SOME GLOBAL SOBOLEV INEQUALITIES RELATED TO KOLMOGOROV-TYPE OPERATORS
SU ALCUNE DISUGUAGLIANZE DI SOBOLEV GLOBALI RELATIVE AD OPERATORI DI TIPO KOLMOGOROV

GIULIO TRALLI

Abstract. In this note we review a recent result in [17] in collaboration with N. Garofalo, where we establish global versions of Hardy-Littlewood-Sobolev inequalities attached to hypoelliptic equations of Kolmogorov type. The relevant Sobolev spaces are defined through the fractional powers of the operator under consideration. We outline the main steps of the semigroup approach we adopt.

Sunto. Viene qui presentato un recente risultato ottenuto in [17] in collaborazione con N. Garofalo, in cui si dimostrano disuguaglianze globali di tipo Hardy-Littlewood-Sobolev relative ad una classe di operatori ipoellittici di tipo Kolmogorov. Nell'approccio adottato gli spazi di Sobolev sono definiti attraverso le potenze frazionarie dell'operatore in questione.

2010 MSC. Primary 35B45; Secondary 46E35.

Keywords. Global a priori estimates, Kolmogorov-Fokker-Planck diffusion, fractional powers of hypoelliptic operators.

1. Introduction

The importance of a priori estimates is well-recognized in various fields. They play, for example, a crucial role in order to establish existence and regularity results for solutions to linear and nonlinear partial differential equations. Sobolev inequalities, among (and more than) others, occupy also a central position in several geometric analysis problems. Typically, the aim is to control a certain L^q norm of a function in terms of a L^p norm of its derivative. Let us recall here the classical Sobolev inequality in $\mathbb{R}^N$ which can be read

Bruno Pini Mathematical Analysis Seminar, Vol. 11-1 (2020) pp. 143–156

Dipartimento di Matematica, Università di Bologna

ISSN 2240-2829.

as follows: for any $1 \leq p < N$ there exists a constant $S_{N,p}$ such that, for any function f in the Schwartz class $\mathscr{S}$, one has

$$\|f\|_q \leq S_{N,p} \, \|\nabla f\|_p \quad \text{if (and only if)} \qquad \frac{1}{p} - \frac{1}{q} = \frac{1}{N}.$$

The relation between the exponents $\frac{1}{p} - \frac{1}{q} = \frac{1}{N}$ is the well-known Hardy-Littlewood-Sobolev condition, and it implies a gain in the exponent of integrability (i.e. $q > p$) in the embedding $W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$. The Hardy-Littlewood-Sobolev condition is also known to be intimately connected with the interplay between the differential operator ∇ and the homogeneous structure of $\mathbb{R}^N$. Furthermore, we keep in mind that the Laplace operator $\Delta = \operatorname{div}(\nabla \cdot)$ appears naturally in the Euler-Lagrange equation of energies involving the Dirichlet term $\|\nabla f\|_2^2$.

We want to describe new global a priori estimates established in [17] which are related to a class of linear diffusion operators. For the purpose of this exposition we want to single out, as particular cases of the class under consideration, the following simple-looking operators in $\mathbb{R}^2$ (with generic point denoted by (v, x))

- $\mathscr{A}_0 = \partial_v^2 + v\partial_x$,
- $\mathscr{A}_+ = \partial_v^2 + v\partial_x + v\partial_v$,
- $\mathscr{A}_\theta = \partial_v^2 + v\partial_x - x\partial_v$.

Notice that these operators are degenerate elliptic since they are missing any type of control on the second derivative ∂_x^2. They also possess a linear first-order drift term, which prevents in general from writing the operators as first variation of reasonable Dirichlet-type energies. We will see in Corollary 1.1 below that the Sobolev-type inequalities we have attached to these three operators are very different from each other. The main reason is that the underlying geometries (in particular the volumes of the intrinsic pseudo-balls), even if they display a similar behavior for small scales, behave very differently in large scales (see Remark 2.2). Before introducing all the relevant notions, let us describe the whole class of equations we want to consider.

Denote by X the generic point in $\mathbb{R}^N$, $N \geq 2$. We use the notations $\operatorname{tr} M$ and M^* to indicate, respectively, the trace and the transpose of a $N \times N$ matrix M. Let Q and B be two given $N \times N$ matrices with real constant coefficients. Assume $Q = Q^* \geq 0$, together with the following two conditions:

$$K(t) := \frac{1}{t}\int_0^t e^{sB} Q e^{sB^\star} ds > 0 \quad \text{for every } t > 0, \tag{1}$$

