ENHANCED BOUNDARY REGULARITY OF PLANAR NONLOCAL MINIMAL GRAPHS AND A BUTTERFLY EFFECT

REGOLARITÀ ACCRESCIUTA AL BORDO DI GRAFICI MINIMI NONLOCALI NEL PIANO E UN EFFETTO FARFALLA

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ABSTRACT. In this note, we showcase some recent results obtained in [DSV19] concerning the stickiness properties of nonlocal minimal graphs in the plane. To start with, the nonlocal minimal graphs in the plane enjoy an enhanced boundary regularity, since boundary continuity with respect to the external datum is sufficient to ensure differentiability across the boundary of the domain.

As a matter of fact, the Hölder exponent of the derivative is in this situation sufficiently high to provide the validity of the Euler-Lagrange equation at boundary points as well.

From this, using a sliding method, one also deduces that the stickiness phenomenon is generic for nonlocal minimal graphs in the plane, since an arbitrarily small perturbation of continuous nonlocal minimal graphs can produce boundary discontinuities (making the continuous case somehow "exceptional" in this framework).

SUNTO. In questa nota, presentiamo alcuni risultati recenti ottenuti in [DSV19] relativi alla proprietà di "appiccicosità" dei grafici minimi nonlocali nel piano. I grafici minimi non locali nel piano godono di una regolarità "accresciuta" al bordo, in quanto la continuità al bordo rispetto al dato esterno è sufficiente a garantire la differenziabilità attraverso il bordo del dominio.

Inoltre, l'esponente di Hölder della derivata è sufficientemente grande da garantire la validità dell'equazione di Eulero-Lagrange anche ai punti di bordo del dominio.

Da ciò, usando un metodo di scivolamento, si ottiene anche che il fenomeno di appiccicosità è generico per grafici minimi non locali nel piano, nel senso che una perturbazione arbitrariamente piccola di i grafici minimi nonlocali continui produce discontinuità al bordo (rendendo quindi il caso continuo in qualche modo " eccezionale ").

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Three questions on planar nonlocal minimal graphs

Nonlocal minimal surfaces are a beautiful – and extremely challenging – topic of research. The novelty of the subject, together with its intrinsic cross-disciplinary nature, requires the combination of techniques from different fields, including calculus of variations, geometric measure theory, geometric analysis, differential geometry, partial differential and integro-differential equations. The solution of the problems posed by this intriguing scenario is usually based on brand new approaches and opens several perspectives in both pure and applied mathematics.

Moreover, nonlocal minimal surfaces offer a number of important, and very often surprising, differences with respect to the classical case. Among these differences, we believe that the ones related to new "boundary effects" are of particular importance, also in view of some "stickiness phenomena" that have been recently discovered and which seem to play a crucial role in the understanding of phenomena relying on long-range interactions. The goal of this note is to recall some recent results in this direction, and to describe the peculiar boundary situation exhibited by planar nonlocal minimal graphs.

To this end, we recall the definition of s-perimeter introduced in [CRS10]. Namely, given $s \in (0,1)$ and two measurable, disjoint sets A, $B \subseteq \mathbb{R}^n$, we define the nonlocal set-interaction as

$$I(A,B) := \iint_{A\times B} \frac{dx\,dy}{|x-y|^{n+s}}.$$

Also, if $\Omega \subset \mathbb{R}^n$ is a bounded set with Lipschitz boundary, and $E \subseteq \mathbb{R}^n$ is a measurable set, we define the s-perimeter of E in Ω as

$$\operatorname{Per}_s(E,\Omega) = I(E\cap\Omega, E^c\cap\Omega) + I(E\cap\Omega, E^c\cap\Omega^c) + I(E\cap\Omega^c, E^c\cap\Omega).$$

The name of s-perimeter for this type of functionals is motivated by the fact that, as $s \nearrow 1$, this functional recovers the classical notion of perimeter (in various forms, including functional estimates, Γ-convergence, density estimates, clean ball conditions, isoperimetric inequalities, etc., see [Bre02, BBM02, Dáv02, ADPM11, CV11]). On the other hand, as $s \searrow 0$, the s-perimeter is related to suitable weighted Lebesgue measures, in which the weights take into account the behavior of the set at infinity (see [MS02, DFPV13]), and these features already somewhat suggest that the problem for s close to 1 may be more "regular" and "close to classical variational problems" than the problem for s close to 0.

We say that E is s-minimal in Ω if

$$\mathrm{Per}_s(E,\Omega) \leqslant \mathrm{Per}_s(F,\Omega)$$

for every $F \subset \mathbb{R}^n$ such that $F \setminus \Omega = E \setminus \Omega$.

If $\tilde{\Omega} \subseteq \mathbb{R}^n$ is unbounded, one can also say that E is s-minimal in $\tilde{\Omega}$ if E is s-minimal in Ω, for all bounded Lipschitz sets $\Omega \Subset \tilde{\Omega}$. We refer to [Lom18] for a comprehensive description of these minimization problems.

The interior regularity theory of s-minimizers is an important topic of contemporary investigation, and complete results are available only in the plane, or when the fractional parameter s is sufficiently close to 1, see [SV13, CV13, BFV14]. See also [CSV19] for quantitative bounds and regularity results of BV type for stable solutions.

The theory of nonlocal minimal surfaces is also related to nonlocal isoperimetric problems (see [FLS08, FS08, FMM11, FFM+15, DCNRV15, CN17, CN18]), to fractional mean curvature equations (see [Imb09, CS10, DdPDV16, CFW18a, CFW18b, CFMN18, CFSMW18]) and to nonlocal geometric flows (see [CMP12, CMP13, CMP15, CNR17, FMP+18, CSV18, SV19, CDNV19, JLM19]).

