{1}{\sqrt{-\ln(2r)}}, \qquad \forall r \in \left(0, \frac{1}{2}\right),$$

where $C > 0$ *only depends on* K.

Similarly, the next lemma corresponds to Lemma 2.2, where we assume that $\nabla u(B_1) \subset V_\Lambda$.

Lemma 2.4. *Let* $K \geq \|\nabla u\|_{L^\infty(B_1)}$. *Assume that there exists* $\Lambda > 0$ *such that* $\nabla u(B_1) \subset V_\Lambda$. *Then*

$$\nu_G\left(\operatorname{osc}_{B_r} \nabla u\right) \leq \frac{C\Lambda}{\sqrt{-\ln(2r)}}, \qquad \forall r \in \left(0, \frac{1}{2}\right),$$

where $C > 0$ *only depends on* K.

We proceed to prove Lemma 2.3 and Lemma 2.4. We strongly rely on the tools introduced by De Silva and Savin in [6] to establish Lemma 2.1 and Lemma 2.2.

In both lemmas, the starting point is the Euler equation $\operatorname{div}\left[\nabla G(\nabla u)\right] = 0$. By differentiation along a unit vector $e \in \mathbb{S}^1$, one gets that $v := \langle \nabla u, e\rangle$ is a solution of the uniformly elliptic equation

$$\operatorname{div}\left[A(x).\nabla v(x)\right] = 0, \tag{5}$$

where $A = \nabla^2 G(\nabla u)$. In order to simplify the notation, we often write u_e instead of $\langle \nabla u, e\rangle$.

A common ingredient in the proofs of Lemma 2.3 and Lemma 2.4 is the maximum principle, see e.g. [9, Theorem 3.1], applied to (5). This implies that for every $r \in (0, 1)$,

$$\operatorname{osc}_{B_r} u_e = \operatorname{osc}_{\partial B_r} u_e. \tag{6}$$

We also observe that for every $r \in (0, 1)$, there exists $e \in \mathbb{S}^1$ such that

$$\operatorname{osc}_{B_r} \nabla u = \operatorname{osc}_{B_r} u_e. \tag{7}$$

Indeed, let $x, y \in \overline{B_r}$ such that $\operatorname{osc}_{B_r} \nabla u = |\nabla u(x) - \nabla u(y)|$. Then, there exists $e \in \mathbb{S}^1$ such that $|\nabla u(x) - \nabla u(y)| = \langle \nabla u(x) - \nabla u(y), e\rangle$. It follows that

$$\operatorname{osc}_{B_r} \nabla u = \langle \nabla u(x) - \nabla u(y), e\rangle = |u_e(x) - u_e(y)| \leq \operatorname{osc}_{\overline{B_r}} u_e = \operatorname{osc}_{B_r} u_e.$$

The opposite inequality follows from the fact that for every $e \in \mathbb{S}^1$ and every $x, y \in B_r$, $|u_e(x) - u_e(y)| \leq |\nabla u(x) - \nabla u(y)|$. This completes the proof of (7).

In the proofs of lemma 2.3 and lemma 2.4, we rely on the weak formulation of (5):

$$\int_{B_1} \langle \nabla^2 G(\nabla u).\nabla u_e, \nabla \phi\rangle \, dx = 0, \qquad \forall \phi \in C_c^\infty(B_1).$$

More precisely, we apply the above identity to the test function $\phi = \xi^2 u_e$, with $\xi \in C_c^\infty(B_1)$, $0 \leq \xi \leq 1$ and $\xi \equiv 1$ on $B_{1/2}$. Then

$$\int_{B_1} \langle \nabla^2 G(\nabla u).\nabla u_e, \nabla u_e\rangle \xi^2 \, dx = -2 \int_{B_1} \langle \nabla^2 G(\nabla u).\nabla u_e, \nabla \xi\rangle \xi u_e \, dx. \tag{8}$$

We will exploit (8) in two different ways in the proofs of Lemma 2.3 and Lemma 2.4.

The last ingredient is a simple estimate which emphasizes the role of the dimension 2 in this problem. We first observe that this calculation is the core of the proof of a lemma due to Lebesgue, which is used by Lindqvist and Ricciotti to establish the explicit modulus of continuity of ∇u; see [10, Lemma 3.1] for the statement and the proof of this lemma. It is a remarkable fact that De Silva and Savin [6, p.497] exploit the very same calculation (even if they do not rely on the full statement of the Lebesgue lemma). Let us be more specific:

Given a continuous function $h : B^1 \to \mathbb{R}^+$, we assume that there exists $\kappa > 0$ such that for almost every $\rho \in (0, \frac{1}{2})$,

$$\kappa \leq \int_{\partial B_\rho} h \, d\sigma.$$

Then by the Cauchy-Schwarz inequality,

$$\kappa^2 \leq 2\pi\rho \int_{\partial B_\rho} h^2 \, d\sigma.$$

Integrating over $\rho \in (r, \frac{1}{2})$, one gets

$$\kappa^2 \ln \frac{1}{2r} \leq 2\pi \int_{B_{\frac{1}{2}}} h^2 \, dx. \tag{9}$$

In spite of its simplicity, the above estimate plays a key role in the proofs of Lemma 2.3 and Lemma 2.4.

Proof of Lemma 2.3. Let $e \in \mathbb{S}^1$. We start from (8). By the assumption $\nabla u(B_1) \subset O_\lambda$, the left hand side is not lower than

$$\lambda \int_{B_1} |\nabla u_e|^2 \xi^2 \, dx.$$

In the right hand side, we observe that $(\nabla G(\nabla u))_e = \nabla^2 G(\nabla u).\nabla u_e$, so that by integration by parts,

$$\begin{aligned} -2\int_{B_1} \langle \nabla^2 G(\nabla u).\nabla u_e, \nabla\xi\rangle \xi u_e \, dx &= -2\int_{B_1} \langle (\nabla G(\nabla u))_e, \xi\nabla\xi\rangle u_e \, dx \\ &= 2\int_{B_1} \langle \nabla G(\nabla u), (\xi\nabla\xi)_e\rangle u_e \, dx + 2\int_{B_1} \langle \nabla G(\nabla u), \xi\nabla\xi\rangle u_{ee} \, dx \\ &\leq C\|\nabla G\|_{L^\infty(B_K)} \left(K + \int_{B_1} \xi |\nabla u_e| \, dx \right), \end{aligned}$$

where C only depends on ξ. Hence,

$$\lambda \int_{B_1} |\nabla u_e|^2 \xi^2 \, dx \leq C\|\nabla G\|_{L^\infty(B_K)} \left(K + \int_{B_1} \xi |\nabla u_e| \, dx \right).$$

Using the Young inequality in the right hand side, we deduce that

$$\int_{B_{\frac{1}{2}}} |\nabla u_e|^2 \, dx \leq C' \left(1 + \frac{\|\nabla G\|^2_{L^\infty(B_K)}}{\lambda^2} \right), \tag{10}$$

where C' only depends on ξ and K.

We can now conclude as follows: let $r \in (0, \frac{1}{2})$. Then for every $\rho \in (r, \frac{1}{2})$,

$$\operatorname{osc}_{B_r} u_e \leq \operatorname{osc}_{B_\rho} u_e \leq \operatorname{osc}_{\partial B_\rho} u_e,$$

where the last inequality follows from the maximum principle, see (6). Next,

$$\operatorname{osc}_{\partial B_\rho} u_e \leq \int_{\partial B_\rho} |\nabla u_e| \, d\sigma.$$

We then apply (9) with $\kappa = \operatorname{osc}_{B_r} u_e$ and $h = |\nabla u_e|$. This gives

$$(\operatorname{osc}_{B_r} u_e)^2 \ln \frac{1}{2r} \leq 2\pi \int_{B_{\frac{1}{2}}} |\nabla u_e|^2 \, dx.$$

Together with (10), this yields

$$\operatorname{osc}_{B_r} u_e \le \frac{\sqrt{2\pi C'}}{\sqrt{-\ln(2r)}}\left(1+\frac{\|\nabla G\|_{L^\infty(B_K)}}{\lambda}\right).$$

The conclusion then follows from (7).

