(NON)LOCAL Γ-CONVERGENCE
Γ-CONVERGENZA (NON)LOCALE

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ABSTRACT. We present some long-range interaction models for phase coexistence which have recently appeared in the literature, recalling also their relation to classical interface and capillarity problems. In this note, the main focus will be on the Γ-convergence methods, emphasizing similarities and differences between the classical theory and the new trends of investigation.

In doing so, we also obtain some new, more precise Γ-convergence results in terms of "interior" and "exterior" contributions. We also discuss the structural differences between Γ-limits and "pointwise" limits, especially concerning the "boundary terms".

SUNTO. In questa nota presentiamo alcuni modelli di interazione a lungo raggio che descrivono problemi di coesistenza di fase e che sono apparsi di recente in letteratura, discutendo anche la loro relazione con questioni classiche riguardanti interfaccia e fenomeni di capillarità. Ci focalizziamo soprattutto su metodi di Γ-convergenza, sottolineando somiglianze e differenze tra la teoria classica e quella riguardante gli scenari nonlocali.

Otteniamo anche alcuni risultati nuovi di Γ-convergenza in termini di contributi energetici "interni" ed "esterni". Discutiamo inoltre le differenze strutturali tra i Γ-limiti e i limiti "puntuali" dell'energia, in particolare per quanto riguarda i "termini di bordo".

2010 MSC. Primary 49J40, 35R11; Secondary 26A33, 76B45.
KEYWORDS. Γ-convergence, pointwise convergence, energy and density estimates, long-range phase transitions, nonlocal perimeter, capillarity, water waves.

1. INTRODUCTION

The goal of this note is to present and discuss some recent developments in the mathematical analysis of phase separation models, with special attention to some problems

Bruno Pini Mathematical Analysis Seminar, Vol. 11-1 (2020) pp. 68–93
Dipartimento di Matematica, Università di Bologna
ISSN 2240-2829.

described in terms of long-range particle interactions, and exploiting methods and techniques related to the classical notion of Γ-convergence.

In 1975 De Giorgi and Franzoni [18, 20] introduced the notion of Γ-convergence as a new type of convergence for functionals, particularly suitable for the study of variational problems. This new tool quickly became popular in the calculus of variations, as it allows one to relate a sequence of minimization problems depending on a parameter (that can be discrete or continuous) with a limit problem, that can possibly have a different nature from the original problems, in terms of energy functionals, functional spaces, physical modelization, etc. In spite of the structural differences between the original functionals and the limit one, this kind of convergence preserves the notion of minimizers in the limit, hence suggesting some relations between the limit problem and the sequence of functionals taken into account.

We now recall one of the possible definitions of Γ-convergence, referring to the monographs [8, 16] for a complete introduction to the subject of Γ-convergence and for all the equivalent definitions of this notion.

Given a family of functionals F_j defined on the function spaces X_j, we are interested in the minimization problems

$$\min \{F_j(u): \; u \in X_j\},$$

depending on a parameter j, and we want to relate this sequence with a limit problem, of the form

$$\min \{F(u): \; u \in X\}.$$

Definition 1.1. We say that F_j converges in the Γ-sense to F if the two following conditions are satisfied:

(i) for every $u \in X$ and every sequence u_j converging[1] to u in X, it holds that

$$\liminf_{j\to\infty} F_j(u_j) \geqslant F(u);$$

[1]Here and in the following we take X such that $X_j \subseteq X$ for every j and we define $F_j \equiv +\infty$ in $X \setminus X_j$.

(ii) for every $u \in X$ there exists a sequence $u_j \in X$ converging to u in X such that

$$\limsup_{j\to\infty} F_j(u_j) \leqslant F(u).$$

These two conditions can be understood by analogy with the direct method of the calculus of variations, keeping in mind that here we have a sequence of functionals instead of a single one. Indeed, on the one hand condition (i) plays the role of the lower semicontinuity, providing a lower bound for the sequence of minimizers. On the other hand, condition (ii) is an upper bound that ensures the optimality of the limit functional F among all the ones satisfying condition (i). Assuming that an equi-coerciveness condition is satisfied by the sequence of functionals F_j, a minimizing sequence $(\overline{u}_j)$ for the family F_j converges to a function $\overline{u} \in X$. Whenever F_j satisfies also (i) and (ii), we then have that

- there exists a minimizer $\overline{u}$ of the limit functional F defined on X;
- the sequence of minimizers $\overline{u}_j$ of F_j converges in X to $\overline{u}$;
- the sequence of minima $F_j(\overline{u}_j)$ converges to $F(\overline{u})$.

These three properties make the Γ-convergence a very useful tool in the study of minimum problems arising in the calculus of variations.

