HILBERT-HAAR COORDINATES AND MIRANDA'S THEOREM IN LIE GROUPS
COORDINATE DI HILBERT-HAAR E TEOREMA DI MIRANDA NEI GRUPPI DI LIE

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ABSTRACT. We study the interior regularity of solutions to a class of quasilinear equations of non-degenerate p-Laplacian type on Lie groups that admit a system of Hilbert-Haar coordinates. These are coordinates with respect to which every linear function has zero symmetrized second order horizontal derivatives. All Carnot groups of rank two are in this category, as well as the Engel group, the Goursat type groups, and those general Carnot groups of step three for which the non-zero commutators of order three are linearly independent.

SUNTO. Studiamo la regolarità interna delle soluzioni di una classe di equazioni quasilineari non degeneri di tipo p-Laplaciano su gruppi di Lie che ammettono un sistema di coordinate di Hilbert-Haar. Si tratta di coordinate rispetto alle quali ogni funzione lineare ha derivate orizzontali simmetrizzate di ordine due nulle. Tutti i gruppi di Carnot di passo due appartengono a questa classe, come anche il gruppo Engel, i gruppi di tipo Goursat e tuti quei gruppi di Carnot di passo tre per i quali i commutatori di ordine tre, diversi da zero, sono linearmente indipendenti.

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1. INTRODUCTION

Let $\mathbb{G}$ be a Lie group and $\mathfrak{g}$ be its Lie algebra of dimension n. Let $\mathfrak{X} = \{X_1, ..., X_{m_1}\}$, $m_1 < n$, be a system of left-invariant vector fields which, together with their commutators up to order $\nu \geq 2$, span $\mathfrak{g}$. Also, consider a domain $\Omega \subset \mathbb{G}$ and $\delta \geq 0$.

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We study the interior regularity of solutions to equations modeled on the p-Laplacian

$$\operatorname{div}_{\mathfrak{X}}\left(\left(\delta^2+|\mathfrak{X}u|^2\right)^{\frac{p-2}{2}}\mathfrak{X}u\right)=0 \text{ in } \Omega\,, \tag{1.1}$$

when we have on Ω a special coordinate system that we call a Hilbert-Haar system, because it allows us to extend the Hilbert-Haar theory to these groups. For a modern description of the Hilbert-Haar theory see [Cla05].

First, we prove that solutions to (1.1) with C^2-boundary values are locally Lipschitz with respect to the Carnot-Carathéodory distance associated to the system $\mathfrak{X}$.

Second, we prove that in Carnot groups that admit Hilbert-Haar coordinates we have interior C^∞-regularity of weak solutions to (1.1) for $\delta > 0$ and p in the range

$$2 \le p < \min\left\{\frac{2\nu}{\nu-1}\,,\ \frac{2Q+8}{Q-2}\right\}\,, \tag{1.2}$$

where Q is the homogeneous dimension of $\mathbb{G}$. See Theorem 3.1 below.

Third, we will show that in many cases, Carnot groups admit Hilbert-Haar coordinates. They include all Carnot groups of rank two, the Engel group, Goursat type groups, and those general Carnot groups of step three for which the non-zero commutators of order three are linearly independent.

The C^∞-regularity is well-known when $p = 2$ (the linear case). To the best of our knowledge, the result presented in this paper is the first regularity result for non-linear equations in some groups of rank 3 or higher with an explicit interval of p. A general Cordes estimates valid for an unspecified interval $p \in [2, 2+\epsilon_{\mathbb{G}})$, and including the degenerate case $\delta = 0$, was established by one of us in [Dom08]. While the Cordes perturbation argument is naturally limited, it is our hope that the new techniques used in this manuscript can be extended to the full range $p \ge 2$.

Miranda [Mir65] established Lipschitz bounds for solutions of the Dirichlet problem with smooth boundary data for a class of elliptic equations on domains satisfying the so-called Bounded Slope Condition (BSC). Key to this argument is that all linear functions are solutions to these equations. While this is automatic in the Euclidean case, it is not obvious in Lie groups. It is not even clear that linear functions have vanishing symmetrized horizontal second derivatives. For the case of the Heisenberg group, that this is the case was first noted by Zhong [Zho17], who proved Lemma 2.2 below in that case. Our starting observation is that linear functions that have vanishing symmetrized horizontal second derivatives also solve equation (1.1) (Lemma 2.1).

We apply Miranda's argument in Lie groups to get Lipschitz bounds in a domain D for solutions with smooth boundary values on $\bar{D}$ (Lemma 2.2). Next, in Carnot groups, given a weak solution u, we approximate it by a sequence of smooth functions ϕ_n, solve the Dirichlet problem with boundary values to get a sequence u_n that converges to u. From Miranda's argument it follows that the functions u_n are C^∞, but we don't have a quantitative control of their Lipschitz bound. Since these are smooth solutions, we can use various integral estimates located in §3.1: Lemma 3.1, Lemma 3.2 of Gagliardo-Niremberg type, Lemma 3.3 of Cacciopoli type for vertical derivatives, and Lemma 3.4 of Cacciopoli type for horizontal derivatives. We remark that the estimates in section §3.1 are valid for all $p > 1$. It is in the first step (Theorem 3.2) of the iteration process, where we have to use difference quotients, that we find the limitation on $2 \le p < \frac{2\nu}{\nu-1}$, part of (1.2). The other part of (1.2), $2 \le p < \frac{2Q+8}{Q-2}$, comes from our current implementation of the Moser iteration in Lemma 3.6. Our final step is a Moser iteration for the horizontal derivatives, Theorem 3.3. We can pass to the limit in these estimates since we have a quantitative control when (1.2) holds.

In Section §4 we discuss Hilbert-Haar coordinates. We show that every Carnot group of step 2 (Theorem 4.1), every Goursat group, which includes the Engel group, (Theorem 4.2), and all groups of step 3 with linearly independent third order commutators (Theorem 4.3) admit a system of Hilbert Haar coordinates.

We finish this introduction by conjecturing that every Carnot group has a system of Hilbert-Haar coordinates. While we present examples of arbitrary step, the general case remains open.

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2. Lipschitz continuity of weak solutions with C^2-boundary condition

In this section let $\mathbb{G}$ be a Lie group endowed with a system of horizontal left invariant vector fields $\mathfrak{X} = \{X_1^1, ..., X_{m_1}^1\}$, $m_1 < n$, generating the Lie algebra $\mathfrak{g}$. From now on, we will use the notation X_k^1 (instead of X_k), to emphasize the first stratum of $\mathfrak{g}$.
Rewrite equation (1.1) as follows:

$$\sum_{i=1}^{m_1} X_i^1 \left(a_i(\mathfrak{X}u)\right) = 0, \tag{2.1}$$

where

$$a_i(\xi) = (\delta^2 + |\xi|^2)^{\frac{p-2}{2}} \xi_i, \text{ for } 1 \le i \le m_1.$$

Note that these functions are differentiable, and there exists $L > 0$ such that the following properties hold for all $\xi, \eta \in \mathbb{R}^{m_1}$:

$$\sum_{i,j=1}^{m_1} \frac{\partial a_i}{\partial \xi_j}(\xi)\ \eta_i \eta_j \ge L\Big(\delta^2 + |\xi|^2\Big)^{\frac{p-2}{2}} |\eta|^2 , \tag{2.2}$$

$$\sum_{i,j=1}^{m_1} \left|\frac{\partial a_i}{\partial \xi_j}(\xi)\right| \le L^{-1}\Big(\delta^2 + |\xi|^2\Big)^{\frac{p-2}{2}} \text{ and,} \tag{2.3}$$

$$|a_i(\xi)| \le L^{-1} \left(\delta^2 + |\xi|^2\right)^{\frac{p-1}{2}} . \tag{2.4}$$

Consider the following Sobolev space adapted to the horizontal system of vector fields $\mathfrak{X}$:

$$W^{1,p}_{\mathfrak{X}}(\Omega) = \Big\{u \in L^p(\Omega) \ : \ X_i^1 u \in L^p(\Omega)\,, \text{ for all } 1 \le i \le m_1\Big\}.$$

Let $W^{1,p}_{\mathfrak{X},0}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,p}_{\mathfrak{X}}(\Omega)$ with respect to its usual norm.

