REGULARITY FOR QUASILINEAR PDES IN CARNOT GROUPS VIA RIEMANNIAN APPROXIMATION
REGOLARITÀ PER PDE QUASILINEARI NEI GRUPPI DI CARNOT TRAMITE APPROSSIMAZIONI RIEMANNIANE

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Abstract. We study the interior regularity of weak solutions to subelliptic quasilinear PDEs in Carnot groups of the form

$$\sum_{i=1}^{m_1} X_i \left(\Phi(|\nabla_{\mathcal{H}} u|^2) X_i u\right) = 0.$$

Here $\nabla_{\mathcal{H}} u = (X_1 u, \ldots, X_{m_1} u)$ is the horizontal gradient, $\delta > 0$ and the exponent $p \in [2, p^*)$, where p^* depends on the step ν and the homogeneous dimension Q of the group, and it is given by

$$p^* = \min\left\{\frac{2\nu}{\nu - 1}\ ,\ \frac{2Q+8}{Q-2}\right\}.$$

Sunto. Studiamo la regolarità interna delle soluzioni deboli di EDP, quasilineari subellittiche in gruppi di Carnot, della forma

$$\sum_{i=1}^{m_1} X_i \left(\Phi(|\nabla_{\mathcal{H}} u|^2) X_i u\right) = 0.$$

Qui $\nabla_{\mathcal{H}} u = (X_1 u, \ldots, X_{m_1} u)$ è il gradiente orizzontale, $\delta > 0$ e l'esponente $p \in [2, p^*)$, dove p^* dipende dal passo ν e dalla dimensione omogenea Q del gruppo ed è dato da

$$p^* = \min\left\{\frac{2\nu}{\nu - 1}\ ,\ \frac{2Q+8}{Q-2}\right\}.$$

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1. Introduction

In this article we consider C^∞-smoothness of weak solutions to equations of non-degenerate p-Laplacian type in Carnot groups of arbitrary step. The basic equation is

$$\Delta_{\mathcal{H},p} u(x) = \sum_{i=1}^{m_1} X_i \left((\delta^2 + |\nabla_{\mathcal{H}} u|^2)^{\frac{p-2}{2}} X_i u \right) = 0,$$

where $\nabla_{\mathcal{H}} u = (X_1 u, \ldots, X_{m_1} u)$ is the horizontal gradient and $p \in [2, p^*)$, where p^* depends on the step ν and the homogeneous dimension Q and it is given by

$$p^* = \min \left\{ \frac{2\nu}{\nu - 1} \, , \, \frac{2Q+8}{Q-2} \right\}.$$

Non-degenerate refers to the fact that $\delta > 0$.

The C^∞-regularity is well-known in the linear case when $p = 2$, [Hör67]. For groups that admit a Hilbert-Haar coordinate system, one can generalize a method of Miranda [Mir65] to established Lipschitz bounds for solutions of the Dirichlet problem with smooth boundary data. For the case of the Heisenberg group this fact was first noted by Zhong [Zho17], and extended by us to the Hilbert-Haar case, [DM20] also presented at this conference. Boundedness of the horizontal gradients is the key step, from which we get higher regularity, see [Cap99] and [DM09]. We also refer to [DM20] for a more detailed history of the problem. The purpose of this note is to remove the Hilbert-Haar condition, in order to prove the regularity result in arbitrary Carnot groups.

Our new ingredient is the method of Riemannian approximations from Capogna-Citti [CC16]. In addition to removing the Hilbert-Haar condition that requires a variational structure, this method allows to treat some non-variational problems

$$\sum_{i=1}^{m_1} X_i \left(a_i(\nabla_{\mathcal{H}} u) \right) = 0, \text{ in } \Omega \,, \tag{1.1}$$

where $\Omega \subset \mathbb{G}$ is a domain and the coefficients $a(\xi) = (a_1(\xi), a_2(\xi), \ldots, a_{m_1}(\xi))$ satisfy the Uhlenbeck structure condition

$$a(\xi) = \Phi(|\xi|^2)\xi,$$

where $\Phi : [0,\infty) \to [0,\infty)$ is an increasing and smooth function such that the ellipticity condition

$$\frac{1}{M}\left(\delta^2+t\right)^{\frac{p}{2}-1} \le \Phi(t)+2t\Phi'(t) \le M\left(\delta^2+t\right)^{\frac{p}{2}-1} \tag{1.2}$$

holds. See [Uhl77].

It is easy to see that (1.2) implies that there exists $L > 0$ such that the following properties hold for all $\xi, \eta \in \mathbb{R}^{m_1}$:

$$\sum_{i,j=1}^{m_1} \frac{\partial a_i}{\partial \xi_j}(\xi)\ \eta_i\eta_j \ge L\Big(\delta^2+|\xi|^2\Big)^{\frac{p-2}{2}}|\eta|^2\,, \tag{1.3}$$

$$\sum_{i,j=1}^{m_1} \left|\frac{\partial a_i}{\partial \xi_j}(\xi)\right| \le L^{-1}\Big(\delta^2+|\xi|^2\Big)^{\frac{p-2}{2}}\,, \tag{1.4}$$

$$|a_i(\xi)| \le L^{-1}\left(\delta^2+|\xi|^2\right)^{\frac{p-1}{2}}\,. \tag{1.5}$$

Also, note that property (1.3) implies that

$$\sum_{i=1}^{m_1} a_i(\xi)\xi_i \ge \frac{L}{p-1}\Big(\delta^2+|\xi|^2\Big)^{\frac{p-2}{2}}|\xi|^2\,. \tag{1.6}$$

Consider the following Sobolev space adapted to the horizontal system of vector fields:

$$W^{1,p}_{\mathcal{H}}(\Omega) = \Big\{u \in L^p(\Omega)\ :\ X_iu \in L^p(\Omega)\,,\ \text{for all } 1 \le i \le m_1\Big\}.$$

A function $u \in W^{1,p}_{\mathcal{H}}(\Omega)$ is a weak solution of the equation (1.1) if

$$\sum_{i=1}^{m_1} \int_\Omega a_i(\nabla_{\mathcal{H}}u(x))\ X_i\phi(x)dx = 0\,,\ \text{for all } \phi \in C_0^\infty(\Omega)\,. \tag{1.7}$$

Our main result is the following.

Theorem 1.1. *Let G be a Carnot group of step ν and homogeneous dimension Q. For values of p satisfying*

$$2 \le p < \min\left\{\frac{2\nu}{\nu-1}\,,\ \frac{2Q+8}{Q-2}\right\}\,,$$

any weak solution $u \in W^{1,p}_{\mathcal{H}}(\Omega)$ to (1.1), with $\delta > 0$, belongs to $C^\infty(\Omega)$.

Given the technical nature of regularity proofs, we will provide full details of the proof for the special case of Goursat groups. The general case is treated similarly, adding only technical modifications that would have made this manuscript longer than what it already is.

The structure of the paper is as follows. In Section §2 we provide background on Carnot groups and Riemannian approximations. We establish integral estimates of Caccioppoli and Gagliardo-Nirenberg type in Section §3. The core of the paper is in Section §4, where we establish the boundedness of the horizontal gradient. The condition $\delta > 0$ is needed for the Moser iteration in Lemma 4.5. The results of §3 and Lemma 4.3 (difference quotient estimates) and Lemma 4.4 (higher integrability for the vertical derivatives) hold uniformly in δ.

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2. Carnot Groups and Riemannian Approximations

Consider a Carnot group $(\mathbb{G}, \cdot) = (\mathbb{R}^n, \cdot)$ and a system of left invariant horizontal vector fields $\{X_1, \ldots, X_{m_1}\}$, $m_1 < n$, which generates the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$. We assume that $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = \bigoplus_{s=1}^{\nu} V^s \,, \tag{2.1}$$

where $\nu \in \mathbb{N}$, $\nu \geq 2$ and

$$(i) \quad \{X_1, ..., X_{m_1}\} \text{ is a basis of } V^1, \tag{2.2}$$

$$(ii) \quad [V^1, V^s] = V^{s+1} \text{ if } \; s \leq \nu - 1, \tag{2.3}$$

$$(iii) \quad [V^1, V^\nu] = \{0\} \,. \tag{2.4}$$

Let us denote $\dim V^s = m_s$ for all $1 \leq s \leq \nu$. The homogeneous dimension of $\mathbb{G}$ is defined as $Q = \sum_{s=1}^{\nu} s\, m_s$.

Definition 2.1. *We say that a Carnot group $\mathbb{G}$ of step ν defined on $\mathbb{R}^{\nu+1}$ is a Goursat group if it admits a system of horizontal vector fields $\{X_1, X_2\}$ and the only non-zero* *commutators are*

$$[X_1, X_i] = X_{i+1},\ 2 \le i \le \nu\,. \tag{2.5}$$

Consider an arbitrary, but fixed $0 < \varepsilon < 1$. Define the following vector fields:

- For $i \in \{1, 2\}$, define $X_i^\varepsilon = X_i$.
- For $i \in \{3, \dots, \nu + 1\}$, define $X_i^\varepsilon = \varepsilon^{i-2} X_i$.