Among all the possible minimization frameworks, the one of the graphs seems to play a special role, since it enjoys a number of structural features and can provide a solid guideline for the general theory. To introduce this setting, given a measurable function $u : \mathbb{R}^{n-1} \to \mathbb{R}$, we use the notation

$$E_u := \big\{(x_1, \dots, x_{n-1}, x_n) \in \mathbb{R}^n \text{ s.t. } x_n < u(x_1, \dots, x_{n-1})\big\}. \tag{1}$$

Then, given a bounded Lipschitz domain $\Omega_o \subset \mathbb{R}^{n-1}$, we say that u is s-minimal in Ω_o if E_u is s-minimal in $\Omega_o \times \mathbb{R}$.

The graphical case constitutes a useful building block for the general theory since it provides a "stable" framework to work with, in the sense that if E is a graph outside $\Omega_o \times \mathbb{R}$, then the s-minimizer in $\Omega_o \times \mathbb{R}$ is a graph as well, see [DSV16].

Also, the graphical structure poses some natural problems of Bernstein type (see [FV17, FV]) and enjoys several special regularity features (see [CC19]). See also [CL] for additional properties of nonlocal minimal surfaces and nonlocal minimal graphs.

In this note, for the sake of concreteness, we will focus on the planar[1] case, with the aim of highlighting the main features of s-minimal graphs in a slab. In this setting, given $u_0 : \mathbb{R} \to \mathbb{R}$, the typical problem is to understand the geometric properties of the minimizer u in $(0, 1)$ with $u = u_0$ in $\mathbb{R} \setminus (0, 1)$.

When $u_0 := 0$, the minimizer u vanishes identically, as proved in [CRS10] using a maximum principle argument (see also [Cab19, Pag19] for recently introduced calibration methods).

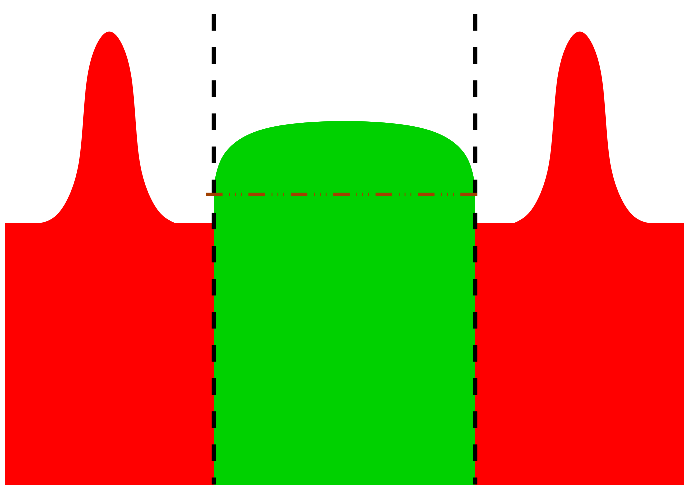

FIGURE 1. *Example of stickiness: initial problem with a datum with two small bumps.*

[1]The higher dimensional situation is structurally more complicated. The first attempt to describe the boundary behavior of higher dimensional nonlocal minimal surfaces can be found in [DSV].

An interesting question to address is what happens for small perturbations of the exterior datum u_0. The case of two small bumps was investigated in [DSV17], where it was established that u remains bounded away from zero in $(0, 1)$, see Figure 1. In particular, nonlocal minimal graphs do not necessarily meet continuously the boundary datum – sometimes they do, as it happens for the case $u_0 := 0$, but small perturbations of such a datum are sufficient to produce boundary discontinuities. Hence, the minimizing problem for nonlocal minimal graphs is well posed in the class of functions, but not in the class of continuous functions, since the s-minimal graph can turn out to be discontinuous at the boundary (and, as a matter of fact, this discontinuity is a jump, since the nonlocal minimal graphs are uniformly continuous inside the domain, see [DSV16]).

This feature is a special case of a general phenomenon that was named "stickiness" in [DSV17], emphasizing that, differently from the classical case, nonlocal minimal surfaces have the tendency to adhere at the domain (this may be also related to a capillarity effect, see also [DMV17, MV17] for a specific analysis of a nonlocal capillarity theory, and [BLV19, BL] for several examples of sticky behaviors of s-minimal surfaces).

The stickiness phenomenon detected in [DSV17], rather than constituting a final goal for the theory of nonlocal minimal surfaces, served as a key to disclose a number of new directions of investigation, including:

(Q1) *How regular are the nonlocal minimal graphs "coming from inside the domain"?*

(Q2) *Is the Euler-Lagrange equation coming from the variation of the s-perimeter satisfied "up to the boundary"?*

(Q3) *How "typical" is the stickiness phenomenon?*

We will show in this note that these questions are intimately correlated and the understanding of each of these problems sheds some light on the others.

The first results addressing (Q1) and (Q2) have been obtained in [CDSS16], in which it is shown that nonlocal minimal graphs, in the vicinity of discontinuity boundary points, can be written as differentiable graphs with respect to the vertical variable. Namely, if u

is s-minimal in $(0,1)$ with respect to a smooth datum $u_0 : \mathbb{R} \to \mathbb{R}$ and

$$u(0) := \lim_{x_1 \searrow 0} u(x_1) > u_0(0), \tag{2}$$

then there exist $\rho > 0$ and $v \in C^{1,\frac{1+s}{2}}(\mathbb{R})$ such that

$$\big\{x_1 \in (0,\rho),\ x_2 = u(x_1)\big\} = \big\{x_2 \in (u(0), u(\rho)),\ x_1 = v(x_2)\big\}, \tag{3}$$

with

$$v'(u(0)) = 0, \tag{4}$$

and a similar statement holds true when (2) is replaced by

$$u(0) < u_0(0). \tag{5}$$

We remark that, in particular, (3) says that u is invertible near the boundary discontinuity, and, in view of (4),

$$\lim_{x_1 \searrow 0} u'(x_1) = +\infty. \tag{6}$$

With respect to question (Q1), this says that at boundary discontinuities the derivative of u blows up, but the graph can be seen as the inverse of a $C^{1,\frac{1+s}{2}}$-function v which has a critical point in correspondence to the jump of u.