A function $u \in W^{1,p}_{\mathfrak{X}}(\Omega)$ is a weak solution of the equation (2.1) if

$$\sum_{i=1}^{m_1} \int_\Omega a_i(\mathfrak{X}u(x))\ X_i^1\phi(x)dx = 0\,, \text{ for all } \phi \in C_0^\infty(\Omega)\,. \tag{2.5}$$

For a function $u : \mathbb{G} \to \mathbb{R}$ we define the matrix of symmetrized second order horizontal derivatives as

$$\left(\nabla^2_{\mathfrak{X}} u\right)^* = \left\{\frac{1}{2}\left(X_k^1\, X_l^1 u + X_l^1\, X_k^1 u\right)\right\}_{1 \le k,l \le m_1} .$$

Lemma 2.1. *Suppose that we have local coordinates in $\Omega \subset \mathbb{G}$ such that for any linear function*

$$L(x_1, ..., x_n) = \sum_{i=1}^{n} a_i x_i$$

we have

$$\left(\nabla^2_{\mathfrak{X}} L\right)^* = 0\,, \ in\,\Omega\,. \tag{2.6}$$

Then L is also a solution to (2.1).

Proof. Let us compute

$$\sum_{i=1}^{m_1} X_i^1(a_i(\mathfrak{X}L)) = \sum_{i,j=1}^{m_1} \frac{\partial a_i(\mathfrak{X}L)}{\partial \xi_j} X_i^1 X_j^1 L$$

$$= \sum_{i=1}^{m_1} \frac{\partial a_i(\mathfrak{X}L)}{\partial \xi_i} X_i^1 X_i^1 L + \sum_{i,j=1, i<j}^{m_1} \frac{\partial a_i(\mathfrak{X}L)}{\partial \xi_j} (X_i^1 X_j^1 L + X_j^1 X_i^1 L) = 0,$$

where we have used that $\frac{\partial a_i}{\partial \xi_j}$ is a symmetric matrix. □

Definition 2.1. *Given a set of horizontal vector fields $\mathfrak{X} = \{X_1^1, ..., X_{m_1}^1\}$, a system of local coordinates $\{x_1, \ldots, x_n\}$ is called a Hilbert-Haar coordinate system if* (2.6) *holds.*

Let us denote by $B_r(x_0)$ (or B_r if the center is clear from context) a Euclidean ball of radius r centered at x_0. Also, $|x-y|$ denotes the Euclidean distance, while $d(x,y)$ denotes the Carnot-Carathéodory distance associated to the horizontal vector fields $X_1^1, ..., X_{m_1}^1$. The notations $\nabla\phi$ and $\nabla^2\phi$ indicate the Euclidean gradient and Euclidean Hessian matrix respectively.

Lemma 2.2. *Suppose that Hilbert-Haar coordinates exist in Ω. Let $p > 1$, $\delta \geq 0$, $B_{3r} \subset \Omega$, $\phi \in C^2(B_{2r})$ and $u \in W^{1,p}_{\mathfrak{X}}(B_r)$ be the unique weak solution of the Dirichlet problem*

$$\begin{cases} \sum_{i=1}^{m_1} X_i^1(a_i(\mathfrak{X}u)) = 0\,, \ in\ B_r \\ u - \phi \in W^{1,p}_{\mathfrak{X},0}(B_r)\,. \end{cases} \tag{2.7}$$

Then, there exists a constant $\lambda > 0$, depending on r, $\|\nabla\phi\|_{L^\infty(\overline{B_r})}$ and $\|\nabla^2\phi\|_{L^\infty(\overline{B_r})}$ such that

$$|u(x) - u(y)| \leq \lambda\, d(x,y)\,, \ for\ all\ x, y \in \overline{B}_r\,. \tag{2.8}$$

Proof. Let us fix an arbitrary $y \in \partial B_r$ and choose the inner normal unit vector

$$\nu_y = \frac{1}{|x_0 - y|}(x_0 - y)\,.$$

Then, we have

$$\langle x - y, \nu_y \rangle \geq \frac{1}{2r}|x - y|^2\,,\ \text{for all}\ x \in \overline{B}_r\,. \tag{2.9}$$

Let $M = \|\nabla^2\phi\|_{L^\infty(\bar{B}_r)}$. Define the linear function

$$L_1(x) = \phi(y) + \langle \nabla\phi(y) + M\,r\,\nu_y\,, x - y\rangle\,.$$

Then, for a point ξ between x and y we have

$$\begin{aligned}\phi(x) &= \phi(y) + \langle \nabla\phi(y), x - y\rangle + \frac{1}{2}\langle \nabla^2\phi(\xi)(x - y), x - y\rangle \\ &\leq \phi(y) + \langle \nabla\phi(y), x - y\rangle + \frac{1}{2}M|x - y|^2 \leq L_1(x)\,.\end{aligned}$$

Hence, we have

$$\phi(x) \leq L_1(x)\,,\ \text{for all}\ x \in \overline{B}_r\,.$$

Since the boundary of B_r is smooth, we know that $u \in C(\overline{B}_r)$ and $u(x) = \phi(x)$, for all $x \in \partial B_r$ (see [TW02] for example). Therefore, $u(x) \leq L_1(x)$ for all $x \in \partial B_r$ and taking into consideration that, by (2.6), L_1 is a solution of (2.1), by the comparison principle we have $u(x) \leq L_1(x)$ for all $x \in \overline{B}_r$. Repeating the above arguments for

$$L_2(x) = \phi(y) + \langle \nabla\phi(y) - M\,r\,\nu_y\,, x - y\rangle\,,$$

we get that

$$L_2(x) \leq u(x) \leq L_1(x)\,,\ \text{for all}\ x \in \overline{B}_r\,.$$

Since we always have $|x - y| \leq c\,d(x, y)$ (see [NSW85]), taking

$$\lambda = c\left(\|\nabla\phi\|_{L^\infty(\overline{B}_r)} + r\|\nabla^2\phi\|_{L^\infty(\overline{B}_r)}\right),$$

we obtain

$$|u(x) - u(y)| \leq \lambda d(x, y)\,,\ \text{for all}\ x \in \overline{B}_r,\ y \in \partial B_r\,. \tag{2.10}$$

Let us fix two arbitrary $x, y \in B_r$. Then, for $g = yx^{-1}$ we define $D = B_r \cap g^{-1} B_r$. Note that $x \in D$ and by the openess of the left multiplication operator, $\operatorname{int} D \neq \emptyset$. Also, if $z \in \partial D$, then $z \in \partial B_r$ or $gz \in \partial B_r$ and hence

$$|u(z) - u(gz)| \leq \lambda d(z, gz) = \lambda d(x, y) \,.$$

Therefore,

$$u(z) \leq u(gz) + \lambda d(x, y)\,, \text{ for all } z \in \partial D \,.$$

Observe that both $u(z)$ and $u(gz) + constant$ are weak solutions of (2.1) on D, and hence by the comparison principle we get that

$$u(z) \leq u(gz) + \lambda d(x, y)\,, \text{ for all } z \in D \,.$$

Similarly, we have

$$u(gz) \leq u(z) + \lambda d(x, y)\,, \text{ for all } z \in D \,,$$

and therefore,

$$|u(z) - u(gz)| \leq \lambda d(x, y)\,, \text{ for all } z \in D \,.$$

Since $x \in D$, we can take $z = x$ to get

$$|u(x) - u(y)| \leq \lambda d(x, y) \,.$$

□

3. Regularity of Weak Solutions in Carnot groups

Consider a Carnot group $(\mathbb{G}, \cdot) = (\mathbb{R}^n, \cdot)$ and a system of left invariant horizontal vector fields $\mathfrak{X} = \{X_1^1, \ldots, X_{m_1}^1\}$, $m_1 < n$, which generates the Lie algebra $\mathfrak{g}$ of $\mathbb{G}$. We assume that $\mathfrak{g}$ admits a stratification

$$\mathfrak{g} = \bigoplus_{s=1}^{\nu} V^s \,, \tag{3.1}$$

where $\nu \in \mathbb{N}$, $\nu \geq 2$ and

$$(i) \quad \{X_1^1, \ldots, X_{m_1}^1\} \text{ is a basis of } V^1, \tag{3.2}$$

$$(ii) \quad [V^1, V^s] = V^{s+1} \text{ if } \; s \leq \nu - 1, \tag{3.3}$$

$$(iii) \quad [V^1, V^\nu] = \{0\}\ . \tag{3.4}$$

Let us denote $\dim V^s = m_s$ for all $1 \le s \le \nu$. Our main result is the following.

Theorem 3.1. *Let G be a Carnot group of step $\nu \ge 2$ and of homogeneous dimension Q, which admits a system of Hilbert-Haar coordinates. For $\delta > 0$ and p in the range*

$$2 \le p < \min\left\{ \frac{2\nu}{\nu - 1}\,,\ \frac{2Q+8}{Q-2} \right\},$$

weak solutions to (2.1) *are in* $C^\infty(\Omega)$.