Formula (2.5) implies the following commutator relations:

$$[X_1^\varepsilon, X_i^\varepsilon] = \frac{1}{\varepsilon} X_{i+1}^\varepsilon,\ 2 \le i \le \nu. \tag{2.6}$$

In the process of the Riemannian approximation the following gradients will be used:

- $\nabla_{\mathcal{H}} = (X_1, X_2)$, the horizontal or sub-Riemannian gradient,
- $\nabla_{\mathcal{V}} = (X_3, ..., X_{\nu+1})$, the vertical gradient,
- $\nabla^\varepsilon = (X_1^\varepsilon, ..., X_{\nu+1}^\varepsilon) = (X_1, X_2, \varepsilon X_3, \dots, \varepsilon^{\nu-1} X_{\nu+1})$, the Riemannian gradient.

We define a Riemannian metric g_ε in $\mathfrak{g}$ by declaring $\{X_1^\varepsilon \dots, X_{\nu+1}^\varepsilon\}$ to be an orthonormal basis. We also note that the adjoint of the vector field X_i with respect to the Haar measure is $X_i^* = -X_i$.

Our equation (1.1) is now

$$X_1 \left(\Phi(|\nabla_{\mathcal{H}} u|^2) X_1 u\right) + X_2 \left(\Phi(|\nabla_{\mathcal{H}} u|^2) X_2 u\right) = 0. \tag{2.7}$$

We approximate a local weak solution u of this equation by the solutions u_ε of the quasi-linear elliptic PDE

$$\sum_{i=1}^{\nu+1} X_i^\varepsilon \left(a_i(\nabla^\varepsilon u)\right) = 0\,,\ \text{in } \Omega\,, \tag{2.8}$$

where

$$\begin{aligned}\sum_{i=1}^{\nu+1} X_i^\varepsilon \left(a_i(\nabla^\varepsilon u)\right) &= \sum_{i=1}^{\nu+1} X_i^\varepsilon \left(\Phi(|\nabla^\varepsilon u|^2) X_i^\varepsilon u\right) \\ &= X_1 \left(\Phi(|\nabla^\varepsilon u|^2) X_1 u\right) + X_2 \left(\Phi(|\nabla^\varepsilon u|^2) X_2 u\right) \\ &+ \sum_{i=3}^{\nu+1} \varepsilon^{2i-4} X_i \left(\Phi(|\nabla^\varepsilon u|^2) X_i u\right).\end{aligned}$$

We have extended the functions $a_i(\xi) = \Phi(|\xi|^2)\xi_i$ from the horizontal layer of dimension $m_1 = 2$ to all vectors $\xi \in \mathfrak{g}$. Note that $a_i(\xi)$ does not depend on ε because of our choice of orthonormal basis.

Since this equation is not degenerate and we are in the elliptic (Riemannian) case, classical regularity theory applies and weak solutions u_ε of the non-degenerate quasilinear elliptic equation (2.8) are smooth in Ω. See for example [LU68]. The task at hand is to find estimates for u_ε that are independent of ε, so that they also apply to u.

We denote by $B_r^\varepsilon(x)$ and $B_r(x)$ respectively, the Riemannian and sub-Riemannian balls centered at x with radius r. We often omit the center x when it is clear from the context.

The first inequality independent of ϵ that we need is the doubling property for concentric balls

$$|B_{2r}^\varepsilon| \leq C|B_r^\varepsilon|, \tag{2.9}$$

for $0 \leq r \leq 1$ with a constant independent of ε. See [CCR13, CC16] where the techniques of [NSW85] are extended to prove (2.9).

The second inequality independent of ϵ that we need, is the Poincaré inequality. It follows from [Jer86] and the doubling property (2.9). See [CC16] for a detailed discussion of Riemannian approximations.

3. Integral Estimates

The following Gagliardo-Nirenberg type inequality depends only on integration by parts, and hence it is true for any function with the necessary differentiability and integrability conditions. A slightly different version was obtained in the case of the Heisenberg group in [MZGZ09, Lemma 4.2]. As the usual notations for arbitrary small numbers, ε and δ, are already used for other things, in proofs we will use $\frac{1}{100}$ for a sufficiently small positive constant, which will help us embedding its term from the right into the left side.

Lemma 3.1. *Let $u \in C^\infty(\Omega)$, $\beta \geq 0$ and $\eta \in C_0^\infty(\Omega)$. Then there exists a constant $c > 0$ depending on p, ν and β and independent of ε and δ, such that*

$$\int_\Omega \eta^2(\delta^2 + |\nabla^\varepsilon u|^2)^{\frac{p+2}{2}+\beta}\, dx \leq c \int_\Omega (\delta^2\eta^2 + |\nabla^\varepsilon \eta|^2 u^2)\,(\delta^2 + |\nabla^\varepsilon u|^2)^{\frac{p}{2}+\beta}\, dx$$

$$+ c\int_\Omega u^2\eta^2\,(\delta^2+|\nabla^\varepsilon u|^2)^{\frac{p-2}{2}+\beta}\,|\nabla^\varepsilon\nabla^\varepsilon u|^2\,dx\,.$$

Proof. For any index $1\le k\le \nu+1$ consider the integral

$$\begin{aligned}
&\int_\Omega \eta^2(\delta^2+|\nabla^\varepsilon u|^2)^{\frac{p}{2}+\beta}\,(X_k^\varepsilon u)^2\,dx\\
&= -\int_\Omega X_k^\varepsilon\left(\eta^2(\delta^2+|\nabla^\varepsilon u|^2)^{\frac{p}{2}+\beta}\,X_k^\varepsilon u\right)\,u\,dx\\
&= -\int_\Omega 2\eta X_k^\varepsilon\eta\,(\delta^2+|\nabla^\varepsilon u|^2)^{\frac{p}{2}+\beta}\,X_k^\varepsilon u\,u\,dx\\
&\quad -\int_\Omega \eta^2\left(\frac{p}{2}+\beta\right)\,(\delta^2+|\nabla^\varepsilon u|^2)^{\frac{p-2}{2}+\beta}\,\sum_{i=1}^{\nu+1}2X_i^\varepsilon u\,X_k^\varepsilon X_i^\varepsilon u\,X_k^\varepsilon u\,u\,dx\\
&\quad -\int_\Omega \eta^2(\delta^2+|\nabla^\varepsilon u|^2)^{\frac{p}{2}+\beta}\,X_k^\varepsilon X_k^\varepsilon u\,u\,dx\\
&= (I_1)+(I_2)+(I_3).
\end{aligned}$$

First, let us estimate (I_1).

$$\begin{aligned}
(I_1) &\le c\int_\Omega |\eta|\,|\nabla^\varepsilon\eta|(\delta^2+|\nabla^\varepsilon u|^2)^{\frac{p+1}{2}+\beta}|u|\,dx\\
&\le \frac{1}{100}\int_\Omega \eta^2\,(\delta^2+|\nabla^\varepsilon u|^2)^{\frac{p+2}{2}+\beta}\,dx\\
&\quad + c\int_\Omega |\nabla^\varepsilon\eta|^2u^2\,(\delta^2+|\nabla^\varepsilon u|^2)^{\frac{p}{2}+\beta}\,dx\,.
\end{aligned}$$

The estimates of (I_2) and (I_3) are similar.

$$\begin{aligned}
(I_2)+(I_3) &\le c\int_\Omega \eta^2\,(\delta^2+|\nabla^\varepsilon u|^2)^{\frac{p}{2}+\beta}\,|\nabla^\varepsilon\nabla^\varepsilon u|\,|u|\,dx\\
&\le \frac{1}{100}\int_\Omega \eta^2\,(\delta^2+|\nabla^\varepsilon u|^2)^{\frac{p+2}{2}+\beta}\,dx\\
&\quad + c\int_\Omega \eta^2u^2\,(\delta^2+|\nabla^\varepsilon u|^2)^{\frac{p-2}{2}+\beta}\,|\nabla^\varepsilon\nabla^\varepsilon u|^2\,dx\,.
\end{aligned}$$

The final estimate of this lemma is obtained by summing all inequalities for $1\le k\le\nu+1$, adding the term

$$\int_\Omega \delta^2\eta^2\,\left(\delta^2+|\nabla^\varepsilon u|^2\right)^{\frac{p}{2}+\beta}$$

to both sides, and embedding the first terms of the right hand sides into the left side. □

The following lemma can be considered a mixed Gagliardo-Nirenberg type inequality.