In particular, given an energy functional depending on a parameter, we can relate it to a new minimum problem by taking its Γ-limit for the parameter going to infinity. This limit problem contains somehow the relevant features of the original one, as its minimizers are the limits of sequences of minimizers of the original variational problem. In this way, through the study of the minimizers of the limit functional, one can recover some important information about the original problem.

The rest of this note is organized as follows. In the forthcoming Section 2 we recall one of the first examples of Γ-convergence, also motivated by the theory of phase coexistence. Then, in Section 3 we discuss some capillarity problems focused at detecting suitable boundary effects.

In Section 4 we present some long-range interaction models describing nonlocal phase separation, nonlocal capillarity and water waves problems. In this section we also provide some new results about the "interior" and "exterior" Γ-convergence of nonlocal energy functionals. Finally, in Section 5 we briefly recall some Γ-convergence results in the fractional parameter and we compare the notions of Γ-convergence and "pointwise" limits, stressing important differences with respect to the boundary contributions obtained via these two alternative approaches.

2. Γ-convergence results for the classical phase coexistence energy functional

A paradigmatic example of Γ-convergence is provided by some classical results for the Allen-Cahn, or Cahn-Hilliard, energy functional, which models the separation of the two phases of a fluid in a container.

In 1958, Cahn and Hilliard [14] proposed a new model for a two-phase fluid in a container, in which the phase transition occurs continuously in a thin layer, instead of discontinuously along an interface. The model is closely related to the minimization of the Helmholtz free energy in a liquid-gas system, as originally proposed by J. D. van der Waals [36] — see also [5].

In this model, one assumes that the configurations of the fluid in a container $\Omega \subset \mathbb{R}^3$ are described by a mass density u that takes values in $[-1,1]$, the pure phases being $A := \{u = -1\}$ and $B := \{u = 1\}$. Then, the energy associated with the configuration of the fluid is the sum of a potential term, in which a nonnegative double-well[2] function W vanishing at -1 and 1 appears, and a Dirichlet term, that penalizes the transitions from one phase to the other. That is, the energy associated to a configuration u is

$$\widetilde{F}_\varepsilon(u,\Omega) := \varepsilon^2 \int_\Omega |\nabla u(x)|^2 \, dx + \int_\Omega W(u(x))\, dx, \tag{2.1}$$

with the parameter ε being representative of the thickness of the layer where the phase transition occurs. In particular, since this length is supposed to be much smaller than the size of the container Ω, it is interesting to study the asymptotic behavior of the configuration, i.e., its limit as $\varepsilon \to 0^+$.

[2] We say that a function $W : \mathbb{R} \to [0,+\infty)$ is a "double-well" with zeros in ± 1 if it satisfies

$$W \in C^2(\mathbb{R}),\ W(\pm 1) = 0,\ W > 0 \text{ in } \mathbb{R} \setminus \{\pm 1\},\ W'(\pm 1) = 0,\ W''(\pm 1) > 0.$$

This type of analysis was initiated in the sense of Γ-convergence by Modica and Mortola [31,33], who considered a suitable rescaling of the energy $\widetilde{F}_\varepsilon$. Namely, they took into account the functional

$$F_\varepsilon(u,\Omega) := \frac{1}{\varepsilon}\,\widetilde{F}_\varepsilon(u,\Omega) = \varepsilon\int_\Omega |\nabla u(x)|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u(x))\,dx,$$

and proved that it Γ-converges to

$$F(u,\Omega) := \begin{cases} c_*\,\mathrm{Per}(E,\Omega) & \text{if } u_{|\Omega} = \chi_E - \chi_{\mathscr{C}E}, \text{ for some set } E\subset\Omega, \\ +\infty & \text{otherwise,}\end{cases} \tag{2.2}$$

where $c_* > 0$ is a normalization constant depending only on n and W, and $\mathrm{Per}(\cdot,\Omega)$ represents the perimeter functional inside the set Ω.

As a consequence, as $\varepsilon\to 0^+$, the minimizers of the functional F_ε converge to the minimal surfaces, i.e., the minimizers of the perimeter functional. We refer to the books [25,28] for a complete introduction to the theory of minimal surfaces and the notion of perimeter.

In particular, the theory of Γ-convergence of phase transitions to minimal surfaces has a geometric counterpart in the convergence of the level sets of the minimizers of F_ε. More precisely, as established in [12], if u is a minimizer of

$$F_1(u,\Omega) := \int_\Omega |\nabla u(x)|^2\,dx + \int_\Omega W(u(x))\,dx,$$

and $u_\varepsilon(x) := u(x/\varepsilon)$, then, up to a subsequence, for every $\vartheta\in(0,1)$, the set $\{u_\varepsilon\in(-\vartheta,\vartheta)\}$ converges locally uniformly as $\varepsilon\to 0^+$ to ∂E, being E a local minimizer of the perimeter functional. That is, for any $R>0$ and any $\delta>0$ there exists $\varepsilon_0\in(0,1)$, possibly depending on R and δ, such that, if $\varepsilon\in(0,\varepsilon_0]$ then

$$\{u_\varepsilon\in(-\vartheta,\vartheta)\}\cap B_R \subseteq \bigcup_{p\in\partial E} B_\delta(p). \tag{2.3}$$