3.1. **Integral estimates for all $p > 1$.** The following Gagliardo-Nirenberg type inequality depends only on integration by parts and hence is true for any function with the necessary integrability conditions. The proof is the same as in the Heisenberg group (see [MZGZ09])

Lemma 3.1. *Let $u \in C^\infty(\Omega)$, $\beta \ge 0$ and $\eta \in C_0^\infty(\Omega)$. Then there exists a constant $c > 0$, depending on β, such that*

$$\int_\Omega \eta^2 (\delta^2 + |\mathfrak{X} u|^2)^{\frac{p+2}{2}+\beta}\, dx \le c \int_\Omega (\delta^2\eta^2 + |\mathfrak{X}\eta|^2 u^2)\, (\delta^2 + |\mathfrak{X} u|^2)^{\frac{p}{2}+\beta}\, dx$$
$$+ c \int_\Omega u^2 \eta^2 \sum_{k=1}^{m_1} (\delta^2 + |X_k^1 u|^2)^{\frac{p-2}{2}+\beta}\, |X_k^1 X_k^1 u|^2\, dx\, .$$

Lemma 3.1 implies the following corollary.

Corollary 3.1. *Let $u \in C^\infty(\Omega)$, $\beta \ge 0$ and $\eta \in C_0^\infty(\Omega)$. Then there exists a constant $c > 0$, depending on β, such that*

$$\int_\Omega \eta^2 (\delta^2 + |\mathfrak{X} u|^2)^{\frac{p+2}{2}+\beta}\, dx \le c \int_\Omega (\delta^2\eta^2 + |\mathfrak{X}\eta|^2 u^2)\, (\delta^2 + |\mathfrak{X} u|^2)^{\frac{p}{2}+\beta}\, dx$$
$$+ c \int_\Omega u^2 \eta^2\, (\delta^2 + |\mathfrak{X} u|^2)^{\frac{p-2}{2}+\beta}\, |\mathfrak{X}\mathfrak{X} u|^2\, dx\, .$$

To prove the next lemma we use the test function $\phi = \eta^2\, u\, (X_k^s u)^{2\beta}$ in the weak form of (2.1). Since we don't need to interchange derivatives, commutators do not appear, and the proof is identical to the proof in the Heisenberg group (see [MZGZ09]).

Lemma 3.2. *Let $u \in C^\infty(\Omega)$ be a solution of (2.1), $\beta \geq 1$ and $\eta \in C_0^\infty(\Omega)$. Then there exists a constant $c > 0$ depending on β and L, such that for all $1 \leq s \leq \nu$ and $1 \leq k \leq m_s$ we have*

$$\int_\Omega \eta^2(\delta^2 + |\mathfrak{X}u|^2)^{\frac{p}{2}} \, |X_k^s u|^{2\beta} \, dx \leq c \int_\Omega (\delta^2\eta^2 + |\mathfrak{X}\eta|^2 u^2) \, (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p-2}{2}} \, |X_k^s u|^{2\beta} \, dx$$
$$+ c\beta^2 \int_\Omega u^2\eta^2 \, (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p-2}{2}} \, |X_k^s u|^{2\beta-2} \, |\mathfrak{X}X_k^s u|^2 \, dx \, .$$

Let us introduce the notation for the vertical gradient:

$$\mathcal{V}u = (X_1^2 u, \ldots, X_{m_2}^2 u, \ldots, X_1^\nu u, \ldots, X_{m_\nu}^\nu u) \, . \tag{3.5}$$

Lemma 3.2 implies the following corollary.

Corollary 3.2. *Let $u \in C^\infty(\Omega)$ be a solution of (2.1), $\beta \geq 1$ and $\eta \in C_0^\infty(\Omega)$. Then there exists a constant $c > 0$ depending on β and L, such that*

$$\int_\Omega \eta^2(\delta^2 + |\mathfrak{X}u|^2)^{\frac{p}{2}} \, |\mathcal{V}u|^{2\beta} \, dx \leq c \int_\Omega (\delta^2\eta^2 + |\mathfrak{X}\eta|^2 u^2) \, (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p-2}{2}} \, |\mathcal{V}u|^{2\beta} \, dx$$
$$+ c \int_\Omega u^2\eta^2 \, (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p-2}{2}} \, |\mathcal{V}u|^{2\beta-2} \, |\mathfrak{X}\mathcal{V}u|^2 \, dx \, .$$

For the following two lemmas we need to use the differentiated forms of (2.1) and track down the commutators. We start with a vertical Caccioppoli type inequality.

Lemma 3.3. *Let $p > 1$, $u \in C^\infty(\Omega)$ be a solution to (2.1), $\beta \geq 0$ and $\eta \in C_0^\infty(\Omega)$. Then there exists a constant $c > 0$ depending on L and β, such that*

$$\int_\Omega \eta^2(\delta^2 + |\mathfrak{X}u|^2)^{\frac{p-2}{2}} \, |\mathcal{V}u|^{2\beta} \, |\mathfrak{X}\mathcal{V}u|^2 dx \, \leq \, c \int_\Omega (\eta^2 + |\mathfrak{X}\eta|^2) \, (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p-2}{2}} \, |\mathcal{V}u|^{2\beta+2} \, dx.$$

Proof. Consider any non-horizontal vector field X_k^s from (3.5) and differentiate (2.1):

$$\sum_i X_k^s X_i^1 a_i = 0 \, .$$

Switch the order of differentiation and get

$$\sum_i X_i^1 X_k^s a_i = \sum_i X_{[i,k]}^{s+1} a_i \, ,$$

where we used the notation $[X_i^1, X_k^s] = X_{[i,k]}^{s+1}$. Hence, for any $\phi \in C_0^\infty(\Omega)$ we have

$$\sum_{i,j} \int_\Omega a_{ij} \cdot X_k^s X_j^1 u \cdot X_i^1 \phi \, dx = -\sum_i \int_\Omega X_{[i,k]}^{s+1} a_i \cdot \phi \, dx \,,$$

where $a_{ij} = \frac{\partial a_i}{\partial \xi_j}$. Another switch in the order of differentiation leads to

$$\sum_{i,j} \int_\Omega a_{ij} \cdot X_j^1 X_k^s u \cdot X_i^1 \phi \, dx \; = \; -\sum_i \int_\Omega X_{[i,k]}^{s+1} a_i \cdot \phi \, dx + \sum_{i,j} \int_\Omega a_{ij} \cdot X_{[j,k]}^{s+1} u \cdot X_i^1 \phi \, dx \,.$$

For any $\eta \in C_0^\infty(\Omega)$, $\eta \geq 0$ and $\beta \geq 0$ let us consider

$$\phi = \eta^2 \cdot |\mathcal{V} u|^{2\beta} \cdot X_k^s u \,,$$

and obtain

$$\sum_{i,j} \int_\Omega a_{ij} \cdot X_j^1 X_k^s u \cdot \eta^2 \cdot |\mathcal{V} u|^{2\beta} \cdot X_i^1 X_k^s u \, dx \tag{3.6}$$

$$+\sum_{i,j} \int_\Omega a_{ij} \cdot X_j^1 X_k^s u \cdot \eta^2 \cdot \beta \cdot |\mathcal{V} u|^{2\beta-2} \cdot X_i^1 \left(|\mathcal{V} u|^2\right) \cdot X_k^s u \, dx \tag{3.7}$$

$$= -\sum_{i,j} \int_\Omega a_{ij} \cdot X_j^1 X_k^s u \cdot 2\eta X_i^1 \eta \cdot |\mathcal{V} u|^{2\beta} \cdot X_k^s u \, dx \tag{3.8}$$

$$-\sum_{i,j} \int_\Omega a_{ij} \cdot X_{[i,k]}^{s+1} X_j^1 u \cdot \eta^2 \cdot |\mathcal{V} u|^{2\beta} \cdot X_k^s u \, dx \tag{3.9}$$

$$+\sum_{i,j} \int_\Omega a_{ij} \cdot X_{[j,k]}^{s+1} u \cdot \eta^2 \cdot |\mathcal{V} u|^{2\beta} \cdot X_i^1 X_k^s u \, dx \tag{3.10}$$

$$+\sum_{i,j} \int_\Omega a_{ij} \cdot X_{[j,k]}^{s+1} u \cdot \eta^2 \cdot \beta \cdot |\mathcal{V} u|^{2\beta-2} \cdot X_i^1 \left(|\mathcal{V} u|^2\right) \cdot X_k^s u \, dx \tag{3.11}$$

$$+\sum_{i,j} \int_\Omega a_{ij} \cdot X_{[j,k]}^{s+1} u \cdot 2\eta X_i^1 \eta \cdot |\mathcal{V} u|^{2\beta} \cdot X_k^s u \, dx \,. \tag{3.12}$$