$$\operatorname{tr} B \geq 0. \tag{2}$$

We can then define

$$\mathscr{A} u = \operatorname{tr}(Q\nabla^2 u) + < BX, \nabla u > . \tag{3}$$

It is well-known that condition (1) is equivalent to Hörmander's finite rank condition for the operators in (3) (see e.g. [20, 25]; see also [15]). The equations under consideration are thus hypoelliptic, and they were in fact discussed by Hörmander in the introduction of his celebrated hypoellipticity paper [20]. Nonetheless, the main examples of degenerate equations in the form (3) were studied in seminal papers in [24, 9] in view of applications in physics, astronomy, probability, kinetic theory of gases. Several aspects of this class of equations (and its parabolic counterpart) have been investigated through the years, and we refer to the survey papers [27, 4] for a detailed account of the literature. We just mention here the results concerning interior pointwise estimates and Harnack inequalities for solutions in [13, 25], and the potential theory issues and boundary estimates addressed in [31, 21, 22, 23]. Moreover, nonlinear equations modeled after the operators in (3) were studied in [26, 29, 11], and regularity estimates for linear operators with variable matrix-coefficients $Q(\cdot)$ with bounded measurable entries were proved under various assumptions in [32, 10, 36, 19, 28, 2, 1, 3].

Besides the classical Laplace operator ($Q = \mathbb{I}_N$, $B = \mathbb{O}_N$), the most relevant example in this class is perhaps the one introduced by Kolmogorov in [24] and it corresponds with the choices

$$N = 2n, \quad Q = \begin{pmatrix} \mathbb{I}_n & \mathbb{O}_n \\ \mathbb{O}_n & \mathbb{O}_n \end{pmatrix}, \quad B = \begin{pmatrix} \mathbb{O}_n & \mathbb{O}_n \\ \mathbb{I}_n & \mathbb{O}_n \end{pmatrix}.$$

Notice that this is a genuinely degenerate-elliptic operator since Q has a n-dimensional kernel, and also that $\operatorname{tr} B = 0$. In the phase-space variables $X = (v, x) \in \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^n$

commonly used in kinetic theory, it can be written as

$$\Delta_v + \langle v, \nabla_x \rangle .$$

Thus, $\mathscr{A}_0$ is just the $N = 2$ (i.e. $n = 1$)-dimensional case of the Kolmogorov operator. If we fix $N = 2$ and the expected notations, the choices

$$Q_0 = Q_+ = Q_\theta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

$$B_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_+ = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \quad B_\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4}$$

define in fact the three operators $\mathscr{A}_0, \mathscr{A}_+, \mathscr{A}_\theta$. Other relevant examples of equations satisfying our conditions can be found in [17, Section 3 (Figure 1)].

As we mentioned, we want to discuss global a priori estimates attached to the operators $\mathscr{A}$ defined in (3) under conditions (1)-(2). The condition (2) allows us, very roughly speaking, to fix the Lebesgue measure as the reference measure for the L^p-spaces in our Sobolev-type estimates. As a matter of fact, one should keep in mind that for the classical Ornstein-Uhlenbeck operator (which corresponds to $Q = \mathbb{I}_N$, $B = -\mathbb{I}_N$ and thus $\operatorname{tr} B = -N < 0$) Sobolev inequalities are known to hold true in L^p-spaces with respect to the standard Gaussian measure (see, e.g., [6]). Throughout this work, instead, the Lebesgue measure has to be considered as fixed. The real focus in our investigation is the understanding of the right replacement for the 'gradient' term in the energy estimates. For instance, if we try to replace the term $|\nabla f(X)|^2$ with a tool like the P.A. Meyer *carré du champ* $\Gamma(f) = \frac{1}{2}[\mathscr{A}(f^2) - 2f\mathscr{A}f]$, we realize that this is not directly effective here since $\Gamma(f) = \langle Q\nabla f, \nabla f\rangle$ which misses all directions of non-ellipticity in the degenerate case (note that $\langle Q\nabla f, \nabla f\rangle$ means for example $(\partial_v f)^2$ for $\mathscr{A}_0, \mathscr{A}_+$ and $\mathscr{A}_\theta$). Furthermore, being independent from the matrix B, $\langle Q\nabla f, \nabla f\rangle$ cannot provide any control on the drift. In this respect, it is worth mentioning that in [32, 10] some localized energy estimates of Sobolev-type have been proved and exploited to get pointwise bounds for solutions to parabolic equations modeled after (3): in these works the authors bound the L^p-norm of f on suitable bounded sets with an energy term exactly of the form $\langle Q\nabla f, \nabla f\rangle$ for all functions f which are nonnegative solutions/subsolutions of the equation (and not for generic functions). Let us also refer the interested reader to the results in [16, Corollary 4.2] where we proved some localized $(2,2)$-Poincaré inequalities for arbitrary smooth functions. Having in mind on one hand this lack of an obvious notion of 'gradient' related to $\mathscr{A}$, and on the other hand the classical result about the equivalence of the L^p-norms

$$\text{for } p>1 \qquad c_p \|\nabla f\|_p \leq \left\|\sqrt{-\Delta} f\right\|_p \leq C_p \|\nabla f\|_p \qquad \forall f \in \mathscr{S}$$