This fact can be used to provide a first answer to (Q2) at boundary discontinuities, since one can equivalently write the Euler-Lagrange equation "along the graph of v", and then pass it to the limit using the regularity of v (roughly speaking, the Euler-Lagrange equation involves a fractional curvature which is an object of order $1+s$; then, since $1 + \frac{1+s}{2} > 1+s$, a control in $C^{1,\frac{1+s}{2}}$ is sufficient to pass the equation to the limit). In this way, one obtains that the Euler-Lagrange equation is satisfied along the closed curve

$$\mathcal{C} := \overline{(\partial E_u) \cap ((0,1) \times \mathbb{R})} \tag{7}$$

provided that the solution has jump discontinuities at $x_1 = 0, 1$, see Theorem B.9 in [BLV19] for a precise statement.

After these preliminary considerations, it remains to address (Q1) and (Q2) at points of boundary continuity (this, as we will see, will also provide an answer to (Q3)).

The main result for (Q1) is that a continuous s-minimal graph is necessarily differentiable across the boundary (and, in fact, of class $C^{1,\frac{1+s}{2}}$). Indeed, as proved in [DSV19], we have that:

Theorem 1 (Enhanced boundary regularity for planar nonlocal minimal graphs: continuity implies differentiability)**.** *Let* $\beta \in (s,1)$ *and* $u_0 : \mathbb{R} \to \mathbb{R}$, *with*

$$u_0 \in C^{1,\beta}([-h,0]) \tag{8}$$

for some $h > 0$*. Assume that* u *is* s*-minimal in* $(0,1)$ *with datum* u_0*, and that*

$$u(0) := \lim_{x_1 \searrow 0} u(x_1) = u_0(0). \tag{9}$$

Then, $u \in C^{1,\gamma}([-h,1/2])$*, with*

$$\gamma := \min\left\{\beta, \frac{1+s}{2}\right\}. \tag{10}$$

When compared to the theory of fractional linear equations, the result in Theorem 1 is quite surprising since it says that continuity is sufficient for differentiability. This is in sharp contrast with the regularity of solutions of fractional Laplace equations such as

$$\begin{cases} (-\Delta)^s u = f & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

which are in general not better than Hölder continuous at the boundary, even when f is as smooth as we wish (see Figure 2, as well as [ROS14] for a thorough discussion of the boundary regularity).

For our purposes, it is interesting to observe that (2), (5) and (9) exhaust all the possible boundary behaviors, and the results in (3) and Theorem 1 always provide a regularity of $C^{1,\frac{1+s}{2}}$-type up to the boundary "in a geometric sense": namely, planar s-minimal graphs corresponding to *smooth external data* are *always* $C^{1,\frac{1+s}{2}}$*-curves* in the domain, *up to the boundary* of the domain, in the sense expressed by the following dichotomy:

- if a boundary discontinuity occurs, then the curve develops a vertical tangent at the boundary, as given in (6),

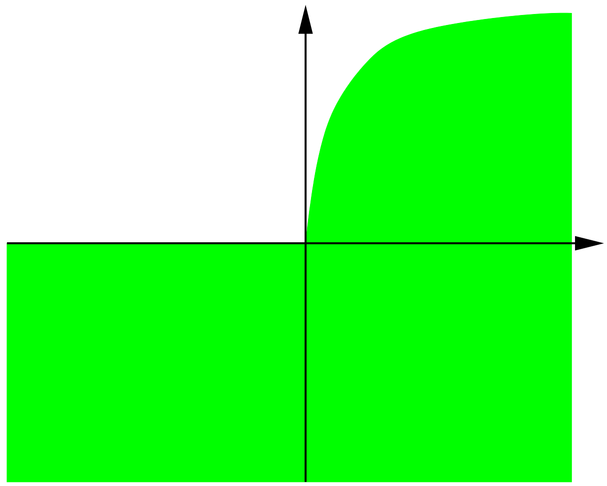

FIGURE 2. *Boundary behavior of an s-harmonic function in $(0,1)$ vanishing in $(-1,0]$, e.g. $u(x_1)=(x_1)_+^s$.*

- if the nonlocal minimal graph happens to be continuous at the boundary, then it is actually $C^{1,\frac{1+s}{2}}$ across the boundary.

More explicitly, we have the following result:

Theorem 2 (Regularity of s-minimal curves). *Let $u_0:\mathbb{R}\to\mathbb{R}$, with $u_0\in C^{1,\frac{1+s}{2}}([-h,0])$ for some $h>0$. Assume that u is s-minimal in $(0,1)$ with datum u_0.*

Then, the set $\mathcal{C}$ in (7) is a $C^{1,\frac{1+s}{2}}$-curve.

We observe that not only Theorem 2 provides a complete answer to question (Q1), but it also answers question (Q2), since one can write the Euler-Lagrange equation at the points in the interior of the domain and then use the regularity of the curve in Theorem 2 in order to reach the boundary of the domain. In this way, one obtains that:

Theorem 3 (Pointwise validity of the Euler-Lagrange equation). *Let $\beta\in(s,1)$ and $u_0:\mathbb{R}\to\mathbb{R}$, with $u_0\in C^{1,\beta}([-h,0])$ for some $h>0$. Assume that u is s-minimal in $(0,1)$ with datum u_0, and let $\mathcal{C}$ be as in (7).*

Then

$$\int_{\mathbb{R}^2}\frac{\chi_{\mathbb{R}^2\setminus E_u}(y)-\chi_{E_u}(y)}{|x-y|^{2+s}}\,dy=0 \tag{11}$$

for all $x=(x_1,x_2)\in\mathcal{C}$.