$\square$

We next turn to the

Proof of Lemma 2.4. Given a nonnegative symmetric matrix A, for every $y\in\mathbb{R}^2$,

$$|A.y|^2 = \langle A.y, A.y\rangle = \langle A^2.y, y\rangle = \alpha^2 r^2 + \beta^2 s^2,$$

where α and β are the eigenvalues of A and (r,s) are the coordinates of y in an orthonormal basis of corresponding eigenvectors. Hence,

$$|A.y|^2 \le \max(\alpha,\beta)(\alpha r^2+\beta s^2) = \max(\alpha,\beta)\langle y, A.y\rangle.$$

We apply this remark to $A=\nabla^2 G(\nabla u)$ and $y=\nabla u_e$, for any $e\in\mathbb{S}^1$. Taking into account the assumption $\nabla u(B_1)\subset V_\Lambda$, this gives

$$\langle \nabla^2 G(\nabla u).\nabla u_e, \nabla u_e\rangle \ge \frac{1}{\Lambda}|\nabla^2 G(\nabla u).\nabla u_e|^2.$$

In view of (8), one gets

$$\begin{aligned}\frac{1}{\Lambda}\int_{B_1}|\nabla^2 G(\nabla u).\nabla u_e|^2\xi^2\,dx &\le -2\int_{B_1}\langle\nabla^2 G(\nabla u).\nabla u_e,\nabla\xi\rangle u_e\xi\,dx\\ &\le 2\int_{B_1}|\nabla^2 G(\nabla u).\nabla u_e||\nabla\xi||u_e|\xi\,dx\\ &\le 2K\|\nabla\xi\|_{L^\infty(B_1)}\int_{B_1}|\nabla^2 G(\nabla u).\nabla u_e|\xi\,dx.\end{aligned}$$

By the Cauchy-Schwarz inequality, this implies

$$\int_{B_1}|\nabla^2 G(\nabla u).\nabla u_e|^2\xi^2\,dx \le 4\pi K^2\Lambda^2\|\nabla\xi\|^2_{L^\infty(B_1)}.$$

Let $w:=\nabla G(\nabla u)$. Then $|||\nabla w|||^2 \le |\nabla^2 G(\nabla u).\nabla u_{x_1}|^2 + |\nabla^2 G(\nabla u).\nabla u_{x_2}|^2$, where we use the notation $|||A||| = \max_{|x|=1}|A.x|$ for any matrix $A\in\mathcal{M}_2(\mathbb{R})$. Hence,

$$\int_{B_{\frac12}}|||\nabla w|||^2\,dx \le C\Lambda^2, \tag{11}$$

where C only depends on ξ and K.

Let $r \in (0, 1/2)$. Then for every $\rho \in (r, 1/2)$, denote by x_ρ^+ and x_ρ^- two points of ∂B_ρ where $u_e|_{\partial B_\rho}$ respectively attains its maximum and its minimum. Consider the sets

$$T^- = \nabla G\left(\{\xi \in \mathbb{R}^2 : \langle \xi, e\rangle = u_e(x_\rho^-)\}\right), \qquad T^+ = \nabla G\left(\{\xi \in \mathbb{R}^2 : \langle \xi, e\rangle = u_e(x_\rho^+)\}\right).$$

For every $\xi, \xi' \in \mathbb{R}^2$ such that $\langle \xi, e\rangle = u_e(x_\rho^+)$, $\langle \xi', e\rangle = u_e(x_\rho^-)$, one has

$$|\xi - \xi'| \geq \langle \xi - \xi', e\rangle = u_e(x_\rho^+) - u_e(x_\rho^-) = \operatorname{osc}_{\partial B_\rho} u_e \geq \operatorname{osc}_{B_r} u_e,$$

where the last inequality follows from the maximum principle, see (6). It follows that

$$\operatorname{dist}\,(T^-, T^+) \geq \nu_G(\operatorname{osc}_{B_r} u_e).$$

Since $w(x_\rho^-) \in T^-$ and $w(x_\rho^+) \in T^+$, the above inequality implies that

$$\nu_G(\operatorname{osc}_{B_r} u_e) \leq |w(x_\rho^+) - w(x_\rho^-)| \leq \int_{\partial B_\rho} |||\nabla w|||\, d\sigma.$$

We then apply (9) with $\kappa = \nu_G(\operatorname{osc}_{B_r} u_e)$ and $h = |||\nabla w|||$. This yields

$$(\nu_G(\operatorname{osc}_{B_r} u_e))^2 \ln \frac{1}{2r} \leq 2\pi \int_{B_{\frac{1}{2}}} |||\nabla w|||^2\, dx.$$

By (11), one gets

$$\nu_G(\operatorname{osc}_{B_r} u_e) \leq \frac{\sqrt{2\pi C}\Lambda}{\sqrt{-\ln(2r)}}.$$

The result now follows from (7). ☐

Remark 2.1. *For every ball $B_r(a)$ with radius $r > 0$ and center $a \in \mathbb{R}^2$ such that $B_r(a) \subset B_1$, the restriction $u|_{B_r(a)}$ is a minimizer to $\mathcal{G}$ on $B_r(a)$. This implies that the map*

$$u_{r,a}(x) := \frac{1}{r} u(a + rx)$$

is a minimizer to $\mathcal{G}$ on B_1. Moreover, $\nabla u(B_r(a)) = \nabla u_{r,a}(B_1)$ and thus $\|\nabla u_{r,a}\|_{L^\infty(B_1)} = \|\nabla u\|_{L^\infty(B_r(a))}$. In particular, if $\nabla u(B_r(a)) \subset O_\lambda$, then one can apply Lemma 2.3 to $u_{r,a}$:

$$\operatorname{osc}_{B_\rho} \nabla u_{r,a} \leq C\left(1 + \frac{\|\nabla G\|_{L^\infty(B_K)}}{\lambda}\right) \frac{1}{\sqrt{-\ln(2\rho)}}, \qquad \forall \rho \in (0, \frac{1}{2}),$$

where C only depends on $K \geq \|\nabla u\|_{L^\infty(B_r(a))}$. This implies that

$$\mathrm{osc}_{B_s(a)} \nabla u \leq C \left(1 + \frac{\|\nabla G\|_{L^\infty(B_K)}}{\lambda} \right) \frac{1}{\sqrt{\ln \frac{r}{2s}}}, \qquad \forall s \in (0, \frac{r}{2}).$$

Similarly, if $\nabla u(B_r(a)) \subset V_\Lambda$, then Lemma 2.4 applied to $u_{r,a}$ gives

$$\nu_G(\mathrm{osc}_{B_s(a)} \nabla u) \leq \frac{C\Lambda}{\sqrt{\ln \frac{r}{2s}}}, \qquad \forall s \in (0, \frac{r}{2}),$$

where C only depends on K.

We will rely on Lemma 2.3 in the *non degenerate* case $p_2 \leq 2$ and on Lemma 2.4 in the *non singular* case $p_1 \geq 2$. In the last case $1 < p_1 < 2 < p_2 < \infty$, we need a new a priori estimate:

Lemma 2.5. *Let $K \geq \|\nabla u\|_{L^\infty(B_1)}$, $L \geq \|\nabla G\|_{L^\infty(B_K)}$ and $\nu : (0, +\infty) \to (0, +\infty)$ such that $\nu \leq \nu_G$ on $(0, +\infty)$. We assume that for every $\varepsilon \in (0,1)$, there exist $\lambda_\varepsilon, \Lambda_\varepsilon > 0$ such that*

$$B_K \cap \left(H^+_{e_1}(\varepsilon) \cup H^+_{-e_1}(\varepsilon) \right) \subset V_{\Lambda_\varepsilon} \tag{12}$$

and

$$B_K \cap \left(H^+_{e_2}(\varepsilon) \cup H^+_{-e_2}(\varepsilon) \right) \subset O_{\lambda_\varepsilon}, \tag{13}$$

where $e_1 = (1,0)$ and $e_2 = (0,1)$.