Let us add the equations considered for each non-horizontal vector field X_k^s and start estimating each term. For simplicity, we use $w = \delta^2 + |\mathfrak{X} u|^2$.

$$(3.6) \geq L \int_\Omega \eta^2 \cdot w^{\frac{p-2}{2}} \cdot |\mathcal{V} u|^{2\beta} \cdot |\mathfrak{X} \mathcal{V} u|^2 \, dx \,.$$

$$(3.7) \geq \frac{\beta L}{2} \int_\Omega \eta^2 \cdot w^{\frac{p-2}{2}} \cdot |\mathcal{V} u|^{2\beta-2} \cdot \left|\mathfrak{X}\left(|\mathcal{V} u|^2\right)\right|^2 \, dx \,.$$

We continue with the terms on the right side.

$$
\begin{aligned}
(3.8) &\leq c \int_\Omega \eta \cdot |\mathfrak{X}\eta| \cdot w^{\frac{p-2}{2}} \cdot |\mathcal{V}u|^{2\beta+1} \cdot |\mathfrak{X}\mathcal{V}u| \, dx \\
&\leq \frac{L}{100} \int_\Omega \eta^2 \cdot w^{\frac{p-2}{2}} \cdot |\mathcal{V}u|^{2\beta} \cdot |\mathfrak{X}\mathcal{V}u|^2 \, dx + c \int_\Omega |\mathfrak{X}\eta|^2 \cdot w^{\frac{p-2}{2}} \cdot |\mathcal{V}u|^{2\beta+2} \, dx \,.
\end{aligned}
$$

For (3.9), initially we have to use $X^{s+1}_{[i,k]} X^1_j u = X^1_j X^{s+1}_{[i,k]} u - X^{s+2}_{[j,[i,k]]} u$, but otherwise the estimates of (3.9) - (3.12) are similar to (3.8), and this finishes the proof. □

The next lemma is a version of the horizontal Caccioppoli inequality.

Lemma 3.4. *Let $p > 1$, $u \in C^\infty(\Omega)$ be a solution to (2.1), $\beta \geq 0$ and $\eta \in C^\infty_0(\Omega)$. Then there exists a constant $c > 0$ depending on β and L such that*

$$
\begin{aligned}
\int_\Omega \eta^2 (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p-2}{2}+\beta} |\mathfrak{X}\mathfrak{X}u|^2 dx &\leq c \int_\Omega \eta^2 \, (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p-2}{2}+\beta} \, |\mathcal{V}u|^2 \, dx \\
&\quad + c \int_\Omega (\eta^2 + |\mathfrak{X}\eta|^2 + \eta |\mathcal{V}\eta|) \, (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p}{2}+\beta} \, dx \,.
\end{aligned}
$$

Proof. Consider any horizontal vector field X^1_k and differentiate (2.1):

$$\sum_i X^1_k X^1_i a_i = 0 \,.$$

Switch the order of differentiation and get

$$\sum_i X^1_i X^1_k a_i = \sum_i X^2_{[i,k]} a_i \,.$$

Hence, for any $\phi \in C^\infty_0(\Omega)$ we have

$$\sum_{i,j} \int_\Omega a_{ij} \cdot X^1_k X^1_j u \cdot X^1_i \phi \, dx = \sum_i \int_\Omega a_i \cdot X^2_{[i,k]} \phi \, dx \,.$$

Another switch in the order of differentiation leads to

$$\sum_{i,j} \int_\Omega a_{ij} \cdot X^1_j X^1_k u \cdot X^1_i \phi \, dx \; = \; \sum_i \int_\Omega a_i \cdot X^2_{[i,k]} \phi \, dx + \sum_{i,j} \int_\Omega a_{ij} \cdot X^2_{[j,k]} u \cdot X^1_i \phi \, dx \,.$$

For any $\eta \in C^\infty_0(\Omega)$, $\eta \geq 0$ and $\beta \geq 0$ let us consider

$$\phi = \eta^2 \cdot \left(\delta^2 + |\mathfrak{X}u|^2\right)^\beta \cdot X^1_k u \,,$$

and use again $w = \delta^2 + |\mathfrak{X}u|^2$. Therefore, we obtain the following equation.

$$\sum_{i,j} \int_\Omega a_{ij} \cdot X_j^1 X_k^1 u \cdot \eta^2 \cdot w^\beta \cdot X_i^1 X_k^1 u \, dx \tag{3.13}$$

$$+ \sum_{i,j} \int_\Omega a_{ij} \cdot X_j^1 X_k^1 u \cdot \eta^2 \cdot \beta \cdot w^{\beta-1} \cdot X_i^1 \left(|\mathfrak{X}u|^2\right) \cdot X_k^1 u \, dx \tag{3.14}$$

$$= - \sum_{i,j} \int_\Omega a_{ij} \cdot X_j^1 X_k^1 u \cdot 2\eta X_i^1 \eta \cdot w^\beta \cdot X_k^1 u \, dx \tag{3.15}$$

$$+ \sum_{i} \int_\Omega a_i \cdot X_{[i,k]}^2 \left(\eta^2 \cdot w^\beta \cdot X_k^1 u\right) \, dx \tag{3.16}$$

$$+ \sum_{i,j} \int_\Omega a_{ij} \cdot X_{[j,k]}^2 u \cdot \eta^2 \cdot w^\beta \cdot X_i^1 X_k^1 u \, dx \tag{3.17}$$

$$+ \sum_{i,j} \int_\Omega a_{ij} \cdot X_{[j,k]}^2 u \cdot \eta^2 \cdot \beta \cdot w^{\beta-1} \cdot X_i^1 \left(|\mathfrak{X}u|^2\right) \cdot X_k^1 u \, dx \tag{3.18}$$

$$+ \sum_{i,j} \int_\Omega a_{ij} \cdot X_{[j,k]}^2 u \cdot 2\eta X_i^1 \eta \cdot w^\beta \cdot X_k^1 u \, dx \, . \tag{3.19}$$

After summing over k, we get the following estimates for the left hand side.

$$(3.13) \geq L \int_\Omega \eta^2 \cdot w^{\frac{p-2}{2}+\beta} \cdot |\mathfrak{X}\mathfrak{X}u|^2 \, dx \, .$$

$$(3.14) \geq \frac{\beta L}{2} \int_\Omega \eta^2 \cdot w^{\frac{p-4}{2}+\beta} \cdot \left|\mathfrak{X}\left(|\mathfrak{X}u|^2\right)\right|^2 \, dx \, .$$

We continue by estimating the right hand side:

$$\begin{aligned} (3.15) &\leq c \int_\Omega \eta \cdot |\mathfrak{X}\eta| \cdot w^{\frac{p-1}{2}+\beta} \cdot |\mathfrak{X}\mathfrak{X}u| \, dx \\ &\leq \frac{L}{100} \int_\Omega \eta^2 \cdot w^{\frac{p-2}{2}+\beta} \cdot |\mathfrak{X}\mathfrak{X}u|^2 \, dx + c \int_\Omega |\mathfrak{X}\eta|^2 \cdot w^{\frac{p}{2}+\beta} \, dx \, . \end{aligned}$$

For (3.16), we first write:

$$(3.16) = \sum_{i,k} \int_\Omega a_i \cdot X_{[i,k]}^2 \left(\eta^2 \cdot w^\beta \cdot X_k^1 u\right) \, dx$$

$$= \sum_{i,k} \int_\Omega a_i \cdot 2\eta X_{[i,k]}^2 \eta \cdot w^\beta \cdot X_k^1 u \, dx \tag{3.20}$$

$$+ \sum_{i,k} \int_\Omega a_i \cdot \eta^2 \cdot \beta w^{\beta-1} X_{[i,k]}^2 (|\mathfrak{X}u|^2) \cdot X_k^1 u \, dx \tag{3.21}$$

$$+\sum_{i,k}\int_\Omega a_i\cdot\eta^2\cdot w^\beta\cdot X^2_{[i,k]}X^1_k u\,dx\,. \tag{3.22}$$