Lemma 3.2. *Let $u_\varepsilon \in C^\infty(\Omega)$ be a solution of (2.8), $\beta \geq 1$ and $\eta \in C_0^\infty(\Omega)$. Then there exists a constant $c > 0$ depending on L, p, ν and β and independent of ε and δ, such that*

$$\begin{aligned}&\int_\Omega \eta^2(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, dx \\ &\leq c\int_\Omega (\delta^2\eta^2 + |\nabla^\varepsilon \eta|^2 u_\varepsilon^2)\,(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p-2}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, dx \\ &+ c\beta^2 \int_\Omega u_\varepsilon^2 \eta^2\,(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p-2}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta-2}\, |\nabla^\varepsilon \nabla_{\mathcal{V}} u_\varepsilon|^2\, dx\,.\end{aligned}$$

Proof. To prove this lemma, in the weak form of (2.8) we substitute the test functions $\phi = \eta^2\, u_\varepsilon\, (X_k u_\varepsilon)^{2\beta}$, $3 \leq k \leq \nu + 1$. Therefore,

$$\begin{aligned}&\sum_i \int_\Omega a_i(\nabla^\varepsilon u_\varepsilon)\, \eta^2\, X_i^\varepsilon u_\varepsilon\, (X_k u_\varepsilon)^{2\beta}\, dx \\ &+ \sum_i \int_\Omega a_i(\nabla^\varepsilon u_\varepsilon)\, 2\eta X_i^\varepsilon \eta\, u_\varepsilon\, (X_k u_\varepsilon)^{2\beta}\, dx \\ &+ \sum_i \int_\Omega a_i(\nabla^\varepsilon u_\varepsilon)\, \eta^2\, u_\varepsilon\, 2\beta (X_k u_\varepsilon)^{2\beta-1}\, X_i^\varepsilon X_k u_\varepsilon\, dx = 0\,.\end{aligned}$$

After summing over $3 \leq k \leq \nu + 1$, adding to both sides the term

$$\frac{1}{(\nu-1)^{\beta-1}}\, \frac{L}{p-1} \int_\Omega \eta^2(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p-2}{2}}\, \delta^2\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, dx\,,$$

and using the properties (1.5) and (1.6), we get the following estimates.

First,

$$\begin{aligned}&\sum_{k\geq 3}\sum_{i\geq 1} \int_\Omega a_i(\nabla^\varepsilon u_\varepsilon)\, \eta^2\, X_i^\varepsilon u_\varepsilon\, (X_k u_\varepsilon)^{2\beta}\, dx \\ &+ \frac{1}{(\nu-1)^{\beta-1}}\, \frac{L}{p-1} \int_\Omega \eta^2(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p-2}{2}}\, \delta^2\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, dx \\ &\geq \frac{1}{(\nu-1)^{\beta-1}}\, \frac{L}{p-1} \int_\Omega \eta^2(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, dx\,.\end{aligned}$$

Second,

$$\begin{aligned}&\sum_{k\geq 3}\sum_{i\geq 1} \int_\Omega a_i(\nabla^\varepsilon u_\varepsilon)\, 2\eta X_i^\varepsilon \eta\, u_\varepsilon\, (X_k u_\varepsilon)^{2\beta}\, dx \\ &\leq c\int_\Omega (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p-1}{2}}\, |\eta|\, |\nabla^\varepsilon \eta|\, |u_\varepsilon|\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, dx\end{aligned}$$

$$\leq \frac{1}{100} \int_\Omega \eta^2(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p}{2}} |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta} \, dx$$
$$+ c \int_\Omega |\nabla^\varepsilon \eta|^2 u_\varepsilon^2 \, (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta} \, dx \, .$$

Third,

$$\sum_{k\geq 3} \sum_{i\geq 1} \int_\Omega a_i(\nabla^\varepsilon u_\varepsilon) \, \eta^2 \, u_\varepsilon \, 2\beta (X_k u_\varepsilon)^{2\beta-1} X_i^\varepsilon X_k u_\varepsilon \, dx$$
$$\leq c \int_\Omega (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p-1}{2}} \eta^2 \, |u_\varepsilon| \, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta-1} \, |\nabla^\varepsilon \nabla_{\mathcal{V}} u_\varepsilon| \, dx$$
$$\leq \frac{1}{100} \int_\Omega \eta^2(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p}{2}} |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta} \, dx$$
$$+ c \int_\Omega \eta^2 u_\varepsilon^2 \, (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta-2} \, |\nabla^\varepsilon \nabla_{\mathcal{V}} u_\varepsilon|^2 \, dx \, .$$

Finishing the proof can be done now by an easy combination of the three estimates from above. □

The next two lemmas contain generalizations of the Cacciopoli-type inequalities, which were developed and gradually refined in the case of Heisenberg group in the papers [MM07, MZGZ09, Zho17, Ric15, CCLDO19].

Lemma 3.3. *Let $0 < \delta < 1$, $\beta \geq 0$ and $\eta \in C_0^\infty(\Omega)$ be such that $0 \leq \eta \leq 1$. Then there exists a constant $c > 0$ depending on ν, p and L and independent of ε and δ, such that for any solution $u_\varepsilon \in C^\infty(\Omega)$ of (2.8) we have*

$$\int_\Omega \eta^2 \, (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta} \, |\nabla^\varepsilon \nabla_{\mathcal{V}} u_\varepsilon|^2 \, dx$$
$$\leq c(\beta+1)^2 \int_\Omega (\eta^2 + |\nabla^\varepsilon \eta|^2) \, (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p-2}{2}} |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta+2} \, dx \, .$$

Proof. In order to accommodate all the terms, we will simplify the writing of (2.8):

$$\sum_i X_i^\varepsilon(a_i) = 0 \, . \tag{3.1}$$

Also, we set

$$\omega_\varepsilon = \delta^2 + |\nabla^\varepsilon u_\varepsilon|^2$$
$$= \delta^2 + (X_1 u_\varepsilon)^2 + (X_2 u_\varepsilon)^2 + \varepsilon^2 (X_3 u_\varepsilon)^2 + \ldots + \varepsilon^{2\nu-2} (X_{\nu+1} u_\varepsilon)^2.$$

By differentiating (3.1) with respect to X_k, $k \geq 3$ and switching X_i^ε and X_k we get

$$\sum_i X_i^\varepsilon(X_k(a_i)) = X_{k+1}a_1.$$

Using the notation $a_{ij} = \frac{\partial a_i}{\partial \xi_j}$, for any $\phi \in C_0^\infty(\Omega)$ we get

$$\sum_{i,j} \int_\Omega a_{ij}\, X_k X_j^\varepsilon u_\varepsilon\, X_i^\varepsilon \phi\, dx = -\int_\Omega X_{k+1}a_1\, \phi\, dx.$$

Another switch between X_j^ε and X_k leads to

$$\begin{aligned}\sum_{i,j} \int_\Omega a_{ij}\, X_j^\varepsilon X_k u_\varepsilon\, X_i^\varepsilon \phi\, dx = &-\int_\Omega X_{k+1}a_1\, \phi\, dx \\ &+ \sum_i \int_\Omega a_{i1}\, X_{k+1}u_\varepsilon\, X_i^\varepsilon \phi\, dx.\end{aligned} \tag{3.2}$$

Let us use $\phi = \eta^2\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, X_k u_\varepsilon$ in (3.2). Then,

$$\begin{aligned}X_i^\varepsilon \phi = &\,2\eta\, X_i^\varepsilon \eta\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, X_k u_\varepsilon + \eta^2\, \beta\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta-2}\, X_i^\varepsilon(|\nabla_{\mathcal{V}} u_\varepsilon|^2)\, X_k u_\varepsilon \\ &+ \eta^2 |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, X_i^\varepsilon X_k u_\varepsilon\,,\end{aligned}$$

and hence

$$\begin{aligned}&\sum_{i,j} \int_\Omega a_{ij}\, X_j^\varepsilon X_k u_\varepsilon\, 2\eta X_i^\varepsilon \eta\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, X_k u_\varepsilon\, dx \\ &+ \sum_{i,j} \int_\Omega a_{ij}\, X_j^\varepsilon X_k u_\varepsilon\, \eta^2\, \beta |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta-2}\, X_i^\varepsilon(|\nabla_{\mathcal{V}} u_\varepsilon|^2)\, X_k u_\varepsilon\, dx \\ &+ \sum_{i,j} \int_\Omega a_{ij}\, X_j^\varepsilon X_k u_\varepsilon\, \eta^2\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, X_i^\varepsilon X_k u_\varepsilon\, dx \\ &= -\int_\Omega X_{k+1}a_1\, \eta^2\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, X_k u_\varepsilon\, dx \\ &+ \sum_i \int_\Omega a_{i1}\, X_{k+1}u_\varepsilon\, 2\eta\, X_i^\varepsilon \eta\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, X_k u_\varepsilon\, dx \\ &+ \sum_i \int_\Omega a_{i1}\, X_{k+1}u_\varepsilon\, \eta^2\, \beta\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta-2}\, X_i^\varepsilon(|\nabla_{\mathcal{V}} u_\varepsilon|^2)\, X_k u_\varepsilon\, dx \\ &+ \sum_i \int_\Omega a_{i1}\, X_{k+1}u_\varepsilon\, \eta^2\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, X_i^\varepsilon X_k u_\varepsilon\, dx\,.\end{aligned}$$

After writing identical equations for each X_k, $k \geq 3$ and summing them, we get an equation in the following form:

$$(L_1) + (L_2) + (L_3) = (R_1) + (R_2) + (R_3) + (R_4).$$

We estimate each term.