Then there exists a function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, which only depends on K, L, ν and the families $\{\lambda_\varepsilon\}_{\varepsilon>0}$ and $\{\Lambda_\varepsilon\}_{\varepsilon>0}$, such that $\lim_{r\to 0} \omega(r) = 0$ and

$$\mathrm{osc}_{B_r} \nabla u \leq \omega(r), \qquad \forall r \in (0,1).$$

Proof. Let $\varepsilon > 0$. By assumption, $\nabla u(B_1) \cap S_{e_2}(\varepsilon/4, \varepsilon/2) \subset O_{\lambda_{\varepsilon/4}}$. By Lemma 2.1, there exists $\delta > 0$ depending on $\varepsilon, \lambda_{\varepsilon/4}, K, L$, and such that either $\nabla u(B_\delta) \subset H^+_{e_2}(\varepsilon/4)$ or $\nabla u(B_\delta) \subset H^-_{e_2}(\varepsilon/2)$. In the first case, $\nabla u(B_\delta) \subset O_{\lambda_{\varepsilon/4}}$. It then follows from Lemma 2.3 (see also Remark 2.1) that one can find $\widetilde{\delta} > 0$, only depending on $\varepsilon, \lambda_{\varepsilon/4}, K, L$, and such that $\mathrm{osc}_{B_{\widetilde{\delta}}} \nabla u \leq \varepsilon$.

Similarly, $\nabla u(B_1) \cap H^+_{e_1}(\varepsilon/4) \subset V_{\Lambda_{\varepsilon/4}}$. Hence, Lemma 2.2 implies that there exists $\delta' > 0$ depending on $\varepsilon, \Lambda_{\varepsilon/4}, K, \nu$, and such that either $\nabla u(B_{\delta'}) \subset H^+_{e_1}(\varepsilon/4)$ or $\nabla u(B_{\delta'}) \subset$

$H^-_{e_1}(\varepsilon/2)$. In the first case, $\nabla u(B_{\delta'}) \subset V_{\Lambda_{\varepsilon/4}}$ and thus, by Lemma 2.4 and Remark 2.1, there exists $\widetilde{\delta}' > 0$, depending on $\varepsilon, \Lambda_{\varepsilon/4}, K, \nu$, and such that $\nu_G(\mathrm{osc}_{B_{\widetilde{\delta}'}}\nabla u) \leq \nu(\varepsilon)/2$. Since $\nu(\varepsilon) \leq \nu_G(\varepsilon)$ and ν_G is nondecreasing, this implies that $\mathrm{osc}_{B_{\widetilde{\delta}'}}\nabla u \leq \varepsilon$.

We can repeat the same arguments for the two other directions, namely on $S_{-e_2}(\varepsilon/4, \varepsilon/2)$ and $H^+_{-e_1}(\varepsilon/4)$.

Let us summarize the current state of the proof as follows: If for one of the four directions, we are in position to apply Lemma 2.3 or Lemma 2.4, then we can conclude that there exists $\delta_\varepsilon > 0$ such that

$$\mathrm{osc}_{B_{\delta_\varepsilon}}\nabla u \leq \varepsilon.$$

Otherwise, one can find $\delta'_\varepsilon > 0$ such that

$$\nabla u(B_{\delta'_\varepsilon}) \subset H^-_{e_2}\left(\frac{\varepsilon}{2}\right) \cap H^-_{e_1}\left(\frac{\varepsilon}{2}\right) \cap H^-_{-e_2}\left(\frac{\varepsilon}{2}\right) \cap H^-_{-e_1}\left(\frac{\varepsilon}{2}\right) = \left(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right)^2.$$

In both cases, one has $\mathrm{osc}_{B_{\delta''_\varepsilon}}\nabla u < 2\varepsilon$ for some δ''_ε which only depends on $\varepsilon, \lambda_{\varepsilon/4}, \Lambda_{\varepsilon/4}, L, K$ and ν.

Finally, we set

$$\omega(r) := \sup\{\mathrm{osc}_{B_r}\nabla u\}, \qquad r \in (0,1),$$

where the supremum is taken over all the minimizers u on B_1 of all the smooth integrands G, such that $\|\nabla u\|_{L^\infty(B_1)} \leq K$ and

$$\nabla^2 G > 0 \text{ on } \mathbb{R}^2, \qquad \|\nabla G\|_{L^\infty(B_K)} \leq L, \qquad \nu_G \geq \nu \text{ on } (0, +\infty),$$

together with (12)–(13).

By definition of ω, for every such minimizer u,

$$\mathrm{osc}_{B_r}\nabla u \leq \omega(r), \quad \forall r \in (0,1).$$

Observe that $0 \leq \omega \leq 2K$ and that ω is nondecreasing as the supremum of nondecreasing functions. The above arguments imply that for every $\varepsilon > 0$, one can find $\delta''_\varepsilon > 0$ such that $\omega(\delta''_\varepsilon) \leq \varepsilon$. It follows that $\lim_{r\to 0}\omega(r) = 0$. The proof is complete. ☐

3. PROOF OF THEOREM 1.1

Given $1 < p_1 \leq p_2 < \infty$, we consider the anisotropic orthotropic integrand

$$F(\xi_1, \xi_2) = \frac{1}{p_1}|\xi_1|^{p_1} + \frac{1}{p_2}|\xi_2|^{p_2},$$

and the associated functional on a bounded open set $\Omega \subset \mathbb{R}^2$:

$$\mathcal{F} : v \in W^{1,1}(\Omega) \mapsto \int_\Omega F(\nabla v(x))\, dx.$$

Let u a minimizer to $\mathcal{F}$ on Ω. We assume that u is Lipschitz on Ω. Let

$$M := \|\nabla u\|_{L^\infty(\Omega)}.$$

Construction of an approximating sequence for F. For every $\varepsilon \in [0,1]$, we introduce the smooth function

$$F_\varepsilon : (\xi_1, \xi_2) \in \mathbb{R}^2 \mapsto \frac{1}{p_1}(\varepsilon^2 + \xi_1^2)^{\frac{p_1}{2}} + \frac{1}{p_2}(\varepsilon^2 + \xi_2^2)^{\frac{p_2}{2}}.$$

We observe that $F = F_0 \leq F_\varepsilon \leq F_1$ on $\mathbb{R}^2$.

We modify F_ε in order to get an integrand which is quadratic outside a large ball. Here, we follow a strategy used in the proof of [5, Theorem 1.1, page 115].

Let $M' := \|F_1\|_{L^\infty(B_{M+2})}$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ a smooth nondecreasing function such that

$$\psi(t) = \begin{cases} t & \text{if } t \in [0, M'+1], \\ M'+2 & \text{if } t \in [M'+2, +\infty), \end{cases}$$

and $\|\psi'\|_{L^\infty(\mathbb{R})} \leq 2$.