As customary, equation (11) can be considered the Euler-Lagrange equation associated with the nonlocal perimeter and its left hand side can be regarded as a nonlocal mean curvature (see e.g. [AV14, PPGS18, MRT19] for further geometric properties of this object).

Having settled questions (Q1) and (Q2) permits us to give an answer to question (Q3) as well, by exploiting a sliding method. Indeed, as proved in [DSV19], we have that the stickiness phenomenon is "generic", in the sense that any small perturbation of any exterior datum is sufficient to produce boundary discontinuities (hence, boundary continuity of planar nonlocal minimal graphs should be considered as an "exception" to the "typical" case in which boundary jumps occur). The precise statement of this result is the following:

Theorem 4 (Genericity of the stickiness phenomenon)**.** *Let u be an s-minimal graph in $(0,1)\times\mathbb{R}$ with smooth external datum u_0. Suppose that*

$$u_0(0) = 0 = \lim_{x_1\searrow 0} u(x_1).$$

Let $\varphi \in C_0^\infty((-2,1),[0,+\infty))$ be not identically zero. For every $t>0$, let $u^{(t)}$ be the s-minimal graph in $(0,1)\times\mathbb{R}$ with external datum $u_0+t\varphi$. Then,

$$\lim_{x_1\searrow 0} u^{(t)}(x_1) > 0. \tag{12}$$

We observe that (12) says that $u^{(t)}$ always presents the stickiness phenomenon for all $t>0$, being the case $t=0$ the only possible exception, namely a small positive bump always pushes up the planar nonlocal minimal graphs in a discontinuous way at the boundary. In other words, if the "unperturbed" minimizer is not sticky, then any positive, small and smooth perturbation of the datum will yield stickiness. In this sense, our answer to question (Q3) is that stickiness is indeed quite a "generic" phenomenon representing the "typical" boundary behavior of nonlocal minimal surfaces (with no counterpart in the theory of classical minimal surfaces).

It is also interesting to observe that (6) and Theorem 1, combined to Theorem 4, showcase a remarkable "butterfly effect" for the derivative of planar nonlocal minimal

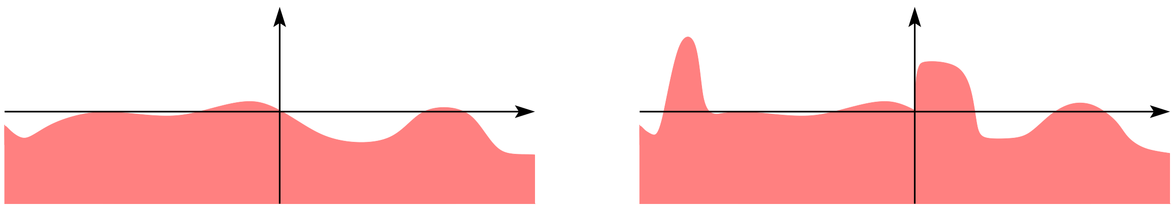

FIGURE 3. *The "butterfly effect" for the derivative of planar nonlocal minimal graphs (the additional external bump on the left in the second picture can be taken arbitrarily small, and arbitrarily far, still making the derivative infinite at the origin, here we have "magnified" this bump to improve the visibility effect).*

graphs: namely, if $u_0'(0^-) = \ell$, for some $\ell \in \mathbb{R}$, and there is no stickiness effect at the origin, then also $u'(0^+) = \ell$; but as soon as a small, and possibly faraway, bump is placed somewhere in the exterior datum, then suddenly $|u'(0^+)| = +\infty$, see Figure 3. In this sense, the stickiness phenomenon also produces generically the sudden divergence of the boundary derivative.

The proof of Theorem 4 relies on a vertical sliding method. Specifically, one slides u and then moves it till u touches $u^{(t)}$ at some point. Then, one reaches a contradiction using the Euler-Lagrange equation associated with the s-perimeter minimization: for this, it is however crucial to know that the Euler-Lagrange equation is indeed satisfied at any point, and this is exactly the step in which Theorem 3 comes into play.

Summarizing, question (Q1) concerning the boundary regularity of planar nonlocal minimal graphs is addressed in [CDSS16] for discontinuous graphs and in Theorem 1 for continuous graphs, thus leading to a general statement, valid both for continuous and discontinuous graphs, as given in Theorem 2, saying that the boundary of planar nonlocal minimal graphs is always a $C^{1,\frac{1+s}{2}}$-curve up to the boundary of the domain (in a geometric sense). This in turn provides an answer for question (Q2), as in Theorem 3, which ensures the validity of the Euler-Lagrange equation at any point of the domain (including points at the boundary of the domain, both in the case of continuous and discontinuous graphs). This fact then allows one to exploit sliding methods, proving the genericity of the stickiness phenomenon, thus answering question (Q3) as in Theorem 4.

To complete the picture, we now provide a sketch of the proof of Theorem 1, from which all the other results heavily depend.

Sketch of the proof of Theorem 1

For simplicity, let us suppose that u_0 is zero on a left neighborhood of the origin, say

$$u_0(x_1) = 0 \quad \text{for every } x_1 \in [-h, 0], \tag{13}$$

for some $h > 0$. We stress that (13) is a slightly simplifying assumption when compared to assumption (8), but the arguments presented here would carry over, up to technical complications, just assuming that u_0 is sufficiently smooth in a left neighborhood of the origin (full details available in [DSV19]).