$$\begin{aligned}
(L_3) &\geq L \int_\Omega \eta^2\, \omega_\varepsilon^{\frac{p-2}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, |\nabla^\varepsilon \nabla_{\mathcal{V}} u_\varepsilon|^2\, dx. \\
(L_2) &= \frac{1}{2} \sum_{i,j} \int_\Omega a_{ij}\, X_j^\varepsilon(|\nabla_{\mathcal{V}} u_\varepsilon|^2)\, \eta^2\, \beta\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta-2}\, X_i^\varepsilon(|\nabla_{\mathcal{V}} u_\varepsilon|^2)\, dx \\
&\geq \frac{\beta L}{2} \int_\Omega \eta^2\, \omega_\varepsilon^{\frac{p-2}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta-2}\, |\nabla^\varepsilon(|\nabla_{\mathcal{V}} u_\varepsilon|^2)|^2\, dx. \\
(L_1) &\leq c \int_\Omega \omega_\varepsilon^{\frac{p-2}{2}}\, |\nabla^\varepsilon \nabla_{\mathcal{V}} u_\varepsilon|\, 2\eta\, |\nabla^\varepsilon \eta|\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta+1}\, dx \\
&\leq \frac{1}{100} \int_\Omega \eta^2 \omega_\varepsilon^{\frac{p-2}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta} |\nabla^\varepsilon \nabla_{\mathcal{V}} u_\varepsilon|^2\, dx \\
&+ c \int_\Omega |\nabla^\varepsilon \eta|^2\, \omega_\varepsilon^{\frac{p-2}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta+2}\, dx. \\
(R_1) &= \sum_k \sum_j \int_\Omega a_{1j}\, X_{k+1} X_j^\varepsilon u_\varepsilon\, \eta^2\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, X_k u_\varepsilon\, dx \\
&= \sum_k \sum_j \int_\Omega a_{1j}\, X_j^\varepsilon X_{k+1} u_\varepsilon\, \eta^2\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, X_k u_\varepsilon\, dx \\
&- \sum_k \int_\Omega a_{11}\, X_{k+2} u_\varepsilon\, \eta^2\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, X_k u_\varepsilon\, dx \\
&\leq c \int_\Omega \omega_\varepsilon^{\frac{p-2}{2}}\, |\nabla^\varepsilon \nabla_{\mathcal{V}} u_\varepsilon|\, \eta^2\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta+1}\, dx + c \int_\Omega \eta^2\, \omega_\varepsilon^{\frac{p-2}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta+2}\, dx \\
&\leq \frac{1}{100} \int_\Omega \eta^2 \omega_\varepsilon^{\frac{p-2}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta} |\nabla^\varepsilon \nabla_{\mathcal{V}} u_\varepsilon|^2\, dx + c \int_\Omega \eta^2\, \omega_\varepsilon^{\frac{p-2}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta+2}\, dx.
\end{aligned}$$

In a similar way we get that

$$\begin{aligned}
(R_2) + (R_3) + (R_4) \leq \frac{1}{100} &\int_\Omega \eta^2\, \omega_\varepsilon^{\frac{p-2}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta}\, |\nabla^\varepsilon \nabla_{\mathcal{V}} u_\varepsilon|^2\, dx \\
&+ c(\beta+1)^2 \int_\Omega (\eta^2 + |\nabla^\varepsilon \eta|^2)\, \omega_\varepsilon^{\frac{p-2}{2}}\, |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta+2}\, dx.
\end{aligned}$$

We finish the proof by combining all the above estimates. $\square$

Lemma 3.4. *Let $0 < \delta < 1$, $\beta \geq 0$ and $\eta \in C_0^\infty(\Omega)$ be such that $0 \leq \eta \leq 1$. Then there exists a constant $c > 0$ depending on ν, p and L and independent of ε and δ, such that for any solution $u_\varepsilon \in C^\infty(\Omega)$ of (2.8) we have*

$$\begin{aligned}
&\int_\Omega \eta^2 \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p-2}{2}+\beta} |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon|^2 \, dx \\
&\leq c(\beta+1)^4 \int_\Omega \eta^2 \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p-2}{2}+\beta} |\nabla_{\mathcal{V}} u_\varepsilon|^2 \, dx \\
&+ c(\beta+1)^2 \int_\Omega \left(\eta^2 + |\nabla^\varepsilon \eta|^2 + \eta |\nabla_{\mathcal{V}} \eta|\right) \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}+\beta} dx.
\end{aligned}$$

Proof. Let's differentiate equation (3.1) with respect to X_1^ε and switch X_1^ε and X_i^ε. In this way we get

$$\sum_i X_i^\varepsilon (X_1^\varepsilon a_i) = -\frac{1}{\varepsilon} \sum_{i \geq 2} X_{i+1}^\varepsilon a_i.$$

The weak form of this equation looks like

$$\sum_{i,j} \int_\Omega a_{ij} \, X_1^\varepsilon X_j^\varepsilon u_\varepsilon \, X_i^\varepsilon \phi \, dx = -\frac{1}{\varepsilon} \sum_{i \geq 2} \int_\Omega a_i \, X_{i+1}^\varepsilon \phi \, dx \, .$$

After switching X_j^ε and X_1^ε we get

$$\begin{aligned}
\sum_{i,j} \int_\Omega a_{ij} \, X_j^\varepsilon X_1^\varepsilon u_\varepsilon \, X_i^\varepsilon \phi \, dx = &- \frac{1}{\varepsilon} \sum_{i \geq 2} \int_\Omega a_i \, X_{i+1}^\varepsilon \phi \, dx \\
&- \frac{1}{\varepsilon} \sum_{i; j \geq 2} \int_\Omega a_{ij} \, X_{j+1}^\varepsilon u_\varepsilon \, X_i^\varepsilon \phi \, dx.
\end{aligned} \tag{3.3}$$

Let us use $\phi = \eta^2 \, \omega_\varepsilon^\beta \, X_1^\varepsilon u_\varepsilon$ in (3.3).

$$\begin{aligned}
&\sum_{i,j} \int_\Omega a_{ij} \, X_j^\varepsilon X_1^\varepsilon u_\varepsilon \, \eta^2 \, \omega_\varepsilon^\beta \, X_i^\varepsilon X_1^\varepsilon u_\varepsilon \, dx \\
&+ \sum_{i,j} \int_\Omega a_{ij} \, X_j^\varepsilon X_1^\varepsilon u_\varepsilon \, \eta^2 \, \beta \omega_\varepsilon^{\beta-1} \, X_i^\varepsilon (|\nabla^\varepsilon u_\varepsilon|^2) \, X_1^\varepsilon u_\varepsilon \, dx \\
&+ \sum_{i,j} \int_\Omega a_{ij} \, X_j^\varepsilon X_1^\varepsilon u_\varepsilon \, 2\eta \, X_i^\varepsilon \eta \, \omega_\varepsilon^\beta \, X_1^\varepsilon u_\varepsilon \, dx \\
&= - \frac{1}{\varepsilon} \sum_{i \geq 2} \int_\Omega a_i \, \eta^2 \, \omega_\varepsilon^\beta \, X_{i+1}^\varepsilon X_1^\varepsilon u_\varepsilon \, dx \\
&\quad - \frac{1}{\varepsilon} \sum_{i \geq 2} \int_\Omega a_i \, \eta^2 \, \beta \omega_\varepsilon^{\beta-1} \, X_{i+1}^\varepsilon (|\nabla^\varepsilon u_\varepsilon|^2) \, X_1^\varepsilon u_\varepsilon \, dx
\end{aligned}$$

$$- \frac{1}{\varepsilon} \sum_{i \geq 2} \int_\Omega a_i \, 2\eta \, X^\varepsilon_{i+1} \eta \, \omega_\varepsilon^\beta \, X_1^\varepsilon u_\varepsilon \, dx$$

$$- \frac{1}{\varepsilon} \sum_{i;j \geq 2} \int_\Omega a_{ij} \, X^\varepsilon_{i+1} u_\varepsilon \, \eta^2 \, \omega_\varepsilon^\beta \, X_i^\varepsilon X_1^\varepsilon u_\varepsilon \, dx$$

$$- \frac{1}{\varepsilon} \sum_{i;j \geq 2} \int_\Omega a_{ij} \, X^\varepsilon_{i+1} u_\varepsilon \, \eta^2 \, \beta \omega_\varepsilon^{\beta-1} \, X_i^\varepsilon (|\nabla^\varepsilon u_\varepsilon|^2) \, X_1^\varepsilon u_\varepsilon \, dx$$

$$- \frac{1}{\varepsilon} \sum_{i;j \geq 2} \int_\Omega a_{ij} \, X^\varepsilon_{i+1} u_\varepsilon \, 2\eta \, X_i^\varepsilon \eta \, \omega_\varepsilon^\beta \, X_1^\varepsilon u_\varepsilon \, dx \, .$$

Repeat the above calculations for $X_2^\varepsilon, ..., X_{\nu+1}^\varepsilon$ and add all equations. In this way we get an equation in the following format

$$\text{(L1)} + \text{(L2)} + \text{(L3)} = \text{(R1)} + \cdots + \text{(R6)}.$$

We estimate each term.