The first term is easy to estimate:

$$(3.20)\le c\int_\Omega \eta|\mathcal{V}\eta|\cdot w^{\frac{p}{2}+\beta}\,dx\,.$$

The term (3.21) is similar to (3.22), only the latter has fewer terms. We will show the estimate of (3.22).

$$\begin{aligned}
(3.22) &= \sum_{i,k}\int_\Omega a_i\cdot\eta^2\cdot w^\beta\cdot X^1_k X^2_{[i,k]}u\,dx+\sum_{i,k}\int_\Omega a_i\cdot\eta^2\cdot w^\beta\cdot X^3_{[k,[i,k]]}u\,dx\\
&= -\sum_{i,k}\int_\Omega X^1_k\left(a_i\cdot\eta^2\cdot w^\beta\right)\cdot X^2_{[i,k]}u\,dx+\sum_{i,k}\int_\Omega a_i\cdot\eta^2\cdot w^\beta\cdot X^3_{[k,[i,k]]}u\,dx\\
&\le c\int_\Omega \eta^2\cdot w^{\frac{p-2}{2}+\beta}\cdot|\mathfrak{X}\mathfrak{X}u|\cdot|\mathcal{V}u|\,dx+c\int_\Omega\eta|\mathfrak{X}\eta|\cdot w^{\frac{p-1}{2}+\beta}\cdot|\mathcal{V}u|\,dx\\
&+c\int_\Omega\eta^2\cdot w^{\frac{p-1}{2}+\beta}\cdot|\mathcal{V}u|\,dx\\
&\le\frac{L}{100}\int_\Omega\eta^2\cdot w^{\frac{p-2}{2}+\beta}\cdot|\mathfrak{X}\mathfrak{X}u|^2\,dx+c\int_\Omega\eta^2\cdot w^{\frac{p-2}{2}+\beta}\cdot|\mathcal{V}u|^2\,dx\\
&+c\int_\Omega\left(\eta^2+|\mathfrak{X}\eta|^2\right)\cdot w^{\frac{p}{2}+\beta}\,dx\,.
\end{aligned}$$

We can finish now the proof by combining the estimates (3.13)-(3.22). □

3.2. C^∞**-interior regularity for the non-degenerate case.** Initially, the following theorem was used in [Dom08, DM09] to provide the differentiability of the weak solutions in the directions of the non-horizontal vector fields. In this section we will apply it to the smooth approximating solutions u_n, in order to find energy and second derivative integrability bounds that are independent of n.

Theorem 3.2. [Dom08, DM09] *Let* $2\le p<\frac{2\nu}{\nu-1}$ *and* $r>0$ *such that* $B_{3r}\subset\Omega$. *If* $u\in X^{1,p}_{\mathfrak{X},\mathrm{loc}}(\Omega)$ *is a weak solution of* (2.1)*, then there exists* $c>0$ *depending on* p, L *and* r *such that for all* $1\le s\le\nu$ *and* $1\le k\le m_s$ *we have*

$$\int_{B_r}|X^s_k u|^p\,dx\le c\int_{B_{2r}}(\delta^2+|\mathfrak{X}u|^2)^{\frac{p}{2}}\,dx\,, \tag{3.23}$$

and

$$\int_{B_r} (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p-2}{2}} (|\mathfrak{X}X_k^s u|^2 + |X_k^s \mathfrak{X}u|^2)\, dx \le c \int_{B_{2r}} (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p}{2}}\, dx\,. \tag{3.24}$$

If we use Theorem 3.2 and Corollary 3.1 with $\beta = 0$, followed by Corollary 3.2 with $\beta = 1$, Lemma 3.4 with $\beta = 1$ and again Corollary 3.1 with $\beta = 1$, we get the following result.

Lemma 3.5. *Let* $2 \le p < \frac{2\nu}{\nu-1}$ *and* $r > 0$ *such that* $B_{3r} \subset \Omega$. *If* $u \in C^\infty(\Omega)$ *is a solution of* (2.1)*, then there exists* $c > 0$ *depending on* p*,* L *and* r *such that*

$$\int_{B_r} (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p+4}{2}}\, dx \le c \left(1 + ||u||^4_{L^\infty(B_{2r})}\right) \int_{B_{2r}} (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p}{2}}\, dx\,, \tag{3.25}$$

The next lemma gives the upper bound for the derivatives in the non-horizontal directions. Notice the loss of homogeneity in (3.26), due to the fact that we will use the inequality $\delta^2 \le \delta^2 + |\mathfrak{X}u|^2$.

Lemma 3.6. *Let us assume that* $2 \le p < \min\{\frac{2\nu}{\nu-1}, \frac{2Q+8}{Q-2}\}$, $\delta > 0$ *and* $u \in C^\infty(\Omega)$ *is a solution of* (2.1)*. Then there exist a constant* c *depending on* p*,* Q*,* L*,* r*,* $||u||_{L^\infty(B_{2r})}$ *and* δ *such that*

$$||\mathcal{V}u||_{L^\infty(B_r)} \le c \left(\int_{B_{2r}} (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p}{2}}\, dx \right)^{\frac{1}{p} + \frac{(p-2)Q}{p(2Q+8-p(Q-2))}}\,. \tag{3.26}$$

Proof. Lemma 3.3 implies the following inequality.

$$\int_\Omega \eta^2 |\mathcal{V}u|^{2\beta} |\mathfrak{X}\mathcal{V}u|^2 dx \le \frac{c}{\delta^{p-2}} \left(1 + ||\mathfrak{X}\eta||^2_{L^\infty(\operatorname{supp}\eta)}\right) \left(\int_{\operatorname{supp}\eta} (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p+4}{2}}\, dx \right)^{\frac{p-2}{p+4}} \cdot \left(\int_{\operatorname{supp}\eta} |\mathcal{V}u|^{(2\beta+2)\frac{p+4}{6}}\, dx \right)^{\frac{6}{p+4}}.$$

By Lemma 3.5 and the Poincaré inequality we obtain

$$\left(\int_\Omega \left(\eta |\mathcal{V}u|^{\beta+1}\right)^{\frac{2Q}{Q-2}}\, dx \right)^{\frac{Q-2}{2Q}} \le c \left(1 + ||\mathfrak{X}\eta||_{L^\infty(\operatorname{supp}\eta)}\right) \left(\int_{B_{2r}} (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p}{2}}\, dx \right)^{\frac{p-2}{2(p+4)}} \cdot \left(\int_{\operatorname{supp}\eta} |\mathcal{V}u|^{(\beta+1)\frac{p+4}{3}}\, dx \right)^{\frac{3}{p+4}}.$$

By defining $a = \frac{2Q}{Q-2}$, $b = \frac{p+4}{3}$, $\chi = \frac{a}{b}$, $\beta_0 = \frac{3p}{p+4}$, $\beta_k + 1 = (\beta_0 + 1)\chi^k$ and $\alpha_k = (\beta_k + 1)b$, we obtain

$$\left(\int_\Omega \eta^a |\mathcal{V}u|^{\alpha_{k+1}}\, dx\right)^{\frac{1}{\alpha_{k+1}}} \le c\left(1 + ||\mathfrak{X}\eta||_{L^\infty(\operatorname{supp}\eta)}\right)^{\frac{b}{\alpha_k}} \left(\int_{B_{2r}} (\delta^2 + |\mathfrak{X}u|^2)^{\frac{p}{2}}\, dx\right)^{\frac{p-2}{6\alpha_k}} \cdot \left(\int_{\operatorname{supp}\eta} |\mathcal{V}u|^{\alpha_k}\, dx\right)^{\frac{1}{\alpha_k}} .$$

Noticing that $\chi > 1$ if $p < \frac{2Q+8}{Q-2}$, estimate (3.26) follows from the standard Moser iteration. □

Theorem 3.3. *Let us assume that* $2 \le p < \min\{\frac{2\nu}{\nu-1}, \frac{2Q+8}{Q-2}\}$, $\delta > 0$ *and* $u \in W^{1,p}_{\mathfrak{X}}(\Omega)$ *is a weak solution to the horizontal quasilinear equation* (2.1)*. Then we have* $\mathfrak{X}u \in L^\infty_{\mathrm{loc}}(\Omega)$*.*

Proof. Let $B_{2r} \subset \Omega$. Let $\{\phi_n\}$ be a sequence of functions in $C_0^\infty(\Omega)$ converging to u in $W^{1,p}_{\mathfrak{X}}(B_r)$. For each $n \in \mathbb{N}$, let $u_n \in W^{1,p}_{\mathfrak{X}}(B_r)$ be the unique weak solution of the Dirichlet problem

$$\begin{cases} \sum_{i=1}^{m_1} X_i^1(a_i(\mathfrak{X}u)) = 0\,, \text{ in } B_r \\ u - \phi_n \in W^{1,p}_{\mathfrak{X},0}(B_r)\,. \end{cases} \tag{3.27}$$