provided by the Calderón-Zygmund theory (in particular by the L^p-continuity of the Riesz transforms), we seek Sobolev-type estimates involving the term

$$\left\|\sqrt{-\mathscr{A}} f\right\|_p .$$

A fractional calculus related to the operators $\mathscr{A}$ in (3) has been developed in [15], and we are going to recall the definition of $\sqrt{-\mathscr{A}} = (-\mathscr{A})^{\frac{1}{2}}$ in Definition 2.1 below. The precise statements of the Sobolev inequalities established in [17] will be given instead in Theorem 2.1 and Theorem 2.2, after having introduced the needed objects coming into play. Here, Theorem 2.1-2.2 are restated in the following corollary in such a way the results are applied to the model operators defined through the choices in (4) and they are specialized to the cases $s = \frac{1}{2}$ and $p > 1$.

Corollary 1.1. *Consider in $\mathbb{R}^2$ the operators $\mathscr{A}_0, \mathscr{A}_+, \mathscr{A}_\theta$ defined above. We have the following:*

- *for any $1 < p < 4$, let $q > p$ be such that $\frac{1}{p} - \frac{1}{q} = \frac{1}{4}$; then*

$$\|f\|_q \leq S_0 \left\|\sqrt{-\mathscr{A}_0} f\right\|_p \qquad \textit{for every } f \in \mathscr{S}.$$

- *for any $D \geq 4$ and $1 < p < D$, let $q > p$ be such that $\frac{1}{p} - \frac{1}{q} = \frac{1}{D}$; then*

$$\|f\|_q \leq S_+ \left\|\sqrt{-\mathscr{A}_+} f\right\|_p \qquad \textit{for every } f \in \mathscr{S}.$$

- *for any $1 < p < 2$, let $\bar{q} > q > p$ be such that $\frac{1}{p} - \frac{1}{\bar{q}} = \frac{1}{2}$ and $\frac{1}{p} - \frac{1}{q} = \frac{1}{4}$; then*

$$\|f\|_{L^q + L^{\bar{q}}} \leq S_\theta \left\|\sqrt{-\mathscr{A}_\theta} f\right\|_p \qquad \textit{for every } f \in \mathscr{S}.$$

Notice that $\mathscr{A}_0$ behaves like the Laplacian in dimension 4 (4 is the homogeneous dimension which $\mathbb{R}^2$ inherits from $\mathscr{A}_0$), whereas the behavior of $\mathscr{A}_+$ resembles the one of a Laplace-Beltrami operator in a 4-dimensional manifold which is negatively curved (see, e.g., [30]). The unusual estimate related to $\mathscr{A}_\theta$ will be clarified in the next section, together with the explanation of the main results.

2. Description of the results

The approach we adopt is a semigroup-approach, and it has been influenced by the ideas of E. Stein in [34] and Varopoulos in [35] in the setting of positive symmetric semigroups. As a matter of fact, if we denote

$$\mathscr{K} = \mathscr{A} - \partial_t \qquad \text{in } \mathbb{R}^{N+1} \ni (X,t), \tag{5}$$

the Cauchy problem related to $\mathscr{K}$ admits a unique solution for any datum $f \in \mathscr{S}$ at initial time $t = 0$. This generates a strongly continuous semigroup $\{P_t\}_{t>0}$ on L^p defined by

$$P_t f(X) = \int_{\mathbb{R}^N} p(X,Y,t) f(Y) dY,$$

where $p(X,Y,t)$ is the transition kernel (i.e. the fundamental solution of $\mathscr{K}$ with pole at $(Y,0)$). An important fact, which makes very effective working with the semigroup associated to $\mathscr{A}$, is that the transition kernel is known explicitly. In [20] the kernel is in fact constructed via a Fourier analysis, and it was already known to Kolmogorov at least for the equation under consideration in [24]. In our notations, for any $X, Y \in \mathbb{R}^N$ and $t > 0$, we have

$$p(X,Y,t) = \frac{(4\pi)^{-\frac{N}{2}}}{\sqrt{\det(tK(t))}} \exp\left(-\frac{\langle K^{-1}(t)\left(Y - e^{tB}X\right), Y - e^{tB}X\rangle}{4t} \right).$$