Now, roughly speaking, the idea of the proof is to "look at the worst possible scenarios" and "rule out all the other possibilities".

To make this strategy concrete, we can consider the prototype situations embodied by the following[2] cases (see Figure 4):

(i) u has a jump discontinuity at the origin, thus exhibiting the stickiness phenomenon – but this occurrence is ruled out in this case by assumption (9);
(ii) u is Lipschitz continuous in $[-h, 1/2]$, but not better than this;
(iii) $u \in C^\alpha([-h, 1/2])$, for some $0 < \alpha < 1$, but not better than this;
(iv) $u \in C^1([-h, 1/2])$, but $u \notin C^{1,\gamma}([-h, 1/2])$.

Hence, to convince ourselves of the validity of Theorem 1, it is necessary to exclude the possibilities described in (ii), (iii) and (iv) (and also to obtain a uniform bound on the Hölder exponent of the derivative of u).

To do so, it is convenient to consider the blow-up limits corresponding to (ii), (iii) and (iv) and try to understand their relations with the original picture.

[2] We remark that these cases do not really exhaust all the possibilities, but they nevertheless provide a very good indication of what's going on in the general situation. For full details on the proof of Theorem 1, we refer to [DSV19].

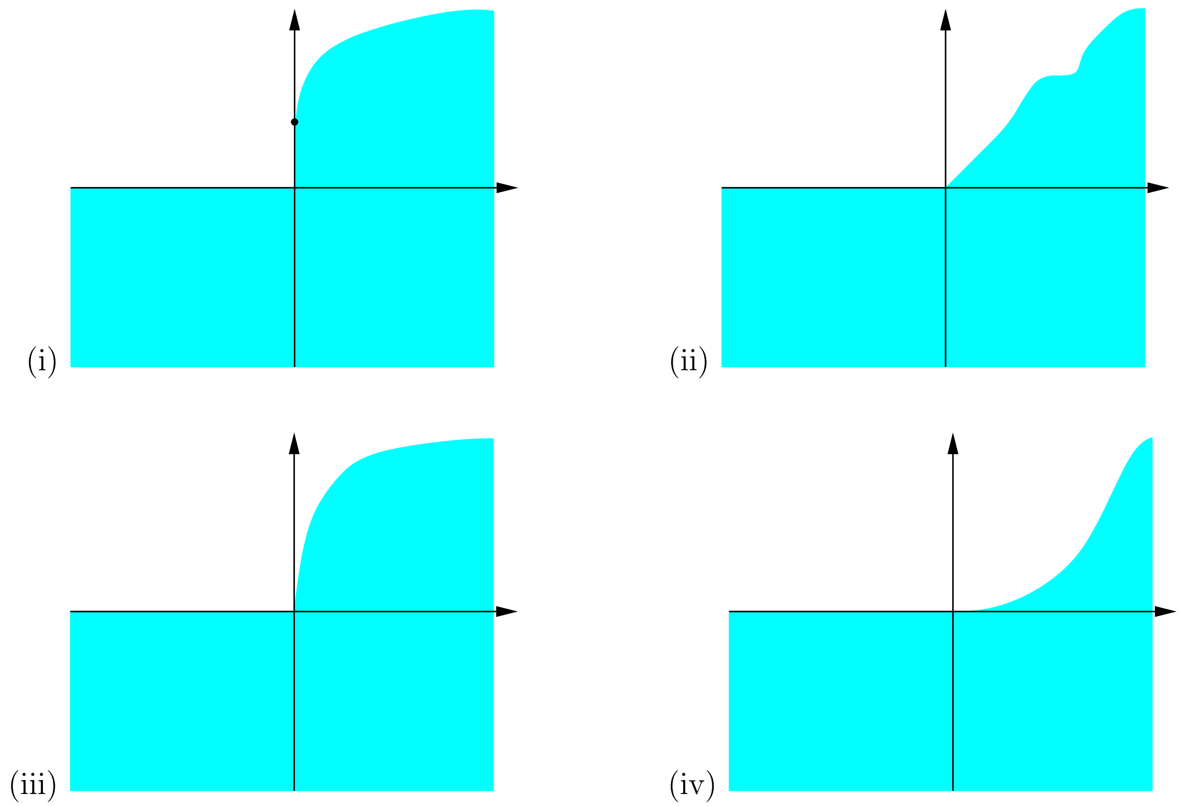

FIGURE 4. *What are the possible boundary behaviors of nonlocal minimal graphs in* $(0,1)$ *vanishing in* $(-1,0]$*?*

If E_u is as in (1), for the sake of shortness we denote it by E, and we define the blow-up sequence E_k of E, with $k \in \mathbb{N}$, as

$$E_k := kE = \{k(x_1, x_2), \ \ (x_1, x_2) \in E\}\,. \tag{14}$$

As a technical remark, we recall that the existence of the blow-up limit, that is the limit as $k \to +\infty$, possibly up to a subsequence, of the set in (14), typically follows from suitable density estimates (in this framework, these density estimates need to be centered at a boundary point, and the setting in (13) allows one to extend the interior estimates to the case under consideration, see Lemma 2.1 in [DSV19] for details).

Now, we would like to say that the blow-up limit is a cone. This usually relies on a specific monotonicity formula, and, in our framework, such a precise monotonicity formula is not available. To circumvent this difficulty, it is convenient to replace the previous blow-up limit with a second blow-up limit (that is, one considers a blow-up sequence obtained from the first blow-up limit, and then takes the limit of this new sequence). The advantage of this second blow-up procedure is that the first blow-up limit is already a halfplane in $\{x_1 < 0\}$, thanks to (13); consequently, every element of the new blow-up sequence is already a halfplane in $\{x_1 < 0\}$. From this, the proof of the monotonicity formula in [CRS10] carries over for the second blow-up sequence, thus ensuring that the second blow-up limit is indeed a cone (full details of this construction can be found in Lemma 2.2 in [DSV19]).