$$\begin{aligned}
\text{(L1)} &= \sum_{i,j,k} \int_\Omega a_{ij} \, X_j^\varepsilon X_k^\varepsilon u_\varepsilon \, \eta^2 \, \omega_\varepsilon^\beta \, X_i^\varepsilon X_k^\varepsilon u_\varepsilon \, dx \\
&\geq L \int_\Omega \eta^2 \, \omega_\varepsilon^{\frac{p-2}{2}+\beta} \, |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon|^2 \, dx. \\
\text{(L2)} &= \sum_{i,j} \int_\Omega a_{ij} \, \sum_k X_j^\varepsilon X_k^\varepsilon u_\varepsilon \, X_k^\varepsilon u_\varepsilon \, \eta^2 \, \beta \omega_\varepsilon^{\beta-1} \, X_i^\varepsilon (|\nabla^\varepsilon u_\varepsilon|^2) \, dx \\
&= \frac{\beta}{2} \sum_{i,j} \int_\Omega a_{ij} \, X_j^\varepsilon (|\nabla^\varepsilon u_\varepsilon|^2) \, \eta^2 \, \omega_\varepsilon^{\beta-1} \, X_i^\varepsilon (|\nabla^\varepsilon u_\varepsilon|^2) \, dx \\
&\geq \frac{\beta L}{2} \int_\Omega \eta^2 \omega_\varepsilon^{\frac{p-2}{2}+\beta-1} \, |\nabla^\varepsilon (|\nabla^\varepsilon u_\varepsilon|^2) \, dx. \\
\text{L(3)} &\leq c \int_\Omega \omega_\varepsilon^{\frac{p-2}{2}} \, |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon| \, \eta \, |\nabla^\varepsilon \eta| \, \omega_\varepsilon^{\beta + \frac{1}{2}} \, dx \\
&\leq \frac{1}{100} \int_\Omega \eta^2 \, \omega_\varepsilon^{\frac{p-2}{2}+\beta} \, |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon|^2 \, dx + c \int_\Omega |\nabla^\varepsilon \eta|^2 \, \omega_\varepsilon^{\frac{p}{2}+\beta} \, dx \, .
\end{aligned}$$

For (R1), each of its terms must be evaluated in the same way. We will show it for $k = 1$. Also, we use the notation that $X_m^\varepsilon = 0$ if $m > \nu + 1$.

$$\begin{aligned}
&- \frac{1}{\varepsilon} \sum_{i \geq 2} \int_\Omega a_i \, \eta^2 \, \omega_\varepsilon^\beta \, X_{i+1}^\varepsilon X_1^\varepsilon u_\varepsilon \, dx \\
&= -\frac{1}{\varepsilon} \sum_{i \geq 2} \int_\Omega a_i \, \eta^2 \, \omega_\varepsilon^\beta \, (X_1^\varepsilon X_{i+1}^\varepsilon u_\varepsilon - \frac{1}{\varepsilon} X_{i+2}^\varepsilon u_\varepsilon) \, dx
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon}\sum_{i\geq 2}\int_\Omega X_1^\varepsilon(a_i\,\eta^2\,\omega_\varepsilon^\beta)\,X_{i+1}^\varepsilon u_\varepsilon\,dx + \frac{1}{\varepsilon^2}\sum_{i\geq 2}\int_\Omega a_i\,\eta^2\,\omega_\varepsilon^\beta\,X_{i+2}^\varepsilon u_\varepsilon\,dx \\
&= \sum_{i\geq 2;j}\int_\Omega a_{ij}\,X_1^\varepsilon X_j^\varepsilon u_\varepsilon\,\eta^2\,\omega_\varepsilon^\beta\,\varepsilon^{i-2}X_{i+1}u_\varepsilon\,dx \\
&+ \sum_{i\geq 2}\int_\Omega a_i\,2\eta\,X_1^\varepsilon\eta\,\omega_\varepsilon^\beta\,\varepsilon^{i-2}X_{i+1}u_\varepsilon\,dx \\
&+ \sum_{i\geq 2}\int_\Omega a_i\,\eta^2\,\beta\omega_\varepsilon^{\beta-1}\,2\langle\nabla^\varepsilon u_\varepsilon, X_1^\varepsilon\nabla^\varepsilon u_\varepsilon\rangle\,\varepsilon^{i-2}X_{i+1}u_\varepsilon\,dx \\
&+ \sum_{i\geq 2}\int_\Omega a_i\,\eta^2\,\omega_\varepsilon^\beta\,\varepsilon^{i-2}X_{i+2}u_\varepsilon\,dx \\
&\leq c\int_\Omega \omega_\varepsilon^{\frac{p-2}{2}+\beta}\,|\nabla^\varepsilon\nabla^\varepsilon u_\varepsilon|\,\eta^2\,|\nabla_{\mathcal{V}} u_\varepsilon|\,dx + c\int_\Omega \omega_\varepsilon^{\frac{p-1}{2}+\beta}\,\eta\,|\nabla^\varepsilon\eta|\,|\nabla_{\mathcal{V}} u_\varepsilon|\,dx \\
&+ c\beta\int_\Omega \omega_\varepsilon^{\frac{p-2}{2}+\beta}\,\eta^2\,|\nabla^\varepsilon\nabla^\varepsilon u_\varepsilon|\,|\nabla_{\mathcal{V}} u_\varepsilon|\,dx \\
&+ c\int_\Omega \omega_\varepsilon^{\frac{p-1}{2}+\beta}\,\eta^2\,|\nabla_{\mathcal{V}} u_\varepsilon|\,dx \\
&\leq \frac{1}{100}\int_\Omega \eta^2\,\omega_\varepsilon^{\frac{p-2}{2}+\beta}\,|\nabla^\varepsilon\nabla^\varepsilon u_\varepsilon|^2\,dx + c(\beta+1)^2\int_\Omega \eta^2\,\omega_\varepsilon^{\frac{p-2}{2}+\beta}\,|\nabla_{\mathcal{V}} u_\varepsilon|^2\,dx \\
&+ c\int_\Omega (\eta^2+|\nabla^\varepsilon\eta|^2)\,\omega_\varepsilon^{\frac{p}{2}+\beta}\,dx\,.
\end{aligned}
$$

The estimate of (R2) depends also on integration by parts.

$$
\begin{aligned}
\text{(R2)} &= -\frac{2\beta}{\varepsilon}\sum_{i\geq 2}\sum_{j\geq 1}\int_\Omega a_i\,\eta^2\,\omega_\varepsilon^{\beta-1}\,X_j^\varepsilon u_\varepsilon\,X_{i+1}^\varepsilon X_j^\varepsilon u_\varepsilon\,X_1^\varepsilon u_\varepsilon\,dx \\
&= -\frac{2\beta}{\varepsilon}\sum_{i\geq 2}\sum_{j\geq 1}\int_\Omega a_i\,\eta^2\,\omega_\varepsilon^{\beta-1}\,X_j^\varepsilon u_\varepsilon\,X_j^\varepsilon X_{i+1}^\varepsilon u_\varepsilon\,X_1^\varepsilon u_\varepsilon\,dx \\
&+ \frac{2\beta}{\varepsilon^2}\sum_{i\geq 2}\int_\Omega a_i\,\eta^2\,\omega_\varepsilon^{\beta-1}\,X_1^\varepsilon u_\varepsilon\,X_{i+2}^\varepsilon u_\varepsilon\,X_1^\varepsilon u_\varepsilon\,dx \\
&= \frac{2\beta}{\varepsilon}\sum_{i\geq 2}\sum_{j\geq 1}\int_\Omega X_j^\varepsilon\left(a_i\,\eta^2\,\omega_\varepsilon^{\beta-1}\,X_j^\varepsilon u_\varepsilon\,X_1^\varepsilon u_\varepsilon\right)X_{i+1}^\varepsilon u_\varepsilon\,dx \\
&+ \frac{2\beta}{\varepsilon^2}\sum_{i\geq 2}\int_\Omega a_i\,\eta^2\,\omega_\varepsilon^{\beta-1}\,X_1^\varepsilon u_\varepsilon\,X_{i+2}^\varepsilon u_\varepsilon\,X_1^\varepsilon u_\varepsilon\,dx.
\end{aligned}
$$

After applying the product rule in the first integral we find that

$$\begin{aligned}(R2) &\le c(\beta+1)^2 \int_\Omega \eta^2\, \omega_\varepsilon^{\frac{p-2}{2}+\beta}\, |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon|\, |\nabla_{\mathcal V} u_\varepsilon|\, dx \\ &\quad + c(\beta+1) \int_\Omega \omega_\varepsilon^{\frac{p-1}{2}+\beta}\, (\eta^2 + \eta\, |\nabla^\varepsilon \eta|)\, |\nabla_{\mathcal V} u_\varepsilon|\, dx \\ &\le \frac{1}{100} \int_\Omega \eta^2\, \omega_\varepsilon^{\frac{p-2}{2}+\beta}\, |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon|^2\, dx \\ &\quad + c(\beta+1)^4 \int_\Omega \eta^2\, \omega_\varepsilon^{\frac{p-2}{2}+\beta}\, |\nabla_{\mathcal V} u_\varepsilon|^2\, dx \\ &\quad + c(\beta+1)^2 \int_\Omega (\eta^2 + |\nabla^\varepsilon \eta|^2)\, \omega_\varepsilon^{\frac{p}{2}+\beta}\, dx.\end{aligned}$$

The estimates for (R3) ... (R6) are similar and left to the reader. □

4. Difference quotients and De Giorgi-Moser iterations

In this section we will construct de Giorgi-Moser type iteration schemes leading to local bounds independent of ε for $\nabla_{\mathcal V} u_\varepsilon$ and $\nabla^\varepsilon u_\varepsilon$.