For every $\varepsilon \in (0,1)$,

$$\nabla F_\varepsilon(\xi_1, \xi_2) = \left(\xi_1(\varepsilon^2 + \xi_1^2)^{\frac{p_1-2}{2}}, \xi_2(\varepsilon^2 + \xi_2^2)^{\frac{p_2-2}{2}}\right)$$

and thus

$$\begin{aligned} |\nabla F_\varepsilon(\xi_1, \xi_2)|^2 &= \xi_1^2(\varepsilon^2 + \xi_1^2)^{p_1-2} + \xi_2^2(\varepsilon^2 + \xi_2^2)^{p_2-2} \\ &\leq (\varepsilon^2 + \xi_1^2)^{p_1-1} + (\varepsilon^2 + \xi_2^2)^{p_2-1} \\ &\leq (1 + \xi_1^2)^{p_1-1} + (1 + \xi_2^2)^{p_2-1}. \end{aligned} \tag{14}$$

Hence,

$$|||\nabla F_\varepsilon(\xi)\otimes\nabla F_\varepsilon(\xi)|||\le(1+\xi_1^2)^{p_1-1}+(1+\xi_2^2)^{p_2-1}.$$

There exists $M''\ge M+2$ such that $\min_{\mathbb{R}^2\setminus B_{M''}}F\ge M'+2$. Let

$$\mu:=1+\|\psi''\|_{L^\infty(\mathbb{R})}\max_{|\xi|\le M''}\left((1+\xi_1^2)^{p_1-1}+(1+\xi_2^2)^{p_2-1}\right).$$

Since $\psi''(F_\varepsilon(\xi))=0$ when $|\xi|\ge M''$, one has

$$|||\psi''(F_\varepsilon(\xi))\nabla F_\varepsilon(\xi)\otimes\nabla F_\varepsilon(\xi)|||\le\mu-1,\qquad\forall\xi\in\mathbb{R}^2. \tag{15}$$

Let $\theta:\mathbb{R}^2\to\mathbb{R}^+$ be a smooth nonnegative convex function such that $\theta\equiv0$ on B_{M+1} and

$$\begin{aligned}\nabla^2\theta(\xi)\ &\ge\ \mu I, && \forall\xi\in\mathbb{R}^2\setminus B_{M+2},\\ \nabla^2\theta(\xi)\ &\le\ ((M+2)\mu+1)\,I, && \forall\xi\in\mathbb{R}^2.\end{aligned}$$

Finally, we set

$$G_\varepsilon=\psi\circ F_\varepsilon+\theta.$$

Then G_ε is a smooth function on $\mathbb{R}^2$ and

$$\nabla G_\varepsilon=\psi'\circ F_\varepsilon\nabla F_\varepsilon+\nabla\theta.$$

From (14) and the fact that $\|\psi'\|_{L^\infty(\mathbb{R})}\le2$, we deduce that

$$\|\nabla G_\varepsilon\|_{L^\infty(B_K)}\le2\sqrt{2}(1+K^2)^{\frac{p_2-1}{2}}+\|\nabla\theta\|_{L^\infty(B_K)},\qquad\forall K>0. \tag{16}$$

We also observe that G_ε is strictly convex on $\mathbb{R}^2$, as a consequence of the following lemma:

Lemma 3.1. *For every $\xi\in B_{M+1}$, $G_\varepsilon(\xi)=F_\varepsilon(\xi)$. Moreover,*

$$\nabla^2G_\varepsilon(\xi)\ge\begin{cases}\nabla^2F_\varepsilon(\xi) & \text{for } \xi\in B_{M+2},\\ I & \text{for } \xi\notin B_{M+2},\end{cases} \tag{17}$$

and

$$\nabla^2G_\varepsilon(\xi)\le\begin{cases}2\nabla^2F_\varepsilon(\xi)+(M+3)\mu I & \text{for } \xi\in B_{M''},\\ (M+3)\mu I & \text{for } \xi\notin B_{M''}.\end{cases} \tag{18}$$

Proof. For every $\xi \in B_{M+2}$, $F_\varepsilon(\xi) \le F_1(\xi) \le M'$ and thus $\psi \circ F_\varepsilon(\xi) = F_\varepsilon(\xi)$. This implies that $G_\varepsilon(\xi) = F_\varepsilon(\xi) + \theta(\xi)$. In view of the properties of θ, this gives $G_\varepsilon = F_\varepsilon$ on B_{M+1} and

$$\nabla^2 F_\varepsilon(\xi) \le \nabla^2 G_\varepsilon(\xi) \le \nabla^2 F_\varepsilon(\xi) + (M+3)\mu I, \qquad \forall \xi \in B_{M+2}.$$

Here, we have also used the fact that $\mu \ge 1$, so that $((M+2)\mu + 1)\, I \le (M+3)\mu I$.

When $\xi \notin B_{M''}$, $F_\varepsilon(\xi) \ge F(\xi) \ge M' + 2$, which implies that $\psi \circ F_\varepsilon(\xi) = M' + 2$. Hence,

$$G_\varepsilon(\xi) = M' + 2 + \theta(\xi).$$

It follows that

$$\mu I \le \nabla^2 G_\varepsilon(\xi) \le (M+3)\mu I, \quad \forall \xi \notin B_{M''}.$$

Finally, when $\xi \in B_{M''} \setminus B_{M+2}$, we write

$$\nabla^2 G_\varepsilon(\xi) = \psi'(F_\varepsilon(\xi))\nabla^2 F_\varepsilon(\xi) + \psi''(F_\varepsilon(\xi))\nabla F_\varepsilon(\xi) \otimes \nabla F_\varepsilon(\xi) + \nabla^2\theta(\xi).$$

In view of (15) and the properties of ψ' and θ, this implies

$$\nabla^2 G_\varepsilon(\xi) \le \|\psi'\|_{L^\infty(\mathbb{R})}\nabla^2 F_\varepsilon(\xi) + (\mu - 1)I + ((M+2)\mu + 1)\, I \le 2\nabla^2 F_\varepsilon(\xi) + (M+3)\mu I.$$

Moreover, relying on the fact that $\psi'(F_\varepsilon(\xi))\nabla^2 F_\varepsilon(\xi) \ge 0$, we also get

$$\nabla^2 G_\varepsilon(\xi) \ge \psi''(F_\varepsilon(\xi))\nabla F_\varepsilon(\xi) \otimes \nabla F_\varepsilon(\xi) + \nabla^2\theta(\xi).$$

Using now that $\nabla^2\theta(\xi) \ge \mu I$ together with (15), this gives

$$\nabla^2 G_\varepsilon(\xi) \ge I, \quad \forall \xi \in B_{M''} \setminus B_{M+2}.$$

The proof is complete.

□

We proceed to derive from the above lemma several uniform estimates on G_ε. By *uniform*, we mean that the constants involved do not depend on ε. Those constants depend on the exponents p_1 and p_2, but we will not explicitly mention this dependence.

First, Lemma 3.1 implies that G_ε has a quadratic growth outside a large ball:

Lemma 3.2. *There exist $\ell_M, L_M > 0$ which only depend on M such that for every $\varepsilon \in (0,1)$,*

$$\frac{1}{8}|\xi|^2 - \ell_M \leq G_\varepsilon(\xi) \leq L_M(1+|\xi|^2), \qquad \forall \xi \in \mathbb{R}^2. \tag{19}$$

Proof. Let us prove the lower bound in (19). By (17), $\nabla^2 G_\varepsilon(\xi) \geq I$ for every $|\xi| \geq M+2$. For such a $\xi \notin B_{M+2}$, let $t := (M+2)/|\xi|$ and $\xi' = t\xi$. Then

$$\begin{aligned} G_\varepsilon(\xi) &\geq G_\varepsilon(\xi') + \langle \nabla G_\varepsilon(\xi'), \xi - \xi' \rangle + \frac{1}{2}|\xi - \xi'|^2 \\ &= G_\varepsilon(\xi') + \frac{|\xi| - (M+2)}{M+2} \langle \nabla G_\varepsilon(\xi'), \xi' \rangle + \frac{1}{2}\left(1 - \frac{M+2}{|\xi|}\right)^2 |\xi|^2. \end{aligned}$$

By convexity of G_ε, $\langle \nabla G_\varepsilon(\xi'), \xi' \rangle \geq \langle \nabla G_\varepsilon(0), \xi' \rangle = 0$. Together with the fact that $G_\varepsilon(\xi') \geq 0$, this gives

$$G_\varepsilon(\xi) \geq \frac{1}{2}\left(1 - \frac{M+2}{|\xi|}\right)^2 |\xi|^2.$$

In particular, for every $|\xi| > 2(M+2)$,

$$G_\varepsilon(\xi) \geq \frac{1}{8}|\xi|^2.$$

Since G_ε only takes nonnegative values, this proves that the lower bound in (19) holds true with $\ell_M = (M+2)^2/2$.