The proof of (2.3) in [12] relies on suitable energy and density estimates. More specifically, it is proved in [12] that if u is a minimizer of F_1 in B_{R+1} with $R>1$, then

$$F_1(u,B_R)\leqslant CR^{n-1}, \tag{2.4}$$

for some constant $C > 0$. Also, if ϑ_1, $\vartheta_2 \in (-1, 1)$ and u is a minimizer of F_1 in B_R with $u(0) > \vartheta_1$, then there exist $R_o(\vartheta_1, \vartheta_2) > 1$ and $c_o > 0$ such that, for all $R \geqslant R_o(\vartheta_1, \vartheta_2)$,

$$|\{u > \vartheta_2\} \cap B_R| \geqslant c_o\, R^n. \tag{2.5}$$

That is, according to (2.4), the energy of the minimizers "mostly arise from a codimension 1 interface", and, in light of (2.5), unless the solution at a given point (say the origin) is very close to a pure phase, we have that the two phases in a large ball occupy a measure which is comparable to the one of the ball itself (i.e., no phase gets lost, at least in a measure theoretic sense).

A very strong connection between phase transition models and minimal surfaces is highlighted by a celebrated conjecture of E. De Giorgi [19] about the rigidity properties of monotone solutions to the Allen-Cahn equation $\Delta u = W'(u)$ in $\mathbb{R}^n$, which can be formulated as follows:

Conjecture 2.1. *Let $u \in C^2(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ be a solution of*

$$\Delta u(x) = W'(u(x)) \qquad \textit{for all } x \in \mathbb{R}^n,$$

and assume also that

$$\frac{\partial u}{\partial x_n}(x) > 0 \qquad \textit{for all } x \in \mathbb{R}^n.$$

Then, is it true that u depends only on one Euclidean variable (i.e., there exist $u_0 : \mathbb{R} \to \mathbb{R}$ and $\omega \in S^{n-1}$ such that $u(x) = u_0(\omega \cdot x)$ for all $x \in \mathbb{R}^n$), at least if $n \leqslant 8$?

Conjecture 2.1 gave rise to several papers about the rigidity of the solutions to the Allen-Cahn equation. We refer to the survey [24] for an introduction to this line of research.

3. Boundary effects and capillarity problems

In the two-phase model in (2.1) the boundary contact energy is assumed to be negligible, since the model mainly focuses on the formation of the phase interfaces inside the domain.

In order to quantitatively take into account the boundary effects of the domain on the phase separation, several other models have been designed.

As a matter of fact, to understand the influence of boundary effects, a classical model is the one describing "capillarity" phenomena in a water-drop problem, in which the boundary contact energy between the fluid and the wall becomes nonnegligible. In this case, the model considers a liquid droplet of constrained mass occupying a small region E in a container Ω, and the energy functional associated to E is of the form

$$G(E) := \mathrm{Per}(E,\Omega) + \sigma\,\mathrm{Per}(E,\partial\Omega), \tag{3.1}$$

where $\sigma \in [-1,1]$ is the "relative adhesion coefficient", that measures the liquid-wall tension with respect to the liquid-air tension. See Chapter 19 in [28] and the references therein for a thorough presentation of classical droplet and capillarity problems.

The functional G in (3.1) shares some obvious similarities with the functional F in (2.2), and therefore, in light of the discussion in Section 2, it is natural to ask whether G can be seen as the Γ-limit of some modification of the phase interface energy functional in (2.1).

In [32], Modica established the Γ-convergence of the energy

$$G_\varepsilon(u,\Omega) := \varepsilon \int_\Omega |\nabla u(x)|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u(x))\,dx + \int_{\partial\Omega} V(u(x))\,d\mathscr{H}^{n-1}(x), \tag{3.2}$$

where V is a nonnegative continuous function, not necessarily of double-well type, and $\mathscr{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure. Specifically, in [32] it is proved that, for problems with prescribed mass, G_ε converges in the Γ-sense to the capillarity energy functional G in (3.1). The relative adhesion coefficient σ appearing in the Γ-limit (3.1) depends only on W and V, and is explicitly computed in [32, Theorem 2.1].