By Lemma 2.2 and [Cap99, DM09], we know that $u_n \in C^\infty(B_r)$. Then we use $\phi = u_n - \phi_n$ in (2.5), and find that the sequence $\{u_n\}$ is bounded in $W^{1,p}_{\mathfrak{X}}(B_r)$ and hence $||u_n||_{L^\infty(B_{r/2})}$ is also bounded. Moreover, there is a subsequence, denoted also by $\{u_n\}$, which converges weakly to $\bar{u} \in W^{1,p}_{\mathfrak{X}}(B_r)$. Then $\bar{u}$ is a weak solution of (2.1) and since $\bar{u} - u \in W^{1,p}_{\mathfrak{X},0}(B_r)$, by the uniqueness of solutions to the Dirichlet problem we conclude that $\bar{u} = u$ on B_r. Lemmas 3.4 and 3.6 imply that for a test function η, with support included in $B_{r/4}$, there exists a constant c depending on δ, p, L, $||u||_{W^{1,p}_{\mathfrak{X}}(B_r)}$, but independent of n, such that

$$\int \eta^2 \left(\delta^2 + |\mathfrak{X}u_n|^2\right)^{\frac{p-2}{2}+\beta} |\mathfrak{X}\mathfrak{X}u_n|^2 \le c \int_{\operatorname{supp}\eta} \left(\delta^2 + |\mathfrak{X}u_n|^2\right)^{\frac{p}{2}+\beta} .$$

By the Moser iteration we obtain that

$$||\mathfrak{X}u_n||_{L^\infty(B_{r/16})} \le c \int_{B_r} \left(\delta^2 + |\mathfrak{X}u_n|^2\right)^{\frac{p}{2}} .$$

Finally, letting $n \to \infty$ we get

$$||\mathfrak{X}u||_{L^\infty(B_{r/16})} \leq c \int_{B_r} \left(\delta^2 + |\mathfrak{X}u|^2\right)^{\frac{p}{2}} .$$

□

Proof of Theorem 3.1: Once Theorem 3.3 gives the local boundedness of the horizontal gradient of the weak solution $u \in W^{1,p}_{\mathfrak{X}}(\Omega)$, Theorem 3.1 follows from the results obtained in [Cap99, DM09].

4. Hilbert-Haar coordinates in Carnot groups

We can identify any left-invariant vector field $X \in \mathfrak{g}$ by its value at the identity element $x = 0$. For any $X \in \mathfrak{g}$ there exists a unique differentiable homomorphism $\varphi_X : \mathbb{R} \to \mathbb{G}$ such that $\varphi_X'(0) = X$. The group exponential $\exp : \mathfrak{g} \to \mathbb{G}$ is defined by $\exp(X) = \varphi_X(1)$. We also use the notation $\varphi_X(t) = \exp(tX)$. The exponential map is a global diffeomorphism in Carnot groups. Hence, we can introduce the exponential coordinates of first kind, by identifying x with X when $x = \exp(X)$. To show how these coordinates are adapted to the stratification of the Lie algebra (3.1), let us use the notation

$$x = (x^1, ..., x^\nu) ,$$

where

$$x^1 = \left(x^1_1, ..., x^1_{m_1}\right) , \ ... \ , \ x^\nu = \left(x^\nu_1, ..., x^\nu_{m_\nu}\right) . \tag{4.1}$$

For all $1 \leq s \leq \nu$ we define degree$(x^s_i) = s$, and the degree of a monomial is

$$\text{degree}\left(\left(x^{s_1}_{i_1}\right)^{p_1} ... \left(x^{s_j}_{i_j}\right)^{p_j}\right) = s_1 p_1 + ... + s_j p_j .$$

We say that a polynomial is of homogeneous degree d if each of its monomial terms has degree d.

We list the horizontal vector fields and some of their non-zero commutators as

$$X^1_1, ..., X^1_{m_1}, \ ... \ , X^\nu_1, ..., X^\nu_{m_\nu} , \tag{4.2}$$

such that $\{X_1^s, ..., X_{m_s}^s\}$ forms a basis for V^s for each $1 \leq s \leq \nu$. By the Baker-Campbell-Hausdorff formula these vector fields can be expressed in terms of exponential coordinates as follows:

$$
\begin{aligned}
X_k^1 &= \frac{\partial}{\partial x_k^1} + \sum_{s=2}^{\nu} \sum_{i=1}^{m_s} P_{k,i}^{1,s}(x^1, ..., x^{s-1}) \frac{\partial}{\partial x_i^s}\,,\ 1 \leq k \leq m_1\,, \\
X_k^2 &= \frac{\partial}{\partial x_k^2} + \sum_{s=3}^{\nu} \sum_{i=1}^{m_s} P_{k,i}^{2,s}(x^1, ..., x^{s-2}) \frac{\partial}{\partial x_i^s}\,,\ 1 \leq k \leq m_2\,, \\
&... \\
X_k^{\nu-1} &= \frac{\partial}{\partial x_k^{\nu-1}} + \sum_{i=1}^{m_\nu} P_{k,i}^{\nu-1,\nu}(x^1) \frac{\partial}{\partial x_i^\nu}\,,\ 1 \leq k \leq m_{\nu-1}\,, \\
X_k^\nu &= \frac{\partial}{\partial x_k^\nu}\,,\ 1 \leq k \leq m_\nu\,,
\end{aligned}
\tag{4.3}
$$

where $P_{k,i}^{j,s}$ are homogeneous polynomials which, if non-zero, have their degree of homogeneity equal to $s - j$.

Definition 4.1. *Given a system of vector fields $\mathfrak{X} = \{X_1^1, ..., X_{m_1}^1\}$, we say that some coordinates in the form of* (4.1) *are privileged, if* (4.3) *holds for some homogeneous polynomials $P_{k,i}^{j,s}$.*

The exponential coordinates of first kind are one example of privileged coordinates. Similar privileged coordinates were studied in [Bel96], in order to obtain explicit distance estimates for the sub-Riemannian structure. However, our privileged coordinates serve only the purpose of starting the process leading to Hilbert-Haar coordinates.

Example 4.1. *The Lie algebra of the first Heisenberg group is defined by the only non-zero commutator $[X_1, X_2] = X_3$. The horizontal vector fields*

$$X_1 = \frac{\partial}{\partial x_1}\,,\ X_2 = \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial x_3}$$

don't satisfy (2.6)*, but after a change of variables*

$$(y_1, y_2,\ y_3) = \left(x_1, x_2,\ x_3 - \frac{1}{2} x_1 x_2 \right)\,, \tag{4.4}$$

we find that the vector fields

$$Y_1 = \frac{\partial}{\partial y_1} - \frac{y_2}{2}\frac{\partial}{\partial y_3}\,,\; Y_2 = \frac{\partial}{\partial y_2} + \frac{y_1}{2}\frac{\partial}{\partial y_3}\,,$$

do satisfy (2.6) *and hence* $\{y_1, y_2, y_3\}$ *are Hilbert-Haar coordinates.*

Example 4.2. *The Lie algebra of the Engel group is defined by the only non-zero commutators* $[X_1^1, X_2^1] = X_1^2$ *and* $[X_1^1, X_1^2] = X_1^3$*. One possible way of expressing these vector fields is*

$$\begin{aligned} X_1^1 &= \tfrac{\partial}{\partial x_1^1}, \\ X_2^1 &= \tfrac{\partial}{\partial x_2^1} + x_1^1 \tfrac{\partial}{\partial x_1^2} + \tfrac{1}{2}(x_1^1)^2 \tfrac{\partial}{\partial x_1^3}, \\ X_1^2 &= \tfrac{\partial}{\partial x_1^2} + x_1^1 \tfrac{\partial}{\partial x_1^3}, \\ X_1^3 &= \tfrac{\partial}{\partial x_1^3}. \end{aligned} \tag{4.5}$$

The horizontal vector fields X_1^1 *and* X_2^1 *don't satisfy* (2.6)*. First we change the variables adapted to the strata one and two:*

$$(y_1^1, y_2^1, y_1^2, y_1^3) = \left(x_1^1, x_2^1,\, x_1^2 - \frac{1}{2}x_1^1 x_2^1,\, x_1^3\right)\,.$$