We can note from the previous formula also the importance of the assumption (1). Sometimes one can find in the literature a slightly different expression for $p(X,Y,t)$, and in particular the following one

$$p(X,Y,t) = \frac{(4\pi)^{-\frac{N}{2}} e^{-t \operatorname{tr} B}}{\sqrt{\det(C(t))}} \exp\left(-\frac{\langle C^{-1}(t)\left(X - e^{-tB}Y\right), X - e^{-tB}Y\rangle}{4} \right),$$

where $C(t) = \int_0^t e^{-sB} Q e^{-sB^\star} ds$. The equivalence of the two expressions can be checked by using the relation $tK(t) = e^{tB} C(t) e^{tB^\star}$. For our purposes it is convenient to rewrite the transition kernel, for $X, Y \in \mathbb{R}^N$ and $t > 0$, as follows

$$p(X, Y, t) = \frac{c_N}{V(t)} \exp\left(-\frac{m_t^2(X, Y)}{4t}\right) \tag{6}$$

where c_N is a positive constant and

$$V(t) = \omega_N \sqrt{\det\left(tK(t)\right)} = \mathrm{Vol}_N(B_t(X, \sqrt{t})), \qquad B_t(X, r) = \{Y \in \mathbb{R}^N \mid m_t(X, Y) < r\},$$

$$m_t(X, Y) = \sqrt{\langle K(t)^{-1}(Y - e^{tB}X), Y - e^{tB}X\rangle}.$$

Following Balakrishnan [5], we studied in [15] the fractional powers of $\mathscr{A}$ by exploiting the properties of the semigroup $\{P_t\}_{t>0}$. We recall here the definition, where we indicate by $\Gamma(\cdot)$ the Euler's gamma-function.

Definition 2.1. *Let $0 < s < 1$. For any $f \in \mathscr{S}$ we define the nonlocal operator $(-\mathscr{A})^s$ by the following pointwise formula*

$$(-\mathscr{A})^s f(X) = -\frac{s}{\Gamma(1-s)} \int_0^\infty t^{-(1+s)} \left(P_t f(X) - f(X)\right) dt, \qquad X \in \mathbb{R}^N.$$

In [15, Section 3] we showed that the previous integral is in fact convergent, and it also defines an L^p-function for any $p \in [1, +\infty]$ whenever (2) holds (see also [17, Lemma 4.3]). Some notable properties of the nonlocal operators driven by the fractional powers of $\mathscr{A}$ and $\mathscr{K}$ have been proved in [15, 16, 8]. In [18] we made use of the operators $(-\mathscr{A})^s$ to introduce a notion of nonlocal perimeter associated to $\mathscr{A}$ and to establish nonlocal isoperimetric inequalities. Fractional powers of hypoelliptic operators of different nature, namely subLaplacians in Carnot groups and sums of squares of Hörmander vector fields, have been treated in [12, 14].

After having introduced $(-\mathscr{A})^s$, we can now take from [17, Section 4] the definition of the Sobolev spaces associated to $\mathscr{A}$ (see also [18, Proposition 2.13]).

Definition 2.2. *Let $1 \le p < +\infty$ and $0 < s < 1$. For $f \in \mathscr{S}$ we denote*

$$\|f\|_{\mathscr{L}^{2s,p}} = \|f\|_{L^p(\mathbb{R}^N)} + \|(-\mathscr{A})^s f\|_{L^p(\mathbb{R}^N)}.$$

We can then define the Banach space $\mathscr{L}^{2s,p}$ as the completion of the space of functions defined in $\mathbb{R}^N$ belonging to the Schwartz class $\mathscr{S}$ with respect to the norm $\|\cdot\|_{\mathscr{L}^{2s,p}}$, i.e.

$$\mathscr{L}^{2s,p} = \overline{\mathscr{S}}^{\|\ \|_{\mathscr{L}^{2s,p}}}.$$

The advantages of adding the time-variable and working with the operator $\mathscr{K}$ go beyond the explicit representation for the solutions of the Cauchy problem via (6). In $\mathbb{R}^{N+1}$ the underlying geometry associated with the class of operators we are considering becomes in fact more clear. It was shown in [25] that $\mathscr{K}$ is left-invariant with respect to the following Lie-group law in $\mathbb{R}^{N+1}$

$$(X,t) \circ (Y,\tau) = \left(Y + e^{-\tau B}X, t+\tau\right). \tag{7}$$

Moreover, Lanconelli and Polidoro identified and characterized in [25] the subclass of operators $\mathscr{K}$ in (5) which are also homogeneous of degree 2 with respect to power-like dilations (Kolmogorov's example falls in particular in such homogeneous class).

Remark 2.1. *Given an operator which is invariant with respect to a homogeneous Lie group structure, one can think to consider a quasi-distance d from a homogeneous quasi-norm and to build a 'gradient' associated to the metric space structure. For the case under consideration, this could be done in $\mathbb{R}^{N+1}$ (and not in $\mathbb{R}^N$) since it would be the operator $\mathscr{K}$ in the space-time variables to be invariant. Even without the homogeneous structure, if we look at the group law in* (7) *we can notice that the projection in the space variables still depends on time: thus, there is a sort of* 1*-parameter family of 'geometries' associated with the operator $\mathscr{A}$ in $\mathbb{R}^N$. This can also be seen in the formula* (6) *where the t-dependent function $m_t(X,Y)$ plays the role of the distance function and defines the family of pseudo-balls $B_t(X,r)$. One way to see the choice of $\sqrt{-\mathscr{A}} = (-\mathscr{A})^{\frac{1}{2}}$ as a replacement of a gradient is to read in Definition 2.1 a weighted average in time of such a* 1*-parameter family of hidden geometries.*