A modification of the energy G_ε defined in (3.2) was considered in [3, 4] by Alberti, Bouchitté, and Seppecher, consisting of the energy functional

$$\mathscr{G}_\varepsilon(u,\Omega) := \varepsilon \int_\Omega |\nabla u(x)|^2\,dx + \frac{1}{\varepsilon}\int_\Omega W(u(x))\,dx + \lambda_\varepsilon \int_{\partial\Omega} V(u(x))\,d\mathscr{H}^{n-1}. \tag{3.3}$$

Here, W is still a double-well potential vanishing in ±1, while V — contrary to (3.2) — is a double-well potential vanishing in α and β, and λ_ε is a parameter that goes to infinity when $\varepsilon \to 0^+$, satisfying

$$\lim_{\varepsilon\to 0^+} \varepsilon \log \lambda_\varepsilon = k \qquad \text{with } k \in (0, +\infty). \tag{3.4}$$

Under these assumptions — which are different in the energy boundary term with respect to [32] — it is established in [4] that the energy functional $\mathscr{G}_\varepsilon$ Γ-converges to the limit energy

$$\mathscr{G}(u) := \begin{cases} \inf\{\phi(u,v) : \ v \in BV(\partial\Omega, \{\alpha;\beta\})\} & \text{if } u \in BV(\Omega, \{-1;1\}), \\ +\infty & \text{otherwise,} \end{cases}$$

where for every $u \in BV(\Omega, \{-1;1\})$ and $v \in BV(\partial\Omega, \{\alpha;\beta\})$ the function $\phi(u,v)$ is defined as

$$\phi(u,v) := \mathscr{H}^{n-1}(Su) + \sigma \int_{\partial\Omega} |H(Tu) - H(v)|\, d\mathscr{H}^{n-1} + c\,\mathscr{H}^{n-2}(Sv). \tag{3.5}$$

Here, Tu denotes the trace of u on the boundary of Ω, H is the primitive function of $2\sqrt{W}$, while the parameters σ and c depend only on W, V, and k, and are explicitly defined in [4].

Also, in (3.5), with Su we denote the set of the points in which u is essentially[3] discontinuous. In this setting, if $u \in BV(\Omega, \{-1;1\})$, then $\mathscr{H}^{n-1}(Su)$ is the measure of the interface between the pure phases $\{u = 1\}$ and $\{u = -1\}$. It is well-known that a function u that belongs to $\{-1;1\}$ almost everywhere has bounded variation if and only if the measure of the jump-set Su is finite.

Similarly, if $v \in BV(\partial\Omega, \{\alpha;\beta\})$, then $\mathscr{H}^{n-2}(Sv)$ denotes the $(n-2)$-dimensional measure of the interface between the boundary phases $\{v = \alpha\}$ and $\{v = \beta\}$. Finally, the second term in the definition (3.5) evaluates the energy of the transition from Tu to v that occurs on the boundary.

The energy functional $\mathscr{G}$ is introduced in [4] as a relaxation[4] of a capillarity functional with line tension energy, which can be seen as a modification of the functional G defined

[3] One says that u is essentially continuous at a point x if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for almost all y, $z \in B_\delta(x)$ one has that $|f(y) - f(z)| < \varepsilon$.

[4] The relaxation procedure outlined in [4] is necessary as the capillarity functional $\mathscr{G}_\sharp$ in (3.6) is not semi-continuous, and this leads to minimum problems which are not well-posed.

in (3.1). If we take $\alpha = -1$ and $\beta = 1$, then the capillarity functional with line tension is

$$\mathscr{G}_\sharp(E) := \mathscr{H}^{n-1}(\Omega \cap \partial E) + \sigma\, \mathscr{H}^{n-1}(\partial\Omega \cap \partial E) + c\, \mathscr{H}^{n-2}\left((\overline{\partial E \cap \Omega}) \cap \partial\Omega\right), \tag{3.6}$$

where $E := \{u(x) = 1\}$. In the three-dimensional case, the so-called "line tension energy", which is the last term in (3.6), models an energy concentrated along the line $(\overline{\partial E \cap \Omega}) \cap \partial\Omega$ where the interface liquid-air $\partial E \cap \Omega$ meets the boundary $\partial\Omega$ of the container.

The results in [4] have later been extended in [26] to the functional

$$\begin{aligned} \mathscr{G}^a_\varepsilon(u, \Omega) \;:=\; & \varepsilon^{1-a} \int_\Omega |\nabla u(x)|^2\, h^a(x)\, dx + \frac{1}{\varepsilon^{1-a}} \int_\Omega W(u(x)) h^{-a}(x)\, dx \\ & + \lambda_\varepsilon \int_{\partial\Omega} V(u(x))\, d\mathscr{H}^{n-1}(x), \end{aligned} \tag{3.7}$$

where $a \in (-1, 0)$, $h : \Omega \to \mathbb{R}$ is the distance function to the boundary of Ω, and $\lambda_\varepsilon \to +\infty$ as $\varepsilon \to 0^+$ with some specific behavior, different from the one in (3.4). More precisely, in [26] it is proved that the energy functional $\mathscr{G}^a_\varepsilon$ achieves the same Γ-limit for every $a \in (-1, 0)$ as the one attained by $\mathscr{G}_\varepsilon$ defined in (3.3).