We next prove the upper bound in (19). Since $\theta(0) = 0$ and $\nabla\theta(0) = 0$, one has

$$\theta(\xi) = \int_0^1 (1-t)\langle \nabla^2\theta(t\xi).\xi, \xi \rangle \, dt, \qquad \forall \xi \in \mathbb{R}^2.$$

Using that $\nabla^2\theta(\zeta) \leq ((M+2)\mu + 1)\, I$ for every $\zeta \in \mathbb{R}^2$, this implies

$$\theta(\xi) \leq \frac{1}{2}\left((M+2)\mu + 1\right) \leq (M+2)\mu|\xi|^2, \qquad \forall \xi \in \mathbb{R}^2.$$

In view of the definition of G_ε and the fact that $\psi \leq M' + 2$ on $\mathbb{R}$, one has

$$G_\varepsilon(\xi) \leq M' + 2 + \theta(\xi), \qquad \forall \xi \in \mathbb{R}^2.$$

The conclusion follows with $L_M := \max\left(M'+2, (M+2)\mu\right)$.

□

We also need a uniform estimate on the modulus of convexity of G_ε. For every $\varepsilon \in (0,1)$,

$$\nabla^2 F_\varepsilon(\xi_1,\xi_2) = \begin{pmatrix} \tau_{1,\varepsilon}(\xi_1) & 0 \\ 0 & \tau_{2,\varepsilon}(\xi_2) \end{pmatrix} \tag{20}$$

where for $i = 1,2$,

$$\tau_{i,\varepsilon}(\xi_i) = (\varepsilon^2 + \xi_i^2)^{\frac{p_i}{2}-2}(\varepsilon^2 + (p_i-1)\xi_i^2).$$

We observe that for every $K > 0$ and every[1] $\xi_i \in [-K,K]$,

$$(p_i-1)(1+K^2)^{\frac{p_i}{2}-1} \le \tau_{i,\varepsilon}(\xi_i) \le |\xi_i|^{p_i-2}, \qquad \text{if } p_i \le 2, \tag{21}$$

$$|\xi_i|^{p_i-2} \le \tau_{i,\varepsilon}(\xi_i) \le (p_i-1)(1+K^2)^{\frac{p_i}{2}-1}, \qquad \text{if } p_i \ge 2. \tag{22}$$

It follows that

$$\nabla^2 G_\varepsilon(\xi) \ge \begin{pmatrix} \mu_1(\xi_1) & 0 \\ 0 & \mu_2(\xi_2) \end{pmatrix}, \qquad \forall \xi \in \mathbb{R}^2, \tag{23}$$

where for $i = 1,2$:

$$\mu_i(\xi_i) = \begin{cases} (p_i-1)(1+(M+2)^2)^{\frac{p_i}{2}-1} & \text{if } p_i \le 2, \\ \min\left(1, |\xi_i|^{p_i-2}\right) & \text{if } p_i \ge 2. \end{cases}$$

Indeed, (23) is a consequence of (17) and (21)–(22) when $\xi \in B_{M+2}$. When $\xi \notin B_{M+2}$, we rely again on (17) and the fact that $\mu_i(\xi_i) \le 1$. This proves (23) in both cases.

Lemma 3.3. *Assume that $p_2 \ge 2$. Then there exists $\gamma_M > 0$ which only depends on M such that the modulus of convexity of G_ε satisfies the following estimate:*

$$\nu_{G_\varepsilon}(t) \ge \gamma_M t \min\left(1, t^{p_2-2}\right), \qquad \forall t \ge 0.$$

Proof. In view of (23), we can apply Lemma 4.2 to $G = G_\varepsilon$ with $\overline{\mu_i} = (p_i-1)(1+(M+2)^2)^{p_i/2-1}$, $i = 1,2$. By (32),

$$\langle \nabla G_\varepsilon(\xi) - \nabla G_\varepsilon(\xi'), \xi - \xi' \rangle \ge \gamma_M |\xi-\xi'|^2 \min\left(1, |\xi-\xi'|^{p_2-2}\right), \qquad \forall \xi, \xi' \in \mathbb{R}^2,$$

---

[1] If $\xi_i = 0$ and $p_i < 2$ in (21), then by $|\xi_i|^{p_i-2}$, we mean $+\infty$.

for some constant γ_M which only depends on M. Then by the Cauchy-Schwarz inequality, one gets

$$|\nabla G_\varepsilon(\xi) - \nabla G_\varepsilon(\xi')| \geq \gamma_M |\xi - \xi'| \min\left(1, |\xi - \xi'|^{p_2 - 2}\right).$$

Hence,

$$\nu_{G_\varepsilon}(t) \geq \gamma_M t \min\left(1, t^{p_2-2}\right), \qquad \forall t \geq 0.$$

$\square$

Construction of an approximating sequence for u. For every $\varepsilon \in (0,1)$, let u_ε be the minimum of

$$v \mapsto \int_\Omega G_\varepsilon(\nabla v)\, dx$$

on the set $W_0^{1,2}(\Omega) + u$.

Lemma 3.4. *For every $\varepsilon > 0$, u_ε is locally Lipschitz on Ω. Moreover, for every $a \in \Omega$ and $r > 0$ such that $B_{2r}(a) \Subset \Omega$,*

$$\sup_{\varepsilon \in (0,1)} \|\nabla u_\varepsilon\|_{L^\infty(B_r(a))} < +\infty.$$

Proof. By the upper bound in (19) and the fact that $\nabla^2 G_\varepsilon \geq I$ on $\mathbb{R}^2 \setminus B_{M''}$, we are in position to apply [8, Theorem 2.7], which yields the desired uniform estimate on the Lipschitz ranks of the u_ε's. More precisely, given $a \in \Omega$ and $r > 0$ such that $B_{2r}(a) \Subset \Omega$,

$$\|\nabla u_\varepsilon\|^2_{L^\infty(B_r(a))} \leq C_0 \left(1 + \frac{1}{r^2}\int_{B_{2r}(a)} |\nabla u_\varepsilon|^2\, dx\right), \tag{24}$$

where C_0 only depends on M (through L_M and M''). By (19),

$$|\nabla u_\varepsilon|^2 \leq 8 G_\varepsilon(\nabla u_\varepsilon) + 8\ell_M \quad \text{and} \quad G_\varepsilon(\nabla u) \leq L_M\left(1 + |\nabla u|^2\right).$$

Since u_ε is a minimizer for G_ε on Ω,

$$\int_\Omega G_\varepsilon(\nabla u_\varepsilon)\, dx \leq \int_\Omega G_\varepsilon(\nabla u)\, dx.$$

Hence,

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx \leq 8|\Omega|(\ell_M + L_M) + 8L_M \int_\Omega |\nabla u|^2 \leq 8|\Omega|\left(\ell_M + L_M(1 + M^2)\right).$$

Inserting this estimate into (24), one gets

$$\|\nabla u_\varepsilon\|_{L^\infty(B_r(a))} \leq K, \tag{25}$$

where K only depends on M, $|\Omega|$ and r. □

The fact that G_ε is smooth and satisfies $\nabla^2 G_\varepsilon > 0$ on $\mathbb{R}^2$ implies by De Giorgi's regularity theorem that u_ε is smooth on Ω.