We denote[3] the second blow-up limit by E_{00}, and we recall that, in view of (13), we have that

$$E_{00} \cap \{x_1 < 0\} = \{x_1 < 0, \quad x_2 < 0\}.$$

See also Figure 5 for a description of the second blow-up limits corresponding to the possibilities depicted in Figure 4. Comparing with the possibilities (ii), (iii) and (iv), that should be ruled out in order to establish Theorem 1, we obtain the following scenarios for the second blow-up's:

(ii)' $E_{00} \cap \{x_1 > 0\} = \{x_2 < bx_1\} \cap \{x_1 > 0\}$, for some $b \in \mathbb{R} \setminus \{0\}$, which is the second blow-up limit corresponding to possibility (ii);

(iii)' $E_{00} \cap \{x_1 > 0\} = \{x_1 > 0\}$, which is the second blow-up limit corresponding to alternative (iii);

(iv)' $E_{00} = \{x_2 < 0\}$, that is E_{00} is a half-plane, which is the second blow-up limit corresponding to possibility (iv).

Hence, our sketch of the proof of Theorem 1 would be completed once we eliminate the possibilities in (ii)', (iii)' and (iv)'.

[3]We observe that, in the simplified framework presented here in Figures 4 and 5, there is no need for a second blow-up (but, in principle, in a more general situation one needs to carefully exploit a suitable monotonicity formula to check the homogeneous structures of the blow-up limits). As a matter of fact, one of the consequences of Theorem 1 is that all blow-up limits are classified (they are halfplanes in case no stickiness occurs, and right angles in case of stickiness) – hence, a posteriori, we will also know that the second blow-up reduces to the first one. Nonetheless, this second blow-up is technically convenient to prove Theorem 1.

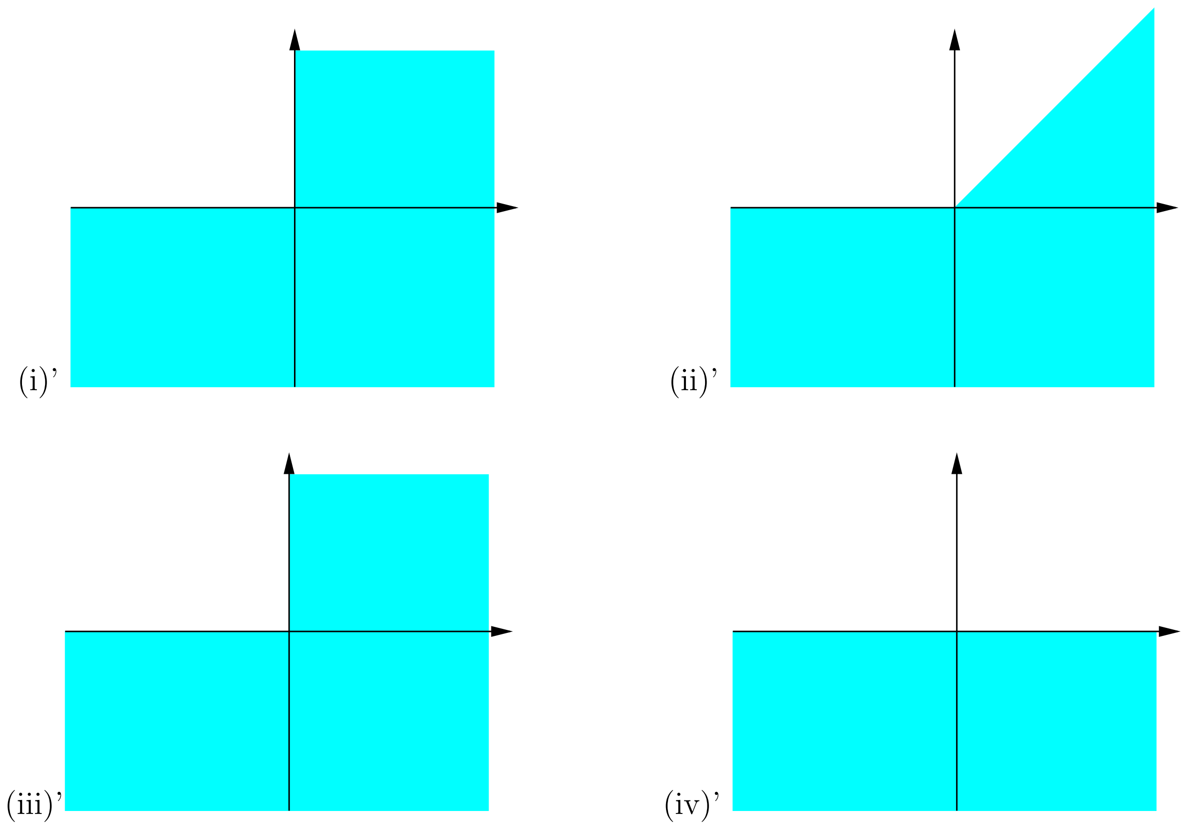

FIGURE 5. *Blow-up's of the four possibilities depicted in Figure 4.*

First, we proceed to exclude possibility (ii)'. For this, using the minimality of E_{00} in $\{x_1 > 0\}$ we have that

$$\int_{\mathbb{R}^2} \frac{\chi_{\mathbb{R}^2\setminus E_{00}}(y) - \chi_{E_{00}}(y)}{|p-y|^{2+s}}\, dy = 0, \tag{15}$$

where $p := (1, b)$.