4. Local and nonlocal contributions in the Γ-limit

In this section, we describe some phase separation models in which the interaction energy is of nonlocal type. For this, we start by presenting the results in [3], focusing on the dimension $n = 1$. In [3], the authors consider an interval $I \subset \mathbb{R}$ and the energy functional

$$\mathscr{G}^1_\varepsilon(v) := \varepsilon \iint_{I \times I} \frac{|v(x) - v(y)|^2}{|x - y|^2}\, dx\, dy + \lambda_\varepsilon \int_I W(v(x))\, dx, \tag{4.1}$$

where W is a double-well potential with zeros in -1 and 1, and λ_ε is a positive parameter depending on ε and satisfying (3.4).

Then, the main result in [3] establishes that the energy functional defined in (4.1) Γ-converges in the L^1-topology to

$$\mathscr{G}^1(v) := \begin{cases} 8k\, \mathscr{H}^0(Sv) & \text{if } v \in BV(I, \{-1; 1\}), \\ +\infty & \text{otherwise,} \end{cases} \tag{4.2}$$

where k is the one in (3.4) and Sv is the set of the points in which v is essentially discontinuous. Since we are assuming $n = 1$, this simply means that at those points the function is discontinuous with the left-hand limit being different from the right-hand limit.

As customary, $\mathscr{H}^0$ denotes the 0-dimensional Hausdorff measure, corresponding to the "counting measure" (hence, $\mathscr{H}^0(Sv)$ is simply the "number of jumps" of the step function v), and $BV(I, \{-1; 1\})$ the space of the functions with bounded variation which are defined on $I \subset \mathbb{R}$ with values in $\{-1; 1\}$ almost everywhere.

In the context of the Γ-convergence of the functional $\mathscr{G}^a_\varepsilon$ defined in (3.7), the study of the Γ-limit of an interaction energy in dimension $n = 1$ was addressed in [26] for $a \in (-1, 0)$, corresponding to the fractional parameter $s \in (1/2, 1)$. Specifically, for an interval $I \subset \mathbb{R}$, in [26] the author considers the energy

$$\mathscr{G}^{1,a}_\varepsilon(v) := \varepsilon^{1-a} \iint_{I \times I} \frac{|v(x) - v(y)|^2}{|x - y|^{1+2s}}\, dx\, dy + \lambda_\varepsilon \int_I V(v(x))\, dx, \tag{4.3}$$

where $1 - a = 2s$, proving that it Γ-converges to $\mathscr{G}^1(v)$ defined in (4.2).

It is interesting to remark that the models presented in (4.1) and (4.3), though of nonlocal nature, converge to a Γ-limit, namely the one in (4.2), which is local and classical.

In the following pages, we present other long-range interaction models for phase transitions and discuss their Γ-limits. Interestingly, the Γ-limits of the following functionals reduce to local limit problems for suitable ranges of a fractional parameter (corresponding to "weakly nonlocal" interactions), but conserves the nonlocal feature of the original problem for other ranges of this parameter (corresponding to "strongly nonlocal" interactions). Observe that the behavior in the strongly nonlocal regime represents a novelty with respect to the previous works [3, 26], in which this range of parameters was not considered.

More precisely, in [37–39] Savin and the third author study the Γ-convergence, as well as the geometric convergence of level sets of the minimizers, for $\varepsilon \to 0^+$ of a proper rescaling of the interaction energy

$$\begin{aligned} J_\varepsilon(u,\Omega) \;:=\; &\varepsilon^{2s}\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,dx\,dy \\ &+2\varepsilon^{2s}\int_{\Omega\times\mathscr{C}\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{1+2s}}\,dx\,dy+\int_\Omega W(u(x))\,dx, \end{aligned} \tag{4.4}$$

where s is a parameter in $(0,1)$, W a double-well potential, and $\Omega\subset\mathbb{R}^n$ a bounded domain whose complement is $\mathscr{C}\Omega:=\mathbb{R}^n\setminus\Omega$.

In order to describe the result in detail, we introduce the setting in [38]. We let $X:=\{u\in L^\infty(\mathbb{R}^n):\|u\|_{L^\infty(\mathbb{R}^n)}\leqslant 1\}$ and we say that a sequence $u_j\in X$ converges to u in X if u_j converges to u in $L^1_{\rm loc}(\mathbb{R}^n)$.

The energy considered in [38] can be seen as a suitable nonlocal analogue of the classical model in (2.1). Indeed, in (4.4) the classical Dirichlet-type energy is replaced by a long-range interaction energy consisting of the Ω-contribution in the H^s-seminorm of u. In the classical case, only local interactions count in the Dirichlet energy, and the state of the fluid outside the container is not taken into account. In this new long-range setting, it is assumed that every particle interacts with all the other ones, inside and outside of the container, carrying a smaller contribution as the distance between the particle increases (and the energy functional in (4.4) takes into account all the particle interactions in which at least one of the particles lies in the container).