Lemma 3.5. *For every $a \in \Omega$ and $r > 0$ such that $B_{2r}(a) \Subset \Omega$, there exists a function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{t\to 0}\omega(t) = 0$ and for every $\varepsilon > 0$,*

$$\operatorname{osc}_{B_s(a)}\nabla u_\varepsilon \leq \omega(s), \qquad \forall s \in \left(0, \frac{r}{2}\right).$$

The function ω may depend on M, r and $|\Omega|$ but not on ε.

Proof. Let $a \in \Omega$, $r > 0$ such that $B_{2r}(a) \Subset \Omega$ and $v_\varepsilon(x) := \frac{1}{r}u_\varepsilon(a + rx), x \in B_1$. Then v_ε is a minimizer for the functional

$$\mathcal{G}_\varepsilon(v) = \int_{B_1} G_\varepsilon(\nabla v)\, dx.$$

Moreover, $\|\nabla v_\varepsilon\|_{L^\infty(B_1)} = \|\nabla u_\varepsilon\|_{L^\infty(B_r(a))}$. By Lemma 3.4, there exists $K > 0$ which does not depend on ε such that

$$\|\nabla v_\varepsilon\|_{L^\infty(B_1)} \leq K.$$

Case 1. If $p_1 \geq 2$, then by (20) and (22),

$$\nabla^2 F_\varepsilon(\xi) \leq (p_2 - 1)(1 + M''^2)^{\frac{p_2}{2}-1} I, \qquad \forall \xi \in B_{M''}.$$

Hence by (18),

$$\nabla^2 G_\varepsilon(\xi) \leq \Lambda I, \qquad \forall \xi \in \mathbb{R}^2,$$

where $\Lambda = \big(2(p_2 - 1)(1 + M''^2)^{p_2/2-1} + (M + 3)\mu\big)$. Then Lemma 2.4 implies that

$$\nu_{G_\varepsilon}(\operatorname{osc}_{B_\rho}\nabla v_\varepsilon) \leq \frac{C\Lambda}{\sqrt{-\ln(2\rho)}}, \qquad \forall \rho \in \left(0, \frac{1}{2}\right), \tag{26}$$

where $C > 0$ only depends on K. In view of Lemma 3.3, $\nu_{G_\varepsilon}(t) \geq \gamma_M t \min(1, t^{p_2-2})$. We also observe that there exists $\delta_K > 0$ which only depends on K such that

$$t \min\left(1, t^{p_2-2}\right) \geq \delta_K t^{p_2-1}, \qquad \forall t \in [0, 2K].$$

Since $\operatorname{osc}_{B_\rho}\nabla v_\varepsilon \in [0,2K]$, it follows from (26) that

$$\operatorname{osc}_{B_\rho}\nabla v_\varepsilon \leq \left(\frac{C\Lambda}{\gamma_M \delta_K \sqrt{-\ln(2\rho)}}\right)^{\frac{1}{p_2-1}}, \qquad \forall \rho \in \left(0, \frac{1}{2}\right).$$

This implies that on every ball $B_s(a)$, with $s \in (0, r/2)$, the map u_ε satisfies

$$\operatorname{osc}_{B_s}\nabla u_\varepsilon \leq \frac{C''}{\left(\ln \frac{r}{2s}\right)^{\frac{1}{2(p_2-1)}}},$$

for some $C'' > 0$ which only depends on M and K.

Case 2. If $p_2 \leq 2$, then by (23),

$$\nabla^2 G_\varepsilon(\xi) \geq \lambda I, \qquad \forall \xi \in \mathbb{R}^2,$$

where $\lambda = (p_1-1)(1+(M+2)^2)^{p_1/2-1} I$. Then (16) and Lemma 2.3 imply that

$$\operatorname{osc}_{B_\rho}\nabla v_\varepsilon \leq \frac{C}{\sqrt{-\ln 2\rho}}, \qquad \forall \rho \in \left(0, \frac{1}{2}\right),$$

where $C > 0$ only depends on M and K. Hence,

$$\operatorname{osc}_{B_s(a)}\nabla u_\varepsilon \leq \frac{C}{\sqrt{\ln \frac{r}{2s}}}, \qquad \forall s \in \left(0, \frac{r}{2}\right).$$

Case 3. If $p_1 \leq 2 \leq p_2$, then for every $K > 0$ and every $\delta > 0$, (21)–(22) imply that

$$\tau_{1,\varepsilon}(\xi_1) \leq \delta^{p_1-2}, \quad \tau_{2,\varepsilon}(\xi_2) \leq (p_2-1)(1+K^2)^{\frac{p_2}{2}-1}, \qquad \forall (\xi_1,\xi_2) \in (H^+_{e_1}(\delta) \cup H^+_{-e_1}(\delta)) \cap B_K,$$

$$\tau_{1,\varepsilon}(\xi_1) \geq (p_1-1)(1+K^2)^{\frac{p_1}{2}-1}, \quad \tau_{2,\varepsilon}(\xi_2) \geq \delta^{p_2-2}, \qquad \forall (\xi_1,\xi_2) \in (H^+_{e_2}(\delta) \cup H^+_{-e_2}(\delta)) \cap B_K.$$

Let us introduce

$$\overline{\lambda_\delta} := \min\left((p_1-1)(1+K^2)^{p_1/2-1}, \delta^{p_2-2}\right) \text{ and } \overline{\Lambda_\delta} := \max\left((p_2-1)(1+K^2)^{p_2/2-1}, \delta^{p_1-2}\right).$$

We deduce therefrom that

$$\nabla^2 F_\varepsilon(\xi) \leq \overline{\Lambda_\delta} I, \qquad \forall \xi \in (H^+_{e_1}(\delta) \cup H^+_{-e_1}(\delta)) \cap B_K,$$

$$\nabla^2 F_\varepsilon(\xi) \geq \overline{\lambda_\delta} I, \qquad \forall \xi \in (H^+_{e_2}(\delta) \cup H^+_{-e_2}(\delta)) \cap B_K.$$

Hence, using (18) and (17), this yields

$$\nabla^2 G_\varepsilon(\xi) \leq \left(2\overline{\Lambda_\delta} + (M+3)\mu\right) I, \qquad \forall \xi \in \left(H^+_{e_1}(\delta) \cup H^+_{-e_1}(\delta)\right) \cap B_K,$$

$$\nabla^2 G_\varepsilon(\xi) \geq \min\left(\overline{\lambda_\delta}, 1\right) I, \qquad \forall \xi \in \left(H^+_{e_2}(\delta) \cup H_{-e_2}(\delta)^+\right) \cap B_K.$$

By Lemma 3.3, $\nu_{G_\varepsilon}(t) \geq \gamma_M t \min(1, t^{p_2-2})$ for every $t \in \mathbb{R}^+$, where $\gamma_M > 0$ only depends on M. We can apply Lemma 2.5 with the parameters K, $L = 2\sqrt{2}(1 + K^2)^{(p_2-1)/2} + \|\nabla\theta\|_{L^\infty(B_K)}$ (see (16)), the function $\nu = \gamma_M t \min(1, t^{p_2-2})$ and the families $\lambda_\delta := \min\left(\overline{\lambda_\delta}, 1\right)$, $\Lambda_\delta := 2\overline{\Lambda_\delta} + (M+3)\mu$.

Then there exists a function $\omega : \mathbb{R}^+ \to \mathbb{R}^+$, which only depends on K, L, ν and the families $\{\lambda_\delta\}_{\delta>0}$ and $\{\Lambda_\delta\}_{\delta>0}$, such that $\lim_{\rho\to 0} \omega(\rho) = 0$ and

$$\operatorname{osc}_{B_\rho} \nabla v_\varepsilon \leq \omega(\rho), \qquad \forall \rho \in (0,1).$$

It follows that

$$\operatorname{osc}_{B_s(a)} \nabla u_\varepsilon \leq \omega\left(\frac{s}{r}\right), \qquad \forall s \in (0,r).$$

The proof is complete.