For any $Z \in \mathfrak{g}$ we denote by e^Z the group exponential. Consider an arbitrary $x \in \Omega$ and $s > 0$ such that $x\, e^{\pm sZ} \in \Omega$. For any function $u : \Omega \to \mathbb{R}$ and $\theta > 0$ we define the difference quotient

$$D_{Z,\pm s,\theta} u(x) = \frac{u(xe^{\pm sZ}) - u(x)}{\pm s^\theta}\,. \tag{4.1}$$

We recall the following lemma about the connection between the differentiability of a function u and the control of the L^p norm of its difference quotients (see, for example, [Hör67] or [Cap97]).

Lemma 4.1. *Let K be a compact set included in Ω, Z be a left invariant vector field and $u \in L^p_{\mathrm{loc}}(\Omega)$. If there exist σ and C, two positive constants, such that*

$$\sup_{0<|s|\le\sigma} \|D_{Z,s,1} u\|_{L^p(K)}\, dx \le C$$

then $Zu \in L^p(K)$ and $||Zu||_{L^p(K)} \le C$.

Conversely, if $Zu \in L^p(K)$ then for some $\sigma > 0$

$$\sup_{0<|s|\le\sigma} \|D_{Z,s,1} u\|_{L^p(K)}\, dx \le 2||Zu||_{L^p(K)}\,.$$

The De Giorgi-Moser type iterations, which later will lead to the boundedness of the horizontal gradient, require the following lemmas providing estimates with constants independent of ε.

The first result is a direct consequence of the Baker-Campbell-Hausdorff formula [Hör67] and the fact that $|\nabla_{\mathcal{H}} u| \leq |\nabla^{\varepsilon} u|$.

Lemma 4.2. *Let $Z \in V_i$, $u \in C^{\infty}(\Omega)$, $\sigma > 0$ and $r > 0$ such that $B^{\varepsilon}_{3r} \subset \Omega$. Then there exists a positive constant c independent of u and ε, such that*

$$\sup_{0<|s|\leq\sigma} \int_{B^{\varepsilon}_r} \left| D_{Z,s,\frac{1}{i}} u(x) \right|^p dx \leq c \int_{B^{\varepsilon}_{2r}} \left(|u|^p + |\nabla^{\varepsilon} u|^p \right) dx \,. \tag{4.2}$$

The following lemma provides a uniform local L^p-bound for the vertical gradient.

Lemma 4.3. *Let $u_{\varepsilon} \in C^{\infty}(\Omega)$ be a solution of (2.8) and $r > 0$ such that $B^{\varepsilon}_{3r} \subset \Omega$. If $2 \leq p < \frac{2\nu}{\nu-1}$, then there exists a positive constant c depending on r, ν, p and L, but independent of ε and δ, such that*

$$\int_{B^{\varepsilon}_r} |\nabla_{\mathcal{V}} u_{\varepsilon}|^p \, dx \leq c \int_{B^{\varepsilon}_{2r}} \left(\delta^2 + |\nabla^{\varepsilon} u_{\varepsilon}|^2 \right)^{\frac{p}{2}} \, dx \,. \tag{4.3}$$

Proof. We start the proof by substituting into the weak form of the equation (2.8) the test function

$$\phi(x) = D_{X_{\nu+1},-s,\frac{1}{\nu}} \left(\eta^2 D_{X_{\nu+1},s,\frac{1}{\nu}} u_{\varepsilon}(x) \right) \,,$$

where η is a cut-off function between $B^{\varepsilon}_{r/2}$ and B^{ε}_r. As $X_{\nu+1}$ commutes with any other vector field X^{ε}_i, similarly to the proof of [Dom08, Theorem 2.1], we can use Lemma 4.2 and the properties of second order difference quotients to obtain that

$$\sup_{0<|s|\leq\sigma} \int_{B^{\varepsilon}_{r/2}} \left| D_{X_{\nu+1},s,\frac{1}{\nu}+\frac{2}{\nu p}} u(x) \right|^p dx \leq c \int_{B^{\varepsilon}_{2r}} \left(\delta^2 + |\nabla^{\varepsilon} u|^2 \right)^{\frac{p}{2}} \, dx \,.$$

We can now restart the estimates with the test function

$$\phi(x) = D_{X_{\nu+1},-s,\frac{1}{\nu}+\frac{2}{\nu p}} \left(\eta^2 D_{X_{\nu+1},s,\frac{1}{\nu}+\frac{2}{\nu p}} u_{\varepsilon}(x) \right) \,,$$

where η is a cut-off function between $B^{\varepsilon}_{r/4}$ and $B^{\varepsilon}_{r/2}$. Continuing in this way we will increase the fractional differentiability order to

$$\gamma_k = \frac{1}{\nu} + \frac{2}{\nu p} + \cdots + \frac{2^{k-1}}{\nu p^{k-1}} \,.$$

Note that, there exists $k \in \mathbb{N}$ such that $\gamma_k > 1$ if and only if

$$2 \le p < \frac{2\nu}{\nu - 1}\,.$$

Therefore, for this interval of p, after some rescaling we obtain that

$$\int_{B_r^\varepsilon} |X_{\nu+1}u_\varepsilon|^p\,dx \le c\int_{B_{2r}^\varepsilon} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}}\,dx\,. \tag{4.4}$$

If we repeat the proof of Lemma 3.3 for $\beta = 0$ and $\phi = \eta^2 X_{\nu+1}u_\varepsilon$ we obtain that

$$\int_{B_r^\varepsilon} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p-2}{2}}\,|\nabla^\varepsilon X_{\nu+1}u_\varepsilon|^2\,dx \le c\int_{B_{2r}^\varepsilon} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}}\,dx\,. \tag{4.5}$$

We continue the proof by considering a test function

$$\phi(x) = D_{X_\nu,-s,\frac{1}{\nu-1}}\left(\eta^2 D_{X_\nu,s,\frac{1}{\nu-1}}u_\varepsilon(x)\right)\,.$$

The only non-zero commutator will show up from the term

$$\begin{aligned} X_1^\varepsilon\phi(x) = {} & D_{X_\nu,-s,\frac{1}{\nu-1}}X_1^\varepsilon\left(\eta^2 D_{X_\nu,s,\frac{1}{\nu-1}}u_\varepsilon(x)\right) \\ & - s^{1-\frac{1}{\nu-1}}X_{\nu+1}\left(\eta^2 D_{X_\nu,s,\frac{1}{\nu-1}}u_\varepsilon\right)(xe^{-sX_\nu})\,. \end{aligned}$$

Therefore, the weak form of (2.8) will look like

$$\begin{aligned} & \sum_i \int_\Omega D_{X_\nu,s,\frac{1}{\nu-1}}\,a_i(\nabla^\varepsilon u_\varepsilon)(x)\cdot\eta^2(x)\,D_{X_\nu,s,\frac{1}{\nu-1}}(X_i^\varepsilon u_\varepsilon)(x)\,dx \\ & = -\sum_i \int_\Omega D_{X_\nu,s,\frac{1}{\nu-1}}\,a_i(\nabla^\varepsilon u_\varepsilon)(x)\cdot 2\eta(x)\,X_i^\varepsilon\eta(x)\,D_{X_\nu,s,\frac{1}{\nu-1}}(u_\varepsilon)(x)\,dx \\ & \quad - s^{1-\frac{1}{\nu-1}}\int_\Omega D_{X_\nu,s,\frac{1}{\nu-1}}\,a_1(\nabla^\varepsilon u_\varepsilon)(x)\cdot\eta^2(x)\,X_{\nu+1}u_\varepsilon(xe^{sX_{\nu+1}})\,dx \\ & \quad - s^{1-\frac{1}{\nu-1}}\int_\Omega a_1(\nabla^\varepsilon u_\varepsilon(x))X_{\nu+1}\left(\eta^2 D_{X_\nu,s,\frac{1}{\nu-1}}u_\varepsilon\right)(xe^{-sX_\nu})\,dx\,. \end{aligned}$$

We can observe that each line from the right hand side can be estimated using Lemma 3.2, Lemma 3.3 and the previously found estimates (4.4) and (4.5). Therefore, as in the case of $X_{\nu+1}$, after k-steps we can increase the order of the fractional differentiability to

$$\gamma_k = \frac{1}{\nu-1} + \frac{2}{(\nu-1)p} + \cdots + \frac{2^{k-1}}{(\nu-1)p^{k-1}}\,.$$