The volume function $V(t)$ in formula (6), being the Lebesgue measure of the pseudo-balls $B_t(X,\sqrt{t})$, plays a big role in our analysis. If we are in presence of a homogeneous structure, $V(t)$ is exactly a power of t. In the general situation, thanks to the analysis performed in [25], we know that $p(X,Y,t)$ behaves for small times as the transition kernel of a specific operator in the homogeneous class. Therefore we can always say that

$$\exists D_0 \geq N \ \text{ such that } \ V(t) \ \cong \ t^{D_0/2} \ \text{ as } t \to 0^+.$$

We call the number D_0 the intrinsic dimension of the semigroup $\{P_t\}_{t>0}$ at zero. We also know that $D_0 > N$ unless Q is strictly positive definite (in which case $\mathscr{A}$ is elliptic and $D_0 = N$). On the other hand, the behavior of $V(t)$ for large times dictates the rate of decay of the semigroup $\{P_t\}$ and is crucial for our purposes (see also [35]). In [17, Section 3] the following notion has been introduced.

Definition 2.3. *Consider the set*

$$\Sigma_\infty = \left\{ \alpha > 0 \ : \ \int_1^\infty \frac{t^{\alpha/2-1}}{V(t)} dt < \infty \right\}.$$

We call the number $D_\infty = \sup \Sigma_\infty$ the intrinsic dimension at infinity of the semigroup $\{P_t\}_{t>0}$.

It is proved in [17, Proposition 3.1] that $D_\infty \geq 2$ if (2) holds. Furthermore, if the matrix B has at least one eigenvalue with strictly positive real part, then $V(t)$ blows up exponentially fast for large t and $D_\infty = +\infty$: in other words, in such situation the drift induces a sort of negative 'curvature' in the ambient space $\mathbb{R}^N$.

Remark 2.2. *Let us detail what happens for the operators $\mathscr{A}_0, \mathscr{A}_+, \mathscr{A}_\theta$ in $\mathbb{R}^2$ discussed in the Introduction. The 2-dimensional Kolmogorov operator $\mathscr{A}_0 - \partial_t$ is homogeneous of degree 2 with respect to the dilations $(v,x,t) \mapsto (rv, r^3x, r^2t)$: the homogeneous dimension attached to $\mathbb{R}^2$ is thus $4 = 1 + 3$. We have in fact*

$$V_0(t) = \frac{\pi}{2\sqrt{3}} t^2 \qquad\qquad \Longrightarrow \qquad\qquad D_0(\mathscr{A}_0) = D_\infty(\mathscr{A}_0) = 4.$$

The operators $\mathscr{A}_+ - \partial_t$ and $\mathscr{A}_\theta - \partial_t$ behave for small times as the reference homogeneous operator $\mathscr{A}_0 - \partial_t$ according to the work by Lanconelli and Polidoro. We can notice that the drift matrix B_+ in (4) has a positive eigenvalue, whereas B_θ has eigenvalues $\pm i$. We can compute, respectively,

$$V_+(t) = \pi \left(2e^t - \frac{t}{2} - 1 + \frac{t}{2} e^{2t} - e^{2t} \right) \qquad \Longrightarrow \ D_0(\mathscr{A}_+) = 4, \ D_\infty(\mathscr{A}_+) = +\infty;$$

$$V_\theta(t) = \pi\left(\frac{t^2}{4} + \frac{1}{8}\left(\cos(2t) - 1\right)\right)^{\frac{1}{2}} \qquad \Longrightarrow \quad D_0\left(\mathscr{A}_\theta\right) = 4,\ D_\infty\left(\mathscr{A}_\theta\right) = 2.$$

We stress that $D_\infty < D_0$ in the special case of $\mathscr{A}_\theta$. A somehow similar situation might occur in a Riemannian setting in presence of a manifold with a cylindrical end.

We have singled out in [17] two possible behaviors for the volume function $V(t)$ under which we can prove very different Sobolev-type embeddings. The first situation is when

$$\exists\, D, \gamma_D > 0 \ \text{ such that } \ V(t) \geq \gamma_D\, t^{D/2} \quad \forall t > 0. \tag{8}$$

Condition (8) imposes a restriction for the relation between D_0 and D_∞: the validity of (8) implies in fact that $D_0 \leq D \leq D_\infty$. Hence, for the operator $\mathscr{A}_\theta$, the condition (8) does not hold. On the other hand, it holds true for $\mathscr{A}_0$ if and only if $D = 4$, and for $\mathscr{A}_+$ for every $D \geq 4$. In [17, Theorem 7.5] it is proved the following result.