In particular, we define the "interior contribution" as

$$\mathscr{K}^{int}(u,\Omega):=\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

and the "exterior contribution" as

$$\mathscr{K}^{ext}(u,\Omega):=2\iint_{\Omega\times\mathscr{C}\Omega}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

Then, we set

$$\mathscr{K}(u,\Omega):=\mathscr{K}^{int}(u,\Omega)+\mathscr{K}^{ext}(u,\Omega).$$

We observe that in this type of energy functionals we omit the contributions for $(x,y)\in\mathscr{C}\Omega\times\mathscr{C}\Omega$, since we are interested in variational problems in which all the admissible competitors are fixed outside of Ω.

Then, the energy in (4.4) can be written as

$$J_\varepsilon(u,\Omega) = \varepsilon^{2s}\mathscr{K}(u,\Omega) + \int_\Omega W(u(x))\,dx.$$

In order to obtain a relevant Γ-limit, in [38] a proper rescaling of the energy J_ε is taken into account. In the present work, we make this rescaling more explicit, by also highlighting the different contributions coming from the interior and the exterior parts of the energy. For this, we define

(4.5)

$$\mathscr{F}_\varepsilon^{int}(u,\Omega) := \begin{cases} \mathscr{K}^{int}(u,\Omega) + \dfrac{\varepsilon^{-2s}}{2}\displaystyle\int_\Omega W(u(x))\,dx & \text{if } s \in \left(0,\dfrac{1}{2}\right), \\ |\log\varepsilon|^{-1}\,\mathscr{K}^{int}(u,\Omega) + \dfrac{|\varepsilon\log\varepsilon|^{-1}}{2}\displaystyle\int_\Omega W(u(x))\,dx & \text{if } s = \dfrac{1}{2}, \\ \varepsilon^{2s-1}\mathscr{K}^{int}(u,\Omega) + \dfrac{\varepsilon^{-1}}{2}\displaystyle\int_\Omega W(u(x))\,dx & \text{if } s \in \left(\dfrac{1}{2},1\right); \end{cases}$$

and

(4.6)

$$\mathscr{F}_\varepsilon^{ext}(u,\Omega) := \begin{cases} \mathscr{K}^{ext}(u,\Omega) + \dfrac{\varepsilon^{-2s}}{2}\displaystyle\int_\Omega W(u(x))\,dx & \text{if } s \in \left(0,\dfrac{1}{2}\right), \\ |\log\varepsilon|^{-1}\,\mathscr{K}^{ext}(u,\Omega) + \dfrac{|\varepsilon\log\varepsilon|^{-1}}{2}\displaystyle\int_\Omega W(u(x))\,dx & \text{if } s = \dfrac{1}{2}, \\ \varepsilon^{2s-1}\mathscr{K}^{ext}(u,\Omega) + \dfrac{\varepsilon^{-1}}{2}\displaystyle\int_\Omega W(u(x))\,dx & \text{if } s \in \left(\dfrac{1}{2},1\right). \end{cases}$$

The sum of $\mathscr{F}_\varepsilon^{int}$ and $\mathscr{F}_\varepsilon^{ext}$ is the object of the Γ-convergence result in [38], i.e., one defines

$$\mathscr{F}_\varepsilon(u,\Omega) := \mathscr{F}_\varepsilon^{int}(u,\Omega) + \mathscr{F}_\varepsilon^{ext}(u,\Omega).$$

The ε-rescaling in the definitions of $\mathscr{F}_\varepsilon^{int}$ and $\mathscr{F}_\varepsilon^{ext}$ can be seen as a convenient one in order to obtain a significant Γ-limit. It is worth observing that for the case $s = 1/2$ the ε-weights in the definitions of $\mathscr{F}_\varepsilon^{int}$ and $\mathscr{F}_\varepsilon^{ext}$ satisfy[5] the limit assumption (3.4) with $k = 1$ that is taken in [3].

[5] This follows from the fact that

$$\lim_{\varepsilon\to 0^+} |\log\varepsilon|^{-1}\log\left(\varepsilon^{-1}\,|\log\varepsilon|^{-1}\right) = 1.$$

In order to define the Γ-limit of the energy functionals studied in [38], we recall the notion of fractional perimeter, as introduced in [13]. Given two measurable and disjoint sets $E, F \subset \mathbb{R}^n$, one defines

$$I_s(E,F) := \iint_{E\times F} \frac{dx\,dy}{|x-y|^{n+2s}},$$

where $s \in (0,1/2)$. Then, we define the "interior contribution" of the fractional perimeter as

$$\mathrm{Per}_s^{int}(E,\Omega) := I_s(E\cap\Omega, \Omega\setminus E) \tag{4.7}$$

and the "exterior contribution" as

$$\mathrm{Per}_s^{ext}(E,\Omega) := I_s(E\cap\mathscr{C}\Omega, \Omega\setminus E) + I_s(E\cap\Omega, \mathscr{C}\Omega\cap\mathscr{C}E). \tag{4.8}$$