In the case of $2 \leq p < \frac{2\nu}{\nu-1}$, for some k, the order γ_k will become greater than one. In conclusion, we obtain that

$$\int_{B_r^\varepsilon} |X_\nu u_\varepsilon|^p \, dx \leq c \int_{B_{2r}^\varepsilon} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}} \, dx \,. \tag{4.6}$$

Once more, if in the proof of Lemma 3.3 we use $\beta = 0$, a test function $\phi = \eta^2 X_\nu u_\varepsilon$ and apply (4.4), we obtain that

$$\int_{B_r^\varepsilon} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p-2}{2}} |\nabla^\varepsilon X_\nu u_\varepsilon|^2 \, dx \leq c \int_{B_{2r}^\varepsilon} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}} \, dx \,. \tag{4.7}$$

We can continue now with the estimates for $X_{\nu-1} u_\varepsilon$, because all the terms of the right hand side of the corresponding equation can be estimated in terms of (4.4) - (4.7). In similar ways, we can estimate the derivatives in the direction of each non-horizontal layer of the Lie algebra and therefore obtain (4.3). □

Lemma 4.4. *Let $2 \leq p < \frac{2\nu}{\nu-1}$ and $r > 0$ such that $B_{3r} \subset \Omega$. If $u_\varepsilon \in C^\infty(\Omega)$ is a solution of* (2.8)*, then there exists $c > 0$, depending on r, ν, p, and L, but independent of ε and δ, such that*

$$\begin{aligned} &\int_{B_r^\varepsilon} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p+4}{2}} \, dx \\ &\leq c \left(1 + ||u_\varepsilon||^4_{L^\infty(B_{2r}^\varepsilon)}\right) \int_{B_{2r}^\varepsilon} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}} \, dx \,. \end{aligned} \tag{4.8}$$

Proof. First we use Lemma 3.4 with $\beta = 0$ and a cut-off function η between $B_{15r/16}^\varepsilon$ and B_r^ε, followed by Lemma 4.3 and Lemma 3.1 for $\beta = 0$ and a cut-off function between $B_{7r/8}^\varepsilon$ and $B_{15r/16}^\varepsilon$ to obtain the following L^{p+2} estimate.

$$\begin{aligned} &\int_{B_{7r/8}^\varepsilon} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p+2}{2}} \, dx \\ &\leq c \left(1 + ||u_\varepsilon||^2_{L^\infty(B_{2r}^\varepsilon)}\right) \int_{B_{2r}^\varepsilon} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}} \, dx \,. \end{aligned} \tag{4.9}$$

Next we apply Lemma 3.3 for $\beta = 0$ and a cut-off function between $B_{15r/16}^\varepsilon$ and B_r^ε to get

$$\int_{B_{15r/16}^\varepsilon} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p-2}{2}} |\nabla^\varepsilon \nabla_\nu u_\varepsilon|^2 \, dx$$

$$\leq c \int_{B_r^\varepsilon} \left(\delta^2 + |u_\varepsilon|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{V}} u_\varepsilon|^2 \, dx$$
$$\leq c \int_{B_{2r}^\varepsilon} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}} \, dx.$$

We continue by applying Lemma 3.2 for $\beta = 1$ and a cut-off function between $B^\varepsilon_{7r/8}$ and $B^\varepsilon_{15r/16}$ to obtain

$$\int_{B^\varepsilon_{7r/8}} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}} |\nabla_{\mathcal{V}} u_\varepsilon|^2 \, dx$$
$$\leq c \left(1 + ||u_\varepsilon||^2_{L^\infty(B^\varepsilon_{2r})}\right) \int_{B^\varepsilon_{15r/16}} \left(\delta^2 + |u_\varepsilon|^2\right)^{\frac{p-2}{2}} |\nabla_{\mathcal{V}} u_\varepsilon|^2 \, dx$$
$$+ c\, ||u_\varepsilon||^2_{L^\infty(B^\varepsilon_{2r})} \int_{B^\varepsilon_{15r/16}} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p-2}{2}} |\nabla^\varepsilon \nabla_{\mathcal{V}} u_\varepsilon|^2 \, dx$$
$$\leq c \left(1 + ||u_\varepsilon||^2_{L^\infty(B^\varepsilon_{2r})}\right) \int_{B^\varepsilon_{2r}} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}} \, dx\,.$$

Then Lemma 3.4 for $\beta = 1$ and a cut-off function between $B^\varepsilon_{3r/4}$ and $B^\varepsilon_{7r/8}$ gives

$$\int_{B^\varepsilon_{3r/4}} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}} |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon|^2 \, dx$$
$$\leq c \int_{B^\varepsilon_{7r/8}} \left(\delta^2 + |u_\varepsilon|^2\right)^{\frac{p}{2}} |\nabla_{\mathcal{V}} u_\varepsilon|^2 \, dx + c \int_{B^\varepsilon_{7r/8}} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p+2}{2}} \, dx$$
$$\leq c \left(1 + ||u_\varepsilon||^2_{L^\infty(B^\varepsilon_{2r})}\right) \int_{B^\varepsilon_{2r}} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}} \, dx\,.$$

We apply once more Lemma 3.1 for $\beta = 1$ and a cut-off function between $B^\varepsilon_{r/2}$ and $B^\varepsilon_{3r/4}$ to get that

$$\int_{B^\varepsilon_{r/2}} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p+4}{2}} \, dx$$
$$\leq c \left(1 + ||u_\varepsilon||^4_{L^\infty(B^\varepsilon_{2r})}\right) \int_{B^\varepsilon_{2r}} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}} \, dx\,.$$

Finally, estimate (4.8) is given by some rescaling arguments. □

The next Moser iteration lemma gives a uniform upper bound for the derivatives in the non-horizontal directions. Notice the loss of homogeneity in (4.10), due to the fact that we have used the inequality $\delta^2 \leq \delta^2 + |\nabla^\varepsilon u_\varepsilon|^2$. It is at this point that $1/\delta$ appears in an essential way in our estimates.

Lemma 4.5. *Let us assume that* $2 \le p < \min\{\frac{2\nu}{\nu-1}, \frac{2Q+8}{Q-2}\}$, $\delta > 0$, $B^{\varepsilon}_{3r} \subset \Omega$ *and* $u_\varepsilon \in C^\infty(\Omega)$ *is a solution of* (2.8). *Then there exist a constant* c *depending on* ν, Q, p, L, r, δ *and* $||u_\varepsilon||_{L^\infty(B^\varepsilon_{2r})}$ *such that for* $a = \frac{2Q}{Q-2}$ *and* $b = \frac{p+4}{3}$ *we have*

$$||\nabla_{\mathcal{V}} u_\varepsilon||_{L^\infty(B^\varepsilon_{r/4})} \le c \left(\int_{B^\varepsilon_{2r}} (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p}{2}}\, dx \right)^{\frac{1}{p} + \frac{(p-2)a}{6p(a-b)}} . \tag{4.10}$$

Proof. Lemma 3.3 implies the following inequality.

$$\int_\Omega \eta^2 |\nabla_{\mathcal{V}} u_\varepsilon|^{2\beta} |\nabla^\varepsilon \nabla_{\mathcal{V}} u_\varepsilon|^2 dx$$
$$\le \frac{c}{\delta^{p-2}} \left(1 + ||\nabla^\varepsilon \eta||^2_{L^\infty(\operatorname{supp}\eta)}\right) \left(\int_{\operatorname{supp}\eta} (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p+4}{2}}\, dx \right)^{\frac{p-2}{p+4}}$$
$$\cdot \left(\int_{\operatorname{supp}\eta} |\nabla_{\mathcal{V}} u_\varepsilon|^{(2\beta+2)\frac{p+4}{6}}\, dx \right)^{\frac{6}{p+4}} .$$

Hence, for any cut-off function η with support included in B^ε_r, by Lemma 4.4 we have that

$$\int_\Omega \left| \nabla^\varepsilon \left(\eta |\nabla_{\mathcal{V}} u_\varepsilon|^{\beta+1} \right) \right|^2 dx$$
$$\le \frac{c}{\delta^{p-2}} \left(1 + ||\nabla^\varepsilon \eta||^2_{L^\infty(\operatorname{supp}\eta)}\right) \left(1 + ||u_\varepsilon||^4_{L^\infty(B^\varepsilon_{2r})}\right)^{\frac{p-2}{p+4}}$$
$$\cdot \left(\int_{B^\varepsilon_{2r}} (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p}{2}}\, dx \right)^{\frac{p-2}{p+4}} \left(\int_{\operatorname{supp}\eta} |\nabla_{\mathcal{V}} u_\varepsilon|^{(2\beta+2)\frac{p+4}{6}}\, dx \right)^{\frac{6}{p+4}} .$$