Theorem 2.1. *Suppose that* (8) *hold. Let* $0 < s < 1$. *Given* $1 \leq p < D/2s$ *let* $q > p$ *be such that* $\frac{1}{p} - \frac{1}{q} = \frac{2s}{D}$.

(a) *If* $p > 1$ *we have* $\mathscr{L}^{2s,p} \hookrightarrow L^{\frac{pD}{D-2sp}}(\mathbb{R}^N)$. *More precisely, there exists a constant* $S_{p,s} > 0$, *depending on* N, D, s, γ_D, p, *such that for any* $f \in \mathscr{S}$ *one has*

$$\|f\|_q \leq S_{p,s}\|(-\mathscr{A})^s f\|_p.$$

(b) *When* $p = 1$ *we have* $\mathscr{L}^{2s,1} \hookrightarrow L^{\frac{D}{D-2s},\infty}(\mathbb{R}^N)$. *More precisely, there exists a constant* $S_{1,s} > 0$, *depending on* N, D, s, γ_D, *such that for any* $f \in \mathscr{S}$ *one has*

$$\sup_{\lambda > 0} \lambda |\{X \in \mathbb{R}^N \,:\, |f(X)| > \lambda\}|^{1/q} \leq S_{1,s}\|(-\mathscr{A})^s f\|_1.$$

We also refer the interested reader to [18, Section 7] for a strong embedding in the geometric case $p = 1$ of a suitable Besov-type space.

As we mentioned, Theorem 2.1 does not cover the situation when $D_0 > D_\infty$. When this happens we have the following substitute result, which applies in particular to $\mathscr{A}_\theta$. By $L^{q_1}(\mathbb{R}^N) + L^{q_2}(\mathbb{R}^N)$ we mean the Banach space of functions f which can be written as $f = f_1 + f_2$ with $f_1 \in L^{q_1}(\mathbb{R}^N)$ and $f_2 \in L^{q_2}(\mathbb{R}^N)$ which is endowed with the norm

$$\|f\|_{L^{q_1}+L^{q_2}} = \inf_{f = f_1 + f_2 \in L^{q_1}+L^{q_2}} \|f_1\|_{L^{q_1}} + \|f_2\|_{L^{q_2}}.$$

The second condition under which we discuss the Sobolev-type embedding is the following

$$\exists\,\gamma>0 \ \text{ such that } \ V(t)\geq\gamma\min\{t^{D_0/2},t^{D_\infty/2}\} \quad \forall t>0. \tag{9}$$

The next result is taken from [17, Theorem 7.7].

Theorem 2.2. *Suppose that* (9) *hold. Let* $0<s<1$. *Given* $1\leq p<D_\infty/2s<D_0/2s$, *let* $q_\infty>q_0>p$ *be such that* $\frac{1}{p}-\frac{1}{q_\infty}=\frac{2s}{D_\infty}$, $\frac{1}{p}-\frac{1}{q_0}=\frac{2s}{D_0}$.

(a) *If* $p>1$ *we have* $\mathscr{L}^{2s,p}\hookrightarrow L^{\frac{pD_\infty}{D_\infty-2sp}}(\mathbb{R}^N)+L^{\frac{pD_0}{D_0-2sp}}(\mathbb{R}^N)$. *More precisely, there exists a constant* $S_{p,s}>0$, *depending on* $N,D_\infty,D_0,s,\gamma,p$, *such that for any* $f\in\mathscr{S}$ *one has*

$$\|f\|_{L^{q_0}+L^{q_\infty}}\leq S_{p,s}\|(-\mathscr{A})^s f\|_p.$$

(b) *If instead* $p=1$, *we have* $\mathscr{L}^{2s,1}\hookrightarrow L^{\frac{D_0}{D_0-2s},\infty}(\mathbb{R}^N)+L^{\frac{D_\infty}{D_\infty-2s},\infty}(\mathbb{R}^N)$. *More precisely, there exists a constant* $S_{1,s}>0$, *depending on* N,D_∞,D_0,s,γ, *such that for any* $f\in\mathscr{S}$ *one has*

$$\min\left\{\sup_{\lambda>0}\lambda\ |\{X\,:\,|f(X)|>\lambda\}|^{\frac{1}{q_0}},\sup_{\lambda>0}\lambda\ |\{X\,:\,|f(X)|>\lambda\}|^{\frac{1}{q_\infty}}\right\}\leq S_{1,s}||(-\mathscr{A})^s f||_1.$$

Similarly to Theorems 2.1-2.2, the two conditions (8)-(9) have allowed us to prove in [18, Theorem 1.1 and Theorem 1.2] two distinct nonlocal isoperimetric inequalities. We remark that all these results do involve the global geometry related to the class of operators $\mathscr{A}$ in (3), as it becomes evident in the examples discussed in Remark 2.2. For other global results related to $\mathscr{A}$ in absence of an underlying homogeneous structure, we refer the reader to [33, 7] where the authors deal respectively with Liouville-type theorems and global L^p-estimates for second derivates. The proofs in [33], in [7] and in [17, 18] rely on completely different methods.