On the other hand, we see that

$$\int_{\mathbb{R}^2} \frac{\chi_{\mathbb{R}^2\setminus E_{00}}(y) - \chi_{E_{00}}(y)}{|p-y|^{2+s}}\, dy \neq 0, \tag{16}$$

since the contribution of the set and the one of its complement do not cancel each other (compare the symmetric regions arising after drawing the tangent line passing through p).

The contradiction arising from (15) and (16) rules out possibility (ii)', and we now want to exclude possibility (iii)'. We observe that, to exclude this possibility, one cannot only rely on blow-up type analysis, since the same blow-up limit as the one in (iii)' is also achieved when u is discontinuous: that is, possibility (i) would produce the same blow-up picture as possibility (iii), but the original nonlocal minimal graphs present obvious structural differences. Therefore, the strategy to eliminate (iii)' has to take into account the original sets with a finer analysis, and indeed we aim at showing that possibility (iii)' can only come from discontinuous nonlocal minimal graph u (and this possibility, corresponding to (i), was already ruled out in light of (9)).

In this sense, the strategy to eliminate possibility (iii)' consists in proving that "thick s-minimal sets are necessarily full" (or, equivalently, considering complement sets, "narrow s-minimal sets are necessarily void"). The precise statement, which is a particular case of Proposition 3.1 in [DSV19], goes as follows:

Proposition 5. *Let $\lambda > 0$. There exist $M_0 > 1$ and $\mu_0 \in (0,1)$ such that if $M \geqslant M_0$ and $\mu \in (0,\mu_0]$ the following claim holds true.*

Let $F \subset \mathbb{R}^2$ be s-minimal in $(0,M) \times (-4,4)$. If

$$F \cap \{x_1 \in (-M,0)\} \;=\; \{x_2 \leqslant 0\}\,, \tag{17}$$

and

$$\Big((0,M) \times (-M,M)\Big) \setminus F \;\subseteq\; \{x_1 \in (0,\mu)\}\,, \tag{18}$$

then

$$\left(0, \frac{M}{2}\right) \times (-1,1) \;\subseteq\; F. \tag{19}$$

The idea to prove Proposition 5 is to argue by contradiction exploiting a sliding method. Namely, if the thesis in (19) were false, one could take a ball inside F and slide it till it touches the complement of F at some point q. In this framework, one obtains the existence of a ball $B \subseteq F$, with $q \in \partial B \cap \partial F$. The strategy is to show that the s-mean curvature of F at q is strictly negative, thus contradicting the minimality of F.

To compute the s-mean curvature of F in q, it is convenient to consider the symmetric ball to B with respect to the tangent plane through q and denote it by B', see Figure 6.

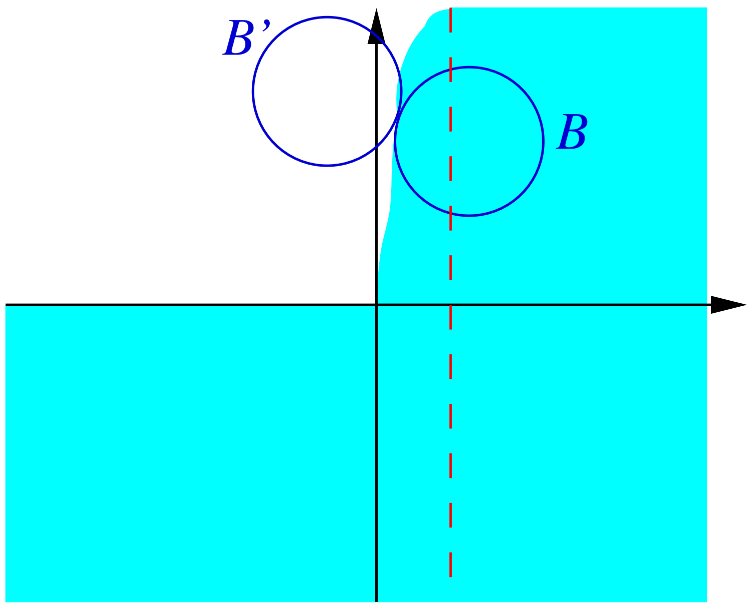

FIGURE 6. *Using symmetric balls to detect integral cancellations.*

By hypothesis (18), we know that the complement of F (in the domain $\{x_1 > 0\}$, up to a large cylinder) is contained in a small slab near the vertical axis. Hence, the integral contributions for the s-mean curvature (recall the left hand side in (11)) are "mostly negative", coming predominantly from points in the set F, with the possible exception of the points in the complement lying in the small slab $\{x_1 \in (0, \mu)\}$. Near the contact point q, the positive contributions coming from these points are "negligible" as long as μ is sufficiently small, since the singularity of the kernel is compensated by the symmetric integration over the balls B and B', with the full ball B providing negative contributions. Similarly, far from q, the negative terms coming from the set F provide a negligible contribution to the s-mean curvature, since the singularity of the kernel plays little role away from q, and the weighted measure of the narrow slab is small with μ.

These quantitative arguments establish Proposition 5 (see again Proposition 3.1 in [DSV19] for full details). With this, one can rule out possibility (iii)' by arguing as follows. From (iii)', one knows that the blow-up limit in $\{x_1 > 0\}$ is either full or void. Let us consider the first case (up to changing a set with its complement), namely suppose that

$$E_{00} \cap \{x_1 > 0\} = (0, +\infty) \times \mathbb{R}. \tag{20}$$

Then, up to a subsequence, a suitable blow-up sequence E_k (recall (14)) would lie locally in the vicinity of E_{00} for a suitably large k. From this and (20), one sees that E_k satisfies (18) for k sufficiently large (possibly depending on the thresholds M_0 and μ_0 in Proposition 5, and notice also that in this setting E_k satisfies (17) as a consequence of (13)). Then, one can apply Proposition 5 to $F := E_k$. Consequently, from (17) and (19), one obtains that the graph describing E_k has a jump discontinuity at the origin. Scaling back, this gives that u has a jump discontinuity at the origin. This is in contradiction with (9), and therefore possibility (iii)' (and hence (iii)) has been excluded.