□

Remark 3.1. *When $p_1 \geq 2$ (case 1) or $p_2 \leq 2$ (case 2), the above proof yields an explicit modulus of continuity for ∇u_ε. More precisely, for every $B_r(a) \subset B_{2r}(a) \Subset \Omega$, one can take*

$$\omega(s) = \frac{C}{\left(\ln \frac{r}{2s}\right)^{\frac{1}{2\max(p_2-1,1)}}}, \qquad \forall s \in \left(0, \frac{r}{2}\right),$$

where $C > 0$ only depends on $M = \|\nabla u\|_{L^\infty(\Omega)}$ and on any number $K \geq \sup_\varepsilon \|\nabla u_\varepsilon\|_{L^\infty(B_r(a))}$ (the existence of such a K is given by Lemma 3.4).

3.1. Completion of the proof.

Since u_ε is a minimizer for G_ε, one gets

$$\int_\Omega G_\varepsilon(\nabla u_\varepsilon)\, dx \leq \int_\Omega G_\varepsilon(\nabla u)\, dx.$$

It follows from (19) that the family $\{u_\varepsilon\}_{\varepsilon>0}$ is bounded in $W_0^{1,2}(\Omega) + u$. Hence, there exists a sequence $(\varepsilon_k)_{k\geq 1}$ converging to 0 such that $(u_{\varepsilon_k})_{k\geq 1}$ weakly converges in $W^{1,2}(\Omega)$ to some $v \in W_0^{1,2}(\Omega) + u$.

By (25), Lemma 3.5 and the Arzela-Ascoli theorem, $v \in C^1(\Omega)$ and up to a subsequence (we do not relabel), $(u_{\varepsilon_k})_{k\geq 1}$ converges to v in $C^1(\Omega)$. In particular, for every $a \in \Omega$ and $r > 0$ such that $B_{2r}(a) \Subset \Omega$, v satisfies

$$\operatorname{osc}_{B_s(a)} \nabla v \leq \omega(s), \qquad \forall s \in \left(0, \frac{r}{2}\right), \tag{27}$$

where ω is the function given by Lemma 3.5.

It remains to prove that

Lemma 3.6. *The map v agrees with u on Ω.*

Proof. Since $\|\nabla u\|_{L^\infty(\Omega)} \leq M$ and $F = G_\varepsilon$ on B_{M+1},

$$\int_\Omega F(\nabla u)\,dx = \int_\Omega G_\varepsilon(\nabla u)\,dx.$$

Using the fact that u_ε is a minimum for G_ε on $W_0^{1,2}(\Omega) + u$, one gets

$$\int_\Omega F(\nabla u)\,dx \geq \int_\Omega G_\varepsilon(\nabla u_\varepsilon)\,dx \geq \int_\Omega \psi \circ F(\nabla u_\varepsilon) + \theta(\nabla u_\varepsilon)\,dx.$$

The last inequality relies on the estimate $F_\varepsilon \geq F$ and the fact that ψ is nondecreasing.

Let $\Omega' \Subset \Omega$. Since ψ and θ are nonnegative, this implies that

$$\int_\Omega F(\nabla u)\,dx \geq \int_{\Omega'} \psi \circ F(\nabla u_\varepsilon) + \theta(\nabla u_\varepsilon)\,dx.$$

Since $(u_{\varepsilon_k})_{k\geq 1}$ converges to v in $C^1(\overline{\Omega'})$, one can let $k \to +\infty$ to get

$$\int_\Omega F(\nabla u)\,dx \geq \int_{\Omega'} \psi \circ F(\nabla v) + \theta(\nabla v)\,dx.$$

The above inequality being true for every $\Omega' \Subset \Omega$, the monotone convergence theorem implies that

$$\int_\Omega F(\nabla u)\,dx \geq \int_\Omega \psi \circ F(\nabla v) + \theta(\nabla v)\,dx. \tag{28}$$

Let us introduce the functional

$$\widetilde{\mathcal{F}}(w) = \int_\Omega (\psi \circ F + \theta)(\nabla w)\,dx, \qquad w \in W^{1,1}(\Omega).$$

Since the sequence of the convex functions G_{ε_k} converges pointwisely to the function $\psi \circ F + \theta$, we deduce that the latter is convex as well. Actually, (23) and Lemma 4.2 imply that

$$\langle \nabla G_\varepsilon(\xi) - \nabla G_\varepsilon(\xi'), \xi - \xi' \rangle \geq C|\xi - \xi'|^2 \min\left(1, |\xi - \xi'|^{p_2-2}\right), \qquad \forall \xi, \xi' \in \mathbb{R}^2,$$

for some constant $C > 0$ which only depends on M. We also observe that $(\nabla F_{\varepsilon_k})_{k\geq 1}$ converges pointwisely to ∇F. It follows that $(\nabla G_{\varepsilon_k})_{k\geq 1}$ converges pointwisely to $\nabla(\psi \circ F + \theta)$ and thus

$$\langle \nabla(\psi\circ F+\theta)(\xi)-\nabla(\psi\circ F+\theta)(\xi'),\xi-\xi'\rangle \geq C|\xi-\xi'|^2 \min\left(1,|\xi-\xi'|^{p_2-2}\right), \qquad \forall \xi,\xi' \in \mathbb{R}^2.$$

We deduce therefrom that $\psi \circ F + \theta$ is strictly convex on $\mathbb{R}^2$.

Next, we claim that u is a minimizer for $\widetilde{\mathcal{F}}$. Indeed, since u is a Lipschitz minimizer for $\mathcal{F}$ and F is at least C^1, one has

$$\int_\Omega \langle \nabla F(\nabla u), \nabla w\rangle \, dx = 0, \qquad \forall w \in C^\infty_c(\Omega).$$

Since $M = \|\nabla u\|_{L^\infty(\Omega)}$ and $F \equiv \psi \circ F + \theta$ on B_{M+1},

$$\nabla F(\nabla u) = \nabla(\psi \circ F + \theta)(\nabla u), \qquad \text{a.e. on } \Omega,$$

and thus

$$\int_\Omega \langle \nabla(\psi \circ F + \theta)(\nabla u), \nabla w\rangle \, dx = 0, \qquad \forall w \in C^\infty_c(\Omega).$$

Since $\nabla(\psi \circ F + \theta)(\nabla u) \in L^\infty(\Omega)$, the above identity remains true for any $w \in W^{1,1}_0(\Omega)$. By convexity of $\psi \circ F + \theta$, we deduce therefrom that

$$\int_\Omega (\psi \circ F + \theta)(\nabla u)\, dx \leq \int_\Omega (\psi \circ F + \theta)(\nabla(u+w))\, dx, \qquad \forall w \in W^{1,1}_0(\Omega).$$

Hence, u is a minimizer for $\widetilde{\mathcal{F}}$ on $W^{1,1}_0(\Omega)$. It follows from (28) that

$$\widetilde{\mathcal{F}}(u) = \widetilde{\mathcal{F}}(v).$$

Since $\widetilde{F}$ is strictly convex, the minimum of $\widetilde{\mathcal{F}}$ on $W^{1,2}_0(\Omega) + u$ is unique. This implies that $v = u$ as desired.

□

This completes the proof of the fact that u is C^1 on Ω.