For the detailed proofs of Theorems 2.1-2.2 we refer to [17, Section 7]. We want to briefly mention few crucial steps of the proof which are again based on a semigroup approach. In [17, Theorem 6.3] we show an inversion formula for the fractional powers of $\mathscr{A}$ in terms of suitable Riesz-type potentials having a semigroup representation, see [17, Definition 6.1 and Lemma 6.2]. In this way the proof of our Sobolev-type embeddings can be deduced from the L^p-L^q mapping properties of the Riesz potentials. The key technical tool to show these mapping properties is the introduction of a maximal function related to $\mathscr{A}$, which we believe has interest in its own. This maximal function, which maps in fact continuously $L^1(\mathbb{R}^N)$ in $L^{1,\infty}(\mathbb{R}^N)$ and any $L^p(\mathbb{R}^N)$ in itself for $p > 1$ ([17, Theorem 5.5]), is defined with the aid of the Poisson semigroup $e^{z\sqrt{-\mathscr{A}}}$ following an idea by Stein in [34]. Depending on p and on the standing assumption (8)/(9), we can prove that the Riesz-potentials are suitably bounded in terms of such a maximal function ([17, equations (7.2) and (7.14)]). As a final remark, we recall that the function $U(X, z) = e^{z\sqrt{-\mathscr{A}}}f(X)$ solves the Poisson problem in the sense that is solution to the equation $\partial_{zz}U + \mathscr{A}U = 0$ in the extended half-space $\mathbb{R}^N \times \{z > 0\}$ with initial condition f at $z = 0$, and its infinitesimal generator $\sqrt{-\mathscr{A}}f$ is retrieved as the Neumann datum at $z = 0$ (see the more general extension problems considered in [15], as well as [17, Lemma 5.2]).

Acknowledgments

The results discussed in this note were presented in the conference '*Something about nonlinear problems*' held in June 2019 in Bologna. The organizers Fausto Ferrari and Fabiana Leoni are kindly acknowledged for the invitation.

References

[1] F. Abedin, G. Tralli. *Harnack inequality for a class of Kolmogorov-Fokker-Planck equations in non-divergence form.* Arch. Rational Mech. Anal., **233** (2019) 867-900.

[2] F. Anceschi, M. Eleuteri, S. Polidoro. *A geometric statement of the Harnack inequality for a degenerate Kolmogorov equation with rough coefficients.* Commun. Contemp. Math., **21** (2019) 1850057.

[3] F. Anceschi, S. Polidoro, M.A. Ragusa. *Moser's estimates for degenerate Kolmogorov equations with non-negative divergence lower order coefficients.* Nonlinear Anal., **189** (2019) 111568, `doi.org/10.1016/j.na.2019.07.001`.

[4] F. Anceschi, S. Polidoro. *A survey on the classical theory for Kolmogorov equation.* ArXiv preprint 1907.05155.

[5] A.V. Balakrishnan. *Fractional powers of closed operators and the semigroups generated by them.* Pacific J. Math., **10** (1960) 419-437.

[6] V.I. Bogachev. *Ornstein-Uhlenbeck operators and semigroups.* Russian Math. Surveys, **73** (2018) 191-260.

[7] M. Bramanti, G. Cupini, E. Lanconelli, E. Priola. *Global L^p estimates for degenerate Ornstein-Uhlenbeck operators.* Math. Z., **266** (2010) 789-816.

[8] F. Buseghin, N. Garofalo. *A chain rule for a class of evolutive nonlocal hypoelliptic equations.* ArXiv preprint 1910.08803.

[9] S. Chandrasekhar. *Stochastic Problems in Physics and Astronomy.* Rev. Mod. Phys., **15** (1943) 1-89.

[10] C. Cinti, A. Pascucci, S. Polidoro. *Pointwise estimates for a class of non-homogeneous Kolmogorov equations.* Math. Ann., **340** (2008) 237-264.

[11] G. Citti, A. Pascucci, S. Polidoro. *Regularity properties of viscosity solutions of a non-Hörmander degenerate equation.* J. Math. Pures Appl. (9), **80** (2001) 901-918.

[12] F. Ferrari, B. Franchi. *Harnack inequality for fractional sub-Laplacians in Carnot groups.* Math. Z., **279** (2015) 435-458.

[13] N. Garofalo, E. Lanconelli. *Level sets of the fundamental solution and Harnack inequality for degenerate equations of Kolmogorov type.* Trans. Amer. Math. Soc., **321** (1990) 775-792.

[14] N. Garofalo. *Some properties of sub-Laplacians.* Proceedings of the Conference "Two nonlinear days in Urbino 2017", Elect. J. Diff. Equations, **25** (2018) 103-131.