It remains to rule out possibility (iv)' (and hence (iv)). To this end, we need to prove that once the blow-up limit is a half-plane, then necessarily the original s-minimal graph was already differentiable at the origin, with a precise estimate on the Hölder exponent of the derivative (we stress that controlling the Hölder exponent of the derivative is a crucial step in order to deduce the results in Theorems 2, 3 and 4 from Theorem 1).

The idea of the proof now consists in using "vertical rescalings" for an "improvement of flatness" (once we know that the solution is sufficiently flat at a large scale, then it is necessarily even flatter at a smaller scale). Differently than other improvement of flatness methods, which were designed in the interior of the domain (see [CRS10]), our setting requires us to achieve this enhanced regularity at boundary points. To this aim, one considers vertical rescalings and proves convergence to some function $\bar{u}$, which satisfies $(-\Delta)^\sigma \bar{u} = 0$ in $(0, +\infty)$, with $\sigma = \frac{1+s}{2}$, and $\bar{u} = 0$ in $(-\infty, 0)$. The linear theory of fractional equations (see e.g. [ROS14]) would only ensure that $\bar{u}$ is Hölder continuous at the origin, but our objective is to prove that in fact $\bar{u}$ is more regular, thus producing the desired enhanced regularity for the original function u, by bootstrapping such improvement of flatness method.

Concretely, one deduces from the linear theory of fractional equations that, for small $x_1 > 0$,

$$\bar{u}(x_1) = a_0 x_1^{\sigma} + O(x_1^{\sigma+1}) \tag{21}$$

for a suitable $a_0 \in \mathbb{R}$. Our goal is to show that

$$a_0 = 0, \tag{22}$$

thus improving the boundary regularity in this specific case. To do this, we construct a suitable corner-like barrier (see Figure 7 here, and Lemma 7.1 in [DSV19] for full details).

Roughly speaking, one can juggle parameters to make the subgraph depicted in Figure 7 have negative fractional mean curvature in the vicinity of the origin. Intuitively, this is possible thanks to a "purely nonlocal effect": indeed, in the classical case, the segment near the origin in Figure 7 would produce a zero curvature (thus making the argument invalid for classical minimal surfaces), while in the nonlocal setting the concave corner at the origin produces a very negative fractional curvature (actually, equal to $-\infty$ at the origin). This negative contribution survives after the bending of the barrier at the side of Figure 7 (which is needed in order to place the barrier "below the solution at infinity").

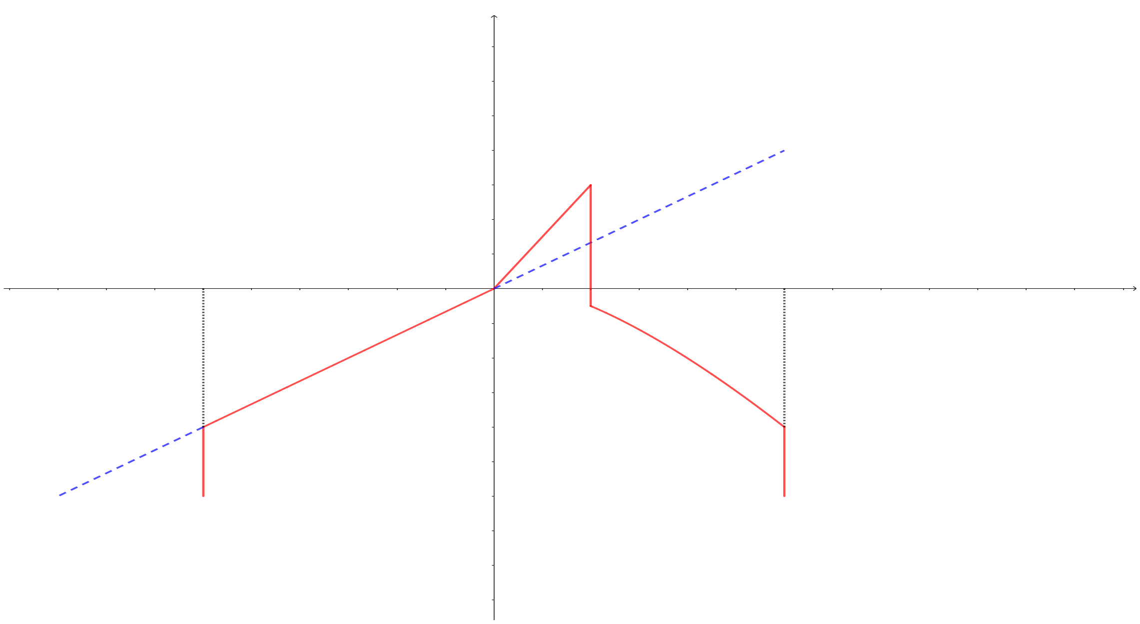

FIGURE 7. *Shape of the corner-like barrier.*

Then, to prove (22), one argues by contradiction, supposing, for instance, that $a_0 > 0$. Then, using (21), one sees that the barrier in Figure 7 can be slided from below the original s-minimal graph u. By maximum principle (and noticing the linear growth of the barrier in Figure 7 for $x_1 > 0$ small), this gives that u lies above a linear function for $x_1 > 0$ small. Consequently, the corresponding blow-up limit would be as in possibility (ii)'. Since this alternative has been already ruled out, we obtain a contradiction, thus establishing (22).

From (22), the improvement of flatness method kicks in, thus producing the desired enhancement regularity result that rules out the last possibility, finally leading to the completion of the proof of Theorem 1.

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