Finally, the full fractional perimeter of a set E in Ω is defined as

$$\mathrm{Per}_s(E,\Omega) := \mathrm{Per}_s^{int}(E,\Omega) + \mathrm{Per}_s^{ext}(E,\Omega).$$

In this setting, the Γ-limit functional $\mathscr{F}$ in [38] is as follows:

$$\mathscr{F}(u,\Omega) := \begin{cases} \mathrm{Per}_s(E,\Omega) & \text{if } s\in(0,1/2) \text{ and } u_{|\Omega} = \chi_E - \chi_{\mathscr{C}E}, \\ c_*\,\mathrm{Per}(E,\Omega) & \text{if } s\in[1/2,1) \text{ and } u_{|\Omega} = \chi_E - \chi_{\mathscr{C}E}, \\ +\infty & \text{otherwise}, \end{cases} \tag{4.9}$$

where c_* is a constant depending only on n, s, and W, which is explicitly determined in [38].

For further reference, it is also convenient, in the case $s \in (0,1/2)$, to reformulate and extend the Γ-convergence result in [38] in terms of "interior" and "exterior" limit functionals:

Theorem 4.1. *Let $s \in (0,1/2)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then,*

(a) *$\mathscr{F}_\varepsilon^{int}$ Γ-converges to the interior contribution in the fractional perimeter, i.e.,*

$$\mathscr{F}^{int}(u,\Omega) := \begin{cases} \mathrm{Per}_s^{int}(E,\Omega) & \textit{if } u_{|\Omega} = \chi_E - \chi_{\mathscr{C}E}, \\ +\infty & \textit{otherwise.} \end{cases}$$

(b) $\mathscr{F}_\varepsilon^{ext}$ Γ*-converges to the exterior contribution in the fractional perimeter, i.e.,*

$$\mathscr{F}^{ext}(u,\Omega) := \begin{cases} \mathrm{Per}_s^{ext}(E,\Omega) & \text{if } u_{|\Omega} = \chi_E - \chi_{\mathscr{C}E}, \\ +\infty & \text{otherwise.} \end{cases}$$

(c) $\mathscr{F}_\varepsilon$ Γ*-converges to the functional* $\mathscr{F}$ *defined in* (4.9).

Sketch of the proof. Since the same strategy works for all three cases, let us deal with point (a). One observes that

$$\mathscr{F}_\varepsilon^{int}(u,\Omega) = \mathscr{K}^{int}(u,\Omega) = \mathscr{F}^{int}(u,\Omega) \qquad \text{if } u_{|\Omega} = \chi_E - \chi_{\mathscr{C}E}. \tag{4.10}$$

First, we want to prove point (i) in Definition 1.1. For every sequence u_ε converging to u in X, we can assume that

$$\liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon^{int}(u_\varepsilon,\Omega) = l < +\infty,$$

otherwise the claim is trivial. Taking a subsequence u_{ε_k} attaining the above limit and a further subsequence (that we still name u_{ε_k}) converging almost everywhere to u, we deduce that

$$l = \lim_{k\to\infty} \mathscr{F}_{\varepsilon_k}^{int}(u_{\varepsilon_k},\Omega) \geqslant \lim_{k\to\infty} \frac{1}{2\varepsilon_k^{2s}} \int_\Omega W(u_{\varepsilon_k}(x))\,dx.$$

Therefore, the integral of $W(u)$ over Ω is zero at the limit and we deduce that $u(x) \in \{-1;1\}$ for almost every $x \in \Omega$, that is $u_{|\Omega} = \chi_E - \chi_{\mathscr{C}E}$ for some set $E \subset \mathbb{R}^n$. Now, by Fatou's lemma and (4.10) we have

$$\liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon^{int}(u_\varepsilon,\Omega) \geqslant \mathscr{F}^{int}(u,\Omega),$$

which is the desired inequality.