By this we obtain the vector fields

$$\begin{aligned} Y_1^1 &= \tfrac{\partial}{\partial y_1^1} - \tfrac{y_2^1}{2} \tfrac{\partial}{\partial y_1^2}, \\ Y_2^1 &= \tfrac{\partial}{\partial y_2^1} + \tfrac{y_1^1}{2} \tfrac{\partial}{\partial y_1^2} + \tfrac{1}{2}(y_1^1)^2 \tfrac{\partial}{\partial y_1^3}\,, \\ Y_1^2 &= \tfrac{\partial}{\partial y_1^2} + y_1^1 \tfrac{\partial}{\partial y_1^3}, \\ Y_1^3 &= \tfrac{\partial}{\partial y_1^3}. \end{aligned} \tag{4.6}$$

The horizontal vector fields Y_1^1 *and* Y_2^1 *satisfy* (2.6) *in the variables* y_1^1, y_2^1, y_1^2*. Another change of variables*

$$(z_1^1, z_2^1, z_1^2, z_1^3) = \left(y_1^1, y_2^1,\, y_1^2,\, y_1^3 - \frac{1}{6}(y_1^1)^2 y_2^1 - \frac{1}{3}y_1^1 y_1^2\right),$$

leads to the vector fields

$$\begin{aligned} Z_1^1 &= \tfrac{\partial}{\partial z_1^1} - \tfrac{z_2^1}{2} \tfrac{\partial}{\partial z_1^2} - \left(\tfrac{z_1^1 z_2^1}{6} + \tfrac{z_1^2}{3}\right) \tfrac{\partial}{\partial z_1^3}, \\ Z_2^1 &= \tfrac{\partial}{\partial z_2^1} + \tfrac{z_1^1}{2} \tfrac{\partial}{\partial z_1^2} + \tfrac{(z_1^1)^2}{6} \tfrac{\partial}{\partial z_1^3}\,, \\ Z_1^2 &= \tfrac{\partial}{\partial z_1^2} + \tfrac{2z_1^1}{3} \tfrac{\partial}{\partial z_1^3}, \\ Z_1^3 &= \tfrac{\partial}{\partial z_1^3}. \end{aligned} \tag{4.7}$$

The horizontal vector fields Z_1^1 and Z_2^1 satisfy the original commutation relations, but also (2.6)*, and hence $\{z_1^1, z_2^1, z_1^2, z_1^3\}$ are Hilbert-Haar coordinates.*

Notice that the Jacobian of each change of variables equals 1.

Theorem 4.1. *Let $\mathbb{G}$ be a Carnot group of step two and $\mathfrak{X} = \{X_1^1, ..., X_{m_1}^1\}$ be a system of horizontal vector fields generating the Lie algebra. Then there exists a system of Hilbert-Haar coordinates.*

Proof. Let us assume that $\mathfrak{g} = V^1 \bigoplus V^2$, the vector fields $X_1^2, ..., X_{m_2}^2$ form a basis of V^2 and we have a set of fixed coefficients $\{b_{j,k}^i\}$ such that:

$$[X_j^1, X_k^1] = \sum_{i=1}^{m_2} b_{j,k}^i \, X_i^2 \,, \quad 1 \le j < k \le m_1 \,. \tag{4.8}$$

As the exponential coordinates are one possible option, without loss of generality, we can consider that we have a system of privileged coordinates such that these vector fields and their commutators have the following form.

$$\begin{aligned} X_k^1 &= \tfrac{\partial}{\partial x_k^1} + \textstyle\sum_{i=1}^{m_2} \left(\sum_{l=1}^{m_1} c_{k,l}^i \, x_l^1 \right) \tfrac{\partial}{\partial x_i^2} \quad , \ 1 \le k \le m_1 \,, \\ X_k^2 &= \tfrac{\partial}{\partial x_k^2} \quad , \ 1 \le k \le m_2 \,. \end{aligned} \tag{4.9}$$

We will show that the coefficients $c_{j,k}^i$ can be redefined in such a way that $\{X_1^1, ..., X_m^1\}$ satisfy (2.6) and (4.8). By (4.9) we get that for $1 \le j < k \le m$ we have

$$[X_j^1, X_k^1] = \sum_{i=1}^{m_2} \left(c_{k,j}^i - c_{j,k}^i \right) \frac{\partial}{\partial x_i^2} \,, \tag{4.10}$$

and for any linear function $L(x) = \sum_{k=1}^{m_1} a_k^1 x_k^1 + \sum_{i=1}^{m_2} a_i^2 x_i^2$ we have

$$\begin{aligned} X_k^1 L(x) &= a_k^1 + \sum_{i=1}^{m_2} \left(\sum_{l=1}^{m_1} c_{k,l}^i \, x_l^1 \right) a_i^2 \,, \\ X_k^1 X_k^1 L(x) &= \sum_{i=1}^{m_2} c_{k,k}^i \, a_i^2 \,, \\ X_j^1 X_k^1 L(x) + X_k^1 X_j^1 L(x) &= \sum_{i=1}^{m_2} \left(c_{k,j}^i + c_{j,k}^i \right) a_i^2 \,. \end{aligned} \tag{4.11}$$

By (4.8), (4.10) and (4.11) we get that

$$
c^i_{k,k} = 0\,,\ \ 1 \le i \le m_2,\ 1 \le k \le m_1
$$

$$
\tag{4.12}
\begin{cases} c^i_{k,j} - c^i_{j,k} = b^i_{j,k} \\ c^i_{k,j} + c^i_{j,k} = 0 \end{cases},\ \ 1 \le j < k \le m_1\,.
$$

For each set of indices, the system (4.12) has a unique solution, and this proves the lemma. ∎

Definition 4.2. *We say that a Carnot group of step ν defined on $\mathbb{R}^{\nu+1}$ is a Goursat group if admits a system of horizontal vector fields $\mathfrak{X} = \{X^1_1, X^1_2\}$ and the only non-zero commutators are*

$$
\tag{4.13}
[X^1_1, X^1_2] = X^2_1,\ [X^1_1, X^s_1] = X^{s+1}_1,\ 2 \le s \le \nu - 1\,.
$$

See Chapter 6 in [Mon02] for more information about Goursat groups.

Theorem 4.2. *Every Goursat group admits a system of Hilbert-Haar coordinates, in which the vector fields have the following expressions:*

$$
\tag{4.14}
\begin{aligned}
X^1_1 &= \frac{\partial}{\partial x^1_1} - \frac{1}{2}x^1_2\frac{\partial}{\partial x^2_1} - \sum_{s=3}^{\nu}\left(\frac{1}{s!}(x^1_1)^{s-2}\,x^1_2 + \frac{2!}{s!}(x^1_1)^{s-3}\,x^2_1 + \cdots + \frac{(s-1)!}{s!}x^{s-1}_1\right)\frac{\partial}{\partial x^s_1}\,,\\
X^1_2 &= \frac{\partial}{\partial x^1_2} + \frac{1}{2}x^1_1\frac{\partial}{\partial x^2_1} + \sum_{s=3}^{\nu}\frac{1}{s!}(x^1_1)^{s-1}\frac{\partial}{\partial x^s_1}\,,\\
X^l_1 &= \frac{\partial}{\partial x^l_1} + \sum_{s=l+1}^{\nu}\frac{l\cdot(s-1)\cdot(s-2)\cdots(s-l+1)}{s!}(x^1_1)^{s-l}\frac{\partial}{\partial x^s_1}\,,\ \ 2 \le l \le \nu - 1\,,\\
X^\nu_1 &= \frac{\partial}{\partial x^\nu_1}\,.
\end{aligned}
$$

Proof. The proof can be done by straightforward calculations, checking that the relations from (4.13) and (2.6) hold. The key for the proof is the following combinatorial identity:

$$
\tag{4.15}
\binom{k}{k} + \binom{k+1}{k} + \cdots + \binom{k+n}{k} = \binom{k+n+1}{k+1},\ k, n \in \mathbb{N}.
$$

∎

We used constructive arguments to find the formulas (4.14). We proceeded by induction on s, assuming that the vector fields $X_1^1, X_2^1, X_1^2, \cdots, X_1^\nu$ have some expressions as in (4.3). When we consider $s = 2$, we look only at the vector fields X_1^1, X_2^1, X_1^2 and only at the variables x_1^1, x_2^1, x_1^2. As in Example 4.1, we can find a polynomial change of variables such that in the new variables, denoted again by x_1^1, x_2^1, x_1^2, we keep the commutator relation $[X_1^1, X_2^1] = X_1^2$, but also (2.6) is satisfied for linear functions in x_1^1, x_2^1, x_1^2. Then, as in Example 4.2, we advance layer by layer, making sure that the formulas (4.13) and (2.6) hold at every step.