[15] N. Garofalo, G. Tralli. *A class of nonlocal hypoelliptic operators and their extensions.* To appear in Indiana Univ. Math. J. (preprint at `www.iumj.indiana.edu/IUMJ/forthcoming.php`).

[16] N. Garofalo, G. Tralli. *Functional inequalities for class of nonlocal hypoelliptic equations of Hörmander type.* To appear in Nonlinear Anal., `doi.org/10.1016/j.na.2019.06.026`, Special Issue 'Fractional and Nonlocal equations'.

[17] N. Garofalo, G. Tralli. *Hardy-Littlewood-Sobolev inequalities for a class of non-symmetric and non-doubling hypoelliptic semigroups.* ArXiv preprint 1904.12982.

[18] N. Garofalo, G. Tralli. *Nonlocal isoperimetric inequalities for Kolmogorov-Fokker-Planck operators.* ArXiv preprint 1907.02281.

[19] F. Golse, C. Imbert, C. Mouhot, A.F. Vasseur. *Harnack inequality for kinetic Fokker-Planck equations with rough coefficients and application to the Landau equation.* Ann. Sc. Norm. Super. Pisa Cl. Sci. (5), **19** (2019) 253-295.

[20] L. Hörmander. *Hypoelliptic second order differential equations.* Acta Math., **119** (1967) 147-171.

[21] A.E. Kogoj, G. Tralli. *Blaschke, Privaloff, Reade and Saks theorems for diffusion equations on Lie groups.* Potential Anal., **38** (2013) 1103-1122.

[22] A.E. Kogoj, E. Lanconelli, G. Tralli. *An inverse mean value property for evolution equations.* Adv. Differential Equations, **19** (2014) 783-804.

[23] A.E. Kogoj, E. Lanconelli, G. Tralli. *Wiener-Landis criterion for Kolmogorov-type operators.* Discrete Contin. Dyn. Syst., **38** (2018) 2467-2485.

[24] A.N. Kolmogorov. *Zufällige Bewegungen (Zur Theorie der Brownschen Bewegung).* Ann. of Math. (2), **35** (1934) 116-117.

[25] E. Lanconelli, S. Polidoro. *On a class of hypoelliptic evolution operators.* Partial differential equations, II (Turin, 1993). Rend. Sem. Mat. Univ. Politec. Torino, **52** (1994) 29-63.

[26] E. Lanconelli, F. Lascialfari. *A boundary value problem for a class of quasilinear operators of Fokker-Planck type.* Proceedings of the Conference "Differential Equations" (Ferrara, 1996), Ann. Univ. Ferrara Sez. VII (N.S.), **41** (1996) 65-84.

[27] E. Lanconelli, A. Pascucci, S. Polidoro. *Linear and nonlinear ultraparabolic equations of Kolmogorov type arising in diffusion theory and in finance.* Nonlinear problems in mathematical physics and related topics, II, 243265, Int. Math. Ser. (N. Y.) **2**, Kluwer/Plenum, New York, 2002.

[28] A. Lanconelli, A. Pascucci, S. Polidoro. *Gaussian lower bounds for non-homogeneous Kolmogorov equations with measurable coefficients.* To appear in J. Evol. Equ. (ArXiv preprint 1704.07307).

[29] F. Lascialfari, D. Morbidelli. *A boundary value problem for a class of quasilinear ultraparabolic equations.* Comm. Partial Differential Equations, **23** (1998) 847-868.

[30] N. Lohoué. *Puissances complexes de l'opérateur de Laplace-Beltrami.* C. R. Acad. Sci. Paris Sér. A-B, **290** (1980) 605-608.

[31] M. Manfredini. *The Dirichlet problem for a class of ultraparabolic equations.* Adv. Differential Equations, **2** (1997) 831-866.

[32] A. Pascucci, S. Polidoro. *The Moser's iterative method for a class of ultraparabolic equations.* Commun. Contemp. Math., **6** (2004) 395-417.

[33] E. Priola, J. Zabczyk. *Liouville theorems for non-local operators.* J. Funct. Anal., **216** (2004) 455-490.

[34] E.M. Stein. Topics in harmonic analysis related to the Littlewood-Paley Theory, Annals of Mathematics Studies no. 63, Princeton University Press, Princeton (1970).

[35] N.Th. Varopoulos. *Hardy-Littlewood theory for semigroups.* J. Funct. Anal., **63** (1985) 240-260.

[36] W. Wang, L. Zhang. *The C^α regularity of weak solutions of ultraparabolic equations.* Discrete Contin. Dyn. Syst., **29** (2011) 1261-1275.

Dipartimento d'Ingegneria Civile e Ambientale (DICEA), Università di Padova, Via Marzolo, 9 - 35131 Padova, Italy.

E-mail address: giulio.tralli@unipd.it