Remark 3.2. *In the cases when $p_1 \geq 2$ or $p_2 \leq 2$, the above proof also yields an explicit modulus of continuity for ∇u. More precisely, by Remark 3.1 and* (27)*, for every $a \in \Omega$*

and $r > 0$ *such that* $B_{2r}(a) \Subset \Omega$,

$$\mathrm{osc}_{B_s(a)} \nabla u \leq \frac{C}{\left(\ln \frac{r}{2s}\right)^{\frac{1}{2\max(1,p_2-1)}}}, \qquad \forall s \in \left(0, \frac{r}{2}\right), \tag{29}$$

where $C > 0$ *only depends on* $M = \|\nabla u\|_{L^\infty(\Omega)}$ *and any number* K *that satisfies* $K \geq \sup_\varepsilon \|\nabla u_\varepsilon\|_{L^\infty(B_r(a))}$. *Since* $(u_{\varepsilon_k})_{k\geq 1}$ *converges to* u *in* $C^1(\Omega)$, *there exists* $k_0 \geq 1$ *such that* $\sup_{k\geq k_0} \|\nabla u_{\varepsilon_k}\|_{L^\infty(B_r(a))} \leq M + 1$. *Hence, in all the calculations above, one can take* $K = M + 1$ *(up to a new extraction if necessary). In particular, the constant* $C > 0$ *in* (29) *can be chosen* a posteriori *as a function of* M *only.*

4. Appendix

We first justify the well-known fact that in the two dimensional case, a minimizer is continuous.

Let $H : \mathbb{R}^2 \to \mathbb{R}^+$ be a nonnegative strictly convex function. We assume that H is superlinear, in the sense that $\lim_{|\xi|\to+\infty} H(\xi)/|\xi| = +\infty$. Given a bounded open set $\Omega \subset \mathbb{R}^2$, we consider the functional

$$\mathcal{H} : v \mapsto \int_\Omega H(\nabla v)\, dx, \qquad v \in W^{1,1}(\Omega).$$

Lemma 4.1. *Let* u *be the minimizer to* $\mathcal{H}$. *Then* $u \in C^0(\Omega)$.

Proof. Let $a \in \Omega$. Since $u \in W^{1,1}(\Omega)$, for a.e. $r > 0$ such that $B_r(a) \Subset \Omega$, the restriction $\varphi := u|_{\partial B_r(a)}$ is in $W^{1,1}(\partial B_r(a))$. By the Morrey embedding, this implies that φ is continuous. Since $u|_{B_r(a)}$ is a minimizer to $\mathcal{H}$ on $B_r(a)$ with respect to the boundary condition given by φ, we deduce that u is continuous on $\overline{B_r(a)}$: this can be seen as in the proof of [11, Theorem 7.1], see also [1, Corollary 1.5] for a more general result. It follows that u is continuous on Ω. $\square$

Given two positive numbers $\overline{\mu_1}, \overline{\mu_2}$, we define for $i = 1, 2$

$$\mu_i(t) := \begin{cases} \overline{\mu_i} & \text{if } p_i \leq 2, \\ \min\left(1, |t|^{p_i-2}\right) & \text{if } p_i \geq 2, \end{cases} \qquad \forall t \in \mathbb{R}.$$

Lemma 4.2. *Let* $G:\mathbb{R}^2\to\mathbb{R}$ *be a smooth function such that*

$$\nabla^2 G(\xi_1,\xi_2)\geq\begin{pmatrix}\mu_1(\xi_1) & 0\\ 0 & \mu_2(\xi_2)\end{pmatrix},\qquad \forall(\xi_1,\xi_2)\in\mathbb{R}^2. \tag{30}$$

Then there exists a constant $C>0$ *which only depends on* $\overline{\mu_1},\overline{\mu_2}$ *such that for every* $\xi,\xi'\in\mathbb{R}^2$*,*

$$\langle\nabla G(\xi)-\nabla G(\xi'),\xi-\xi'\rangle\geq C|\xi-\xi'|^2,\qquad \textit{if } p_2\leq 2, \tag{31}$$

$$\langle\nabla G(\xi)-\nabla G(\xi'),\xi-\xi'\rangle\geq C|\xi-\xi'|^2\min\left(1,|\xi-\xi'|^{p_2-2}\right),\qquad \textit{if } p_2\geq 2. \tag{32}$$

Proof. Let $\xi,\xi'\in\mathbb{R}^2$ with $\xi\neq\xi'$. Then

$$\langle\nabla G(\xi)-\nabla G(\xi'),\xi-\xi'\rangle=\int_0^1\langle\nabla^2 G(\xi'+t(\xi-\xi')).(\xi-\xi'),\xi-\xi'\rangle\,dt.$$

Using (30), we thus obtain

$$\langle\nabla G(\xi)-\nabla G(\xi'),\xi-\xi'\rangle\geq\int_0^1\mu_1(\xi_1'+t(\xi_1-\xi_1'))(\xi_1-\xi_1')^2+\mu_2(\xi_2'+t(\xi_2-\xi_2'))(\xi_2-\xi_2')^2\,dt.$$

Let $i\in\{1,2\}$ such that $|\xi_i-\xi_i'|=\max(|\xi_1-\xi_1'|,|\xi_2-\xi_2'|)$. Then

$$\begin{aligned}\langle\nabla G(\xi)-\nabla G(\xi'),\xi-\xi'\rangle&\geq(\xi_i-\xi_i')^2\int_0^1\mu_i(\xi_i'+t(\xi_i-\xi_i'))\,dt\\ &\geq\frac{1}{2}|\xi-\xi'|^2\int_0^1\mu_i(\xi_i'+t(\xi_i-\xi_i'))\,dt.\end{aligned} \tag{33}$$

We first consider the case when $p_i\leq 2$. Then $\mu_i(\xi_i'+t|\xi_i-\xi_i'|)\geq\overline{\mu_i}$ for every $t\in[0,1]$, and thus

$$\langle\nabla G(\xi)-\nabla G(\xi'),\xi-\xi'\rangle\geq\frac{\overline{\mu_i}}{2}|\xi-\xi'|^2.$$

If $p_2\leq 2$, then (31) follows at once. If $p_2>2$, then one uses that $\min\left(1,|\xi-\xi'|^{p_2-2}\right)\leq 1$ to get (32).

We next consider the case when $p_i>2$ (and thus necessarily $p_2>2$). We claim that

$$\int_0^1\mu_i(\xi_i'+t(\xi_i-\xi_i'))\,dt\geq C\min\left(1,|\xi-\xi'|^{p_2-2}\right), \tag{34}$$

for some $C>0$ which only depends on p_1,p_2.

Indeed, if $|\xi_i-\xi_i'|\leq 2|\xi_i'|$, then for every $t\in[0,1/4]$,

$$|\xi_i'+t(\xi_i-\xi_i')|\geq|\xi_i'|-\frac{1}{4}|\xi_i-\xi_i'|\geq\frac{1}{4}|\xi_i-\xi_i'|$$

and thus

$$\int_0^1 \mu_i(\xi_i' + t(\xi_i - \xi_i'))\,dt \geq \frac{1}{4}\min\left(1, \frac{1}{4^{p_i-2}}|\xi_i - \xi_i'|^{p_i-2}\right) \geq \frac{1}{4}\min\left(1, \frac{1}{(\sqrt{24})^{p_i-2}}|\xi - \xi'|^{p_i-2}\right)$$
$$\geq \frac{1}{4^{p_i-1}(\sqrt{2})^{p_i-2}}\min\left(1, |\xi - \xi'|^{p_i-2}\right).$$

Since $p_2 \geq p_i \geq 2$, the claim (34) follows in that case.

If instead $|\xi_i - \xi_i'| \geq 2|\xi_i'|$, then for every $t \in [3/4, 1]$,

$$|\xi_i' + t(\xi_i - \xi_i')| \geq \frac{3}{4}|\xi_i - \xi_i'| - |\xi_i'| \geq \frac{1}{4}|\xi_i - \xi_i'|,$$

which implies (34), by a similar calculation. Inserting (34) into (33), one eventually gets (32).

□

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