Notice that the homogeneous dimension of a Goursat group of step ν is

$$Q = \frac{\nu^2 + \nu + 2}{2} .$$

Let us define

$$P(\nu) = \min\left\{ \frac{2\nu}{\nu - 1} , \frac{2Q + 8}{Q - 2} \right\} = \begin{cases} \frac{2\nu}{\nu-1} & \text{if } 3 \le \nu < 10 \\ \frac{2\nu^2+2\nu+20}{\nu^2+\nu-2} & \text{if } \nu \ge 10 . \end{cases}$$

In a Goursat group, Theorem 3.1 implies the following result.

Corollary 4.1. *In a Goursat group of step ν, if $2 \le p < P(\nu)$, $\delta > 0$ and $u \in W^{1,p}_{\mathfrak{X}}(\Omega)$ is a weak solution to (2.1), then $u \in C^\infty(\Omega)$.*

We finish with an example of a Carnot group of step 3 admitting Hilbert-Haar coordinates.

Theorem 4.3. *Let $\mathbb{G}$ be a Carnot group of step 3 and $\mathfrak{X} = \{X_1^1, ..., X_{m_1}^1\}$ be a system of horizontal vector fields generating the Lie algebra. If the non-zero commutators*

$$[X_i^1, [X_j^1, X_k^1]], \ 1 \le i, j, k \le m_1$$

are linearly independent, then there exists a system of Hilbert-Haar coordinates associated to $\mathfrak{X}$.

Proof. We list the the horizontal vector fields and their non-zero commutators as

$$X_1^1, ..., X_{m_1}^1, \ X_1^2, ..., X_{m_2}^2, \ X_1^3, ..., X_{m_3}^3 . \tag{4.16}$$

We can assume that, with respect to some privileged coordinates, formula (4.3) holds. By Theorem 4.1, Hilbert-Haar coordinates exist for any Carnot groups of step 2, so we can change coordinates $x^1_1, ..., x^1_{m_1}, x^2_1, ...x^2_{m_2}$ into Hilbert-Haar coordinates.

We will add to the list of Hilbert-Haar coordinates one of x^3_i at a time, in such a way to preserve the commutator relations. We can assume for now that

$$\begin{aligned} X^1_k &= Y^1_k + \left(P^1_{k,1}(x^1) + P^1_{k,2}(x^2)\right)\frac{\partial}{\partial x^3_1},\ 1 \le k \le m_1, \\ X^2_l &= Y^2_l + P^2_{l,1}(x^1)\frac{\partial}{\partial x^3_1},\ 1 \le l \le m_2, \\ X^3_1 &= \frac{\partial}{\partial x^3_1}\,, \end{aligned} \tag{4.17}$$

where the homogeneous polynomials $P^s_{k,j}$ are of order $3-s$ and the vector fields $Y^1_1, ..., Y^2_{m_2}$ depend only on the Hilbert-Haar coordinates $x^1_1, ..., x^2_{m_2}$ and span a Lie algebra of step 2.

We will use the notation

$$\left[X^1_k, X^1_j\right] = X^2_{[k,j]}\,, \tag{4.18}$$

with the note that $X^2_{[k,l]}$ is a member of (4.16) or 0. Also, by the linear independence of the commutators of order 3 and the Jacobi identity, we can assume that $[X^1_1, X^1_2] = X^2_1$, $[X^1_1, X^2_1] = X^3_1$ and for all other cases $[X^1_k, X^2_l] = 0$.

For $1 \le k \le m_1$ and $1 \le l \le m_2$ we have:

$$\begin{aligned} \left[X^1_k, X^2_l\right] &= \left[Y^1_k + \left(P^1_{k,1} + P^1_{k,2}\right)\frac{\partial}{\partial x^3_1}\ ,\ Y^2_l + P^2_{l,1}\frac{\partial}{\partial x^3_1}\right] \\ &= \left(Y^1_k P^2_{l,1} - Y^2_l P^1_{k,2}\right)\frac{\partial}{\partial x^3_1} = \left(\frac{\partial}{\partial x^1_k}P^2_{l,1} - \frac{\partial}{\partial x^2_l}P^1_{k,2}\right)\frac{\partial}{\partial x^3_1}. \end{aligned}$$

Hence, we can define $P^2_{1,1}(x^1) = b_1 x^1_1$, $P^1_{1,2}(x^2) = (b_1 - 1)x^2_1$ and $P^1_{k,2} = 0$ if $k > 1$.

Next, we continue with the commutators of order 2.

$$\begin{aligned} \left[X^1_k, X^1_l\right] &= \left[Y^1_k + \left(P^1_{k,1} + P^1_{k,2}\right)\frac{\partial}{\partial x^3_1}\ , Y^1_l + \left(P^1_{l,1} + P^1_{l,2}\right)\frac{\partial}{\partial x^3_1}\right] \\ &= \left[Y^1_k, Y^1_l\right] + \left(Y^1_k\left(P^1_{l,1} + P^1_{l,2}\right) - Y^1_l\left(P^1_{k,1} + P^1_{k,2}\right)\right)\frac{\partial}{\partial x^3_1}\,. \end{aligned}$$

Therefore,

$$Y^1_k(P^1_{l,1} + P^1_{l,2}) - Y^1_l(P^1_{k,1} + P^1_{k,2}) = P^2_{[k,l],1}\,. \tag{4.19}$$

Let us consider a linear function

$$\mathcal{L}\left(x_1^1, ..., x_{m_2}^2, x_1^3\right) = \sum_{j=1}^{2}\sum_{i=1}^{m_j} a_i^j\, x_i^j + a_1^3\, x_1^3$$
$$= \mathcal{L}^Y(x_1^1, ..., x_{m_2}^2) + a_1^3\, x_1^3\,.$$

Therefore, we have

$$X_k^1\mathcal{L} = Y_k^1\mathcal{L}^Y + (P_{k,1}^1 + P_{k,2}^1)a_1^3\,,$$

and

$$X_l^1 X_k^1\mathcal{L} = Y_l^1 Y_k^1\mathcal{L}^Y + Y_l^1(P_{k,1}^1 + P_{k,2}^1)a_1^3.$$

By the fact that $Y_k^1 Y_l^1\mathcal{L}^Y + Y_l^1 Y_k^1\mathcal{L}^Y = 0$, we get that

$$Y_k^1(P_{l,1}^1 + P_{l,2}^1) + Y_l^1(P_{k,1}^1 + P_{k,2}^1) = 0\,. \tag{4.20}$$

By equations (4.19) and (4.20), for each fixed $1 \leq l \leq m_1$, we have

$$Y_k^1(P_{l,1}^1 + P_{l,2}^1) = \frac{1}{2}P_{[k,l],1}^2\,,\ \text{for all}\ \ 1 \leq k \leq m_1.$$

Therefore,

$$Y_k^1 P_{l,1}^1 = -Y_k^1 P_{l,2}^1 + \frac{1}{2}P_{[k,l],1}^2\,,\ \text{for all}\ \ 1 \leq k \leq m_1. \tag{4.21}$$

Noticing that the polynomials $P_{l,2}^1$ and $P_{j,1}^2$ are already determined in terms of b_1, equation (4.21) will determine the polynomials $P_{l,1}^1$, once the consistency with the commutators is checked.

By equation (4.21), for $j < k$ we have that

$$Y_{[j,k]}^2 P_{1,1}^l = -Y_{[j,k]}^2 P_{l,2}^1 + \frac{1}{2}(Y_j^1 P_{[k,l],1}^2 - Y_k^1 P_{[j,l],1}^2)\,. \tag{4.22}$$

Consider first the case of $l = 1$.

If $j = 1$ and $k = 2$, then we get

$$Y_1^2 P_{1,1}^1 = -Y_1^2((b_1 - 1)x_1^2) + \frac{1}{2}(Y_1^1(-b_1 x_1^1)\,.$$

This leads to

$$0 = -(b_1 - 1) - \frac{1}{2}b_1\,,$$

and therefore $b_1 = \frac{2}{3}$. For all other $(j, k) \neq (1, 2)$, equation (4.21) leads to $0 = 0$. Also, we get $0 = 0$ for all cases if $l > 1$. Once, the consistency is checked, by equation (4.21) and [BLU07, Theorem 20.2.1], all polynomials $P^1_{l,1}$ are uniquely defined and this finishes the proof. □

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