Noticing that the Poincaré inequality with exponent $\frac{2Q}{Q-2}$ can be used uniformly for all sufficiently small $\varepsilon > 0$ [Jer86, Theorem 2.1], we obtain that

$$\left(\int_\Omega \left(\eta |\nabla_{\mathcal{V}} u_\varepsilon|^{\beta+1} \right)^{\frac{2Q}{Q-2}} dx \right)^{\frac{Q-2}{2Q}}$$
$$\le \frac{c}{\delta^{\frac{p-2}{2}}} \left(1 + ||\nabla^\varepsilon \eta||_{L^\infty(\operatorname{supp}\eta)}\right) \left(1 + ||u_\varepsilon||^4_{L^\infty(B^\varepsilon_{2r})}\right)^{\frac{p-2}{2(p+4)}}$$
$$\cdot \left(\int_{B^\varepsilon_{2r}} (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p}{2}}\, dx \right)^{\frac{p-2}{2(p+4)}} \left(\int_{\operatorname{supp}\eta} |\nabla_{\mathcal{V}} u_\varepsilon|^{(\beta+1)\frac{p+4}{3}}\, dx \right)^{\frac{3}{p+4}} .$$

Let us define $\chi = \frac{a}{b}$, and notice that $\chi > 1$ iff $p < \frac{2Q+8}{Q-2}$. Also, by defining $\beta_0 + 1 = \frac{3p}{p+4}$, $\beta_k + 1 = (\beta_0 + 1)\chi^k$ and $\alpha_k = (\beta_k + 1)b$ we get that

$$\left(\int_\Omega \eta^a |\nabla_{\mathcal{V}} u_\varepsilon|^{\alpha_{k+1}}\, dx\right)^{\frac{1}{\alpha_{k+1}}}$$
$$\leq \left(\frac{c}{\delta^{\frac{p-2}{2}}} \left(1 + ||\nabla^\varepsilon \eta||_{L^\infty(\operatorname{supp}\eta)}\right)\right)^{\frac{b}{\alpha^k}} \left(1 + ||u_\varepsilon||^4_{L^\infty(B^\varepsilon_{2r})}\right)^{\frac{p-2}{6\alpha_k}}$$
$$\cdot \left(\int_{B^\varepsilon_{2r}} (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p}{2}}\, dx\right)^{\frac{p-2}{6\alpha_k}} \left(\int_{\operatorname{supp}\eta} |\nabla_{\mathcal{V}} u_\varepsilon|^{\alpha_k}\, dx\right)^{\frac{1}{\alpha_k}}.$$

Estimate (4.10) follows now from the standard Moser iteration. □

Lemma 4.6. *Let us assume that* $2 \leq p < \min\{\frac{2\nu}{\nu-1}, \frac{2Q+8}{Q-2}\}$, $\delta > 0$, $B^\varepsilon_{3r} \subset \Omega$ *and* $u_\varepsilon \in C^\infty(\Omega)$ *is a solution of* (2.8). *Then there exist a constant* c *depending on* ν, Q, p, L, r, δ *and* $||u_\varepsilon||_{L^\infty(B^\varepsilon_{2r})}$ *such that for* $a = \frac{2Q}{Q-2}$ *and* $b = \frac{p+4}{3}$ *we have*

$$||\nabla^\varepsilon u_\varepsilon||_{L^\infty(B^\varepsilon_{r/16})} \leq c \left(\int_{B^\varepsilon_{2r}} (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p}{2}}\, dx\right)^{\left(\frac{2}{p} + \frac{(p-2)a}{3p(a-b)}\right)\frac{Q}{p} + \frac{2}{p}}. \tag{4.11}$$

Proof. For any cut off function η with support included in $B^\varepsilon_{r/4}$, Lemma 3.4 and Lemma 4.5 imply that

$$\int \eta^2 \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p-2}{2}+\beta} |\nabla^\varepsilon \nabla^\varepsilon u_\varepsilon|^2\, dx$$
$$\leq c||\nabla_{\mathcal{V}} u_\varepsilon||^2_{L^\infty(B^\varepsilon_{r/4})} \int_{\operatorname{supp}\eta} (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p-2}{2}+\beta}\, dx + c\int_{\operatorname{supp}\eta} (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p}{2}+\beta}\, dx$$
$$\leq c\left(\frac{1}{\delta^2}\left(\int_{B^\varepsilon_{2r}} (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p}{2}}\, dx\right)^{\frac{2}{p} + \frac{(p-2)a}{3p(a-b)}} + 1\right) \int_{\operatorname{supp}\eta} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\frac{p}{2}+\beta}\, dx.$$

We define $\chi = \frac{Q}{Q-2}$, $\beta_0 = 0$ and $\alpha_k = \frac{p}{2} + \beta_k = \frac{p}{2}\chi^k$ for $k = 0, 1, 2, \cdots$. As we allow the constant c to depend on δ, we get that

$$\left(\int \eta^{2\chi} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\alpha_{k+1}}\, dx\right)^{\frac{1}{\alpha_{k+1}}}$$
$$\leq \quad c^{\frac{1}{\alpha_k}} \left(\left(\int_{B^\varepsilon_{2r}} (\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2)^{\frac{p}{2}}\, dx\right)^{\frac{2}{p} + \frac{(p-2)a}{3p(a-b)}}\right)^{\frac{1}{\alpha^k}} \left(\int_{\operatorname{supp}\eta} \left(\delta^2 + |\nabla^\varepsilon u_\varepsilon|^2\right)^{\alpha_k}\, dx\right)^{\frac{1}{\alpha_k}}.$$

Therefore, the standard Moser iteration leads to

$$||\nabla^{\varepsilon} u_{\varepsilon}||_{L^{\infty}(B^{\varepsilon}_{r/16})} \leq c \left(\int_{B^{\varepsilon}_{2r}} (\delta^2 + |\nabla^{\varepsilon} u_{\varepsilon}|^2)^{\frac{p}{2}} \, dx \right)^{\left(\frac{2}{p} + \frac{(p-2)a}{3p(a-b)}\right)\frac{Q}{p} + \frac{2}{p}} .$$

□

Theorem 4.1. *Let us assume that* $2 \leq p < \min\{\frac{2\nu}{\nu-1}\,,\ \frac{2Q+8}{Q-2}\}$, $\delta > 0$ *and* $u \in W^{1,p}(\Omega)$ *is a weak solution to the horizontal quasi-linear equation* (1.1).
Then we have $\nabla_{\mathcal{H}} u \in L^{\infty}_{\text{loc}}(\Omega)$.

Proof. Let $B_{3r} \subset \Omega$. Notice that for each small $\varepsilon > 0$, $B^{\varepsilon}_r \subset B_r$. Let $u_{\varepsilon} \in C^{\infty}(B_{2r})$ be the unique weak solution of the Dirichlet problem

$$\begin{cases} \displaystyle\sum_{i=1}^{\nu+1} X^{\varepsilon}_i (a_i(\nabla^{\varepsilon} u_{\varepsilon})) = 0\,, \text{ in } B^{\varepsilon}_{2r} \\ u_{\varepsilon} - u \in W^{1,p}_0(B^{\varepsilon}_{2r})\,. \end{cases} \tag{4.12}$$

Since the constants arising in the doubling property (2.9) and Poincaré inequalities are independent of ε, the Harnack inequality give bounds independent of ε for the local L^{∞} and local Hölder norms of weak solutions. A standard Cacciopoli inequality for (4.12) using the test function $\eta^p u_{\varepsilon}$, where η is the cut-off function between B^{ε}_r and B^{ε}_{2r}, gives bounds for $\int_{B^{\varepsilon}_r} (|\nabla^{\varepsilon} u_{\varepsilon}|^2)^{\frac{p}{2}} \, dx$ in terms of $\int_{B^{\varepsilon}_{2r}} (\delta^2 + |u_{\varepsilon}|^2)^{\frac{p}{2}} \, dx$ with constant independent of ε. Therefore, in estimate (4.11) we can let let $\varepsilon \to 0$ and get

$$||\nabla_{\mathcal{H}} u||_{L^{\infty}(B_{r/16})} \leq c \left(\int_{B_{2r}} (\delta^2 + |\nabla_{\mathcal{H}} u|^2)^{\frac{p}{2}} \, dx \right)^{\left(\frac{2}{p} + \frac{(p-2)a}{3p(a-b)}\right)\frac{Q}{p} + \frac{2}{p}} .$$

□

Proof of Theorem 1.1: Once Theorem 4.1 gives the local boundedness of the horizontal gradient of the weak solution $u \in W^{1,p}(\Omega)$, Theorem 1.1 follows from the results obtained in [Cap99, DM09].

Notice that the homogeneous dimension of a Goursat group of step ν is

$$Q = \frac{\nu^2 + \nu + 2}{2}\,.$$

Let us define

$$P(\nu) = \min\left\{\frac{2\nu}{\nu-1}\,,\ \frac{2Q+8}{Q-2}\right\} = \begin{cases} \frac{2\nu}{\nu-1} & \text{if } 3 \le \nu < 10 \\ \frac{2\nu^2+2\nu+20}{\nu^2+\nu-2} & \text{if } \nu \ge 10\,. \end{cases}$$

In a Goursat group, Theorem 1.1 implies the following result.

Corollary 4.1. *In a Goursat group of step ν, if $2 \le p < P(\nu)$, $\delta > 0$ and $u \in W^{1,p}(\Omega)$ is a weak solution to* (1.1)*, then $u \in C^\infty(\Omega)$.*

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