Then, to prove point (ii) in Definition 1.1, we assume that $u_{|\Omega} = \chi_E - \chi_{\mathscr{C}E}$ for some set $E \subset \mathbb{R}^n$, otherwise the claim is trivial. Then, by taking a constant sequence $u_\varepsilon := u$ and using (4.10), it follows that

$$\limsup_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon^{int}(u_\varepsilon,\Omega) \leqslant \mathscr{F}^{int}(u,\Omega),$$

concluding the proof of point (a). □

As a consequence of Theorem 4.1, we can obtain a new result about the Γ-convergence to a nonlocal capillarity functional. Indeed, a fractional analogue of the capillarity functional G defined in (3.1) is studied in [21, 29]. For a bounded container $\Omega \subset \mathbb{R}^n$ and for every set $E \subset \Omega$, one takes into account the energy functional

$$\mathscr{E}_s(E,\Omega) := I_s(E, \Omega \setminus E) + \sigma I_s(E, \mathscr{C}\Omega),$$

where σ is the relative adhesion coefficient that we introduced for the classical capillarity energy — see (3.1). For every $s \in (0, 1/2)$ we define the energy

$$\mathscr{J}_{\varepsilon,s}(u,\Omega) := \mathscr{K}^{int}(u,\Omega) + \sigma \mathscr{K}^{ext}(u,\Omega) + \varepsilon^{-2s} \int_\Omega W(u(x))\, dx.$$

Then, we have the following result for the Γ-convergence of the energy $\mathscr{J}_{\varepsilon,s}$.

Corollary 4.2. *Let $s \in (0, 1/2)$ and Ω be a bounded domain. Then, $\mathscr{J}_{\varepsilon,s}$ converges in the Γ-sense to the fractional capillarity energy defined as*

$$\mathscr{J}_s(E,\Omega) := \begin{cases} \mathscr{E}_s(E,\Omega) & \text{if } u_{|\Omega} = \chi_E - \chi_{\mathscr{C}E}, \\ +\infty & \text{otherwise.} \end{cases}$$

Proof. The result follows from Theorem 4.1, the subadditivity of the lim sup, and the superadditivity of the lim inf. □

Now we focus instead on the case $s \in [1/2, 1)$. In this setting, and using the tools in [38], we can prove a Γ-convergence result for the functionals $\mathscr{F}_\varepsilon^{int}$ and $\mathscr{F}_\varepsilon$ which is similar to, but slightly stronger than, the claim in [38, Theorem 1.4]. We state it in the following theorem and we then sketch its proof, which is obtained by adapting the arguments in [38].

Theorem 4.3. *Let $s \in [1/2, 1)$ and $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary. Then, for any $u \in X$,*

(i) *for every u_ε that converges to u in X,*

$$\liminf_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon^{int}(u_\varepsilon,\Omega) \geqslant \mathscr{F}(u,\Omega);$$

(ii) *there exists u_ε that converges to u in X and such that*

$$\limsup_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon,\Omega) \leqslant \mathscr{F}(u,\Omega).$$

Sketch of the proof. We start with the proof of point (i). We recall that in [38] it is proved that, for every u_ε converging to u in X,

$$\liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon,\Omega) \geqslant \mathscr{F}(u,\Omega). \tag{4.11}$$

Actually, the proof in [38] can be adapted to show point (i) in Theorem 4.3, which is slightly stronger than (4.11), as $\mathscr{F}_\varepsilon(u_\varepsilon,\Omega) \geqslant \mathscr{F}_\varepsilon^{int}(u_\varepsilon,\Omega)$.

To prove point (i), we can assume that

$$\liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon^{int}(u_\varepsilon,\Omega) = l < +\infty, \tag{4.12}$$

otherwise the claim in (i) is trivial. From (4.12), it follows the existence of a subsequence of u_ε, that we still name u_ε, such that u_ε converges to $\chi_E - \chi_{\mathscr{C}E}$ in $L^1(\Omega)$ for some set $E \subset \mathbb{R}^n$ with finite perimeter in Ω. This is proved in [38, Proposition 3.3], under the hypothesis that the lim inf of $\mathscr{F}_\varepsilon(u_\varepsilon,\Omega)$ is finite. However, one can weaken this hypothesis and assume (4.12) instead, from which one can deduce that $\mathscr{F}_\varepsilon^{int}(u_\varepsilon,\Omega)$ is uniformly bounded, by eventually passing to a subsequence, and carry out the whole proof.

Since E has finite perimeter in Ω, by classical results in Geometric Measure Theory — see [25, Theorem 4.4] — we have

$$\mathrm{Per}(E,\Omega) = \mathscr{H}^{n-1}\left(\partial^* E \cap \Omega\right),$$

where $\partial^* E$ is the "reduced boundary" of the set E. We refer again to [25, 28] for the theory of sets with finite perimeter and in particular for the definition of the reduced boundary. Then, by the rectifiability of the reduced boundary, for every $\alpha > 0$ we can find a collection of balls B_j with radii $\rho_j > 0$, whose smallness depends from α, such that

$$\mathrm{Per}(E,\Omega) \leqslant \alpha + \omega_{n-1} \sum_{j=0}^{+\infty} \rho_j^{n-1},$$

where ω_{n-1} is the measure of the $(n-1)$-dimensional unit ball. By Vitali's covering theorem we can assume that these balls are disjoint, hence

$$\mathscr{F}_\varepsilon^{int}(u_\varepsilon,\Omega) \geqslant \sum_{j=0}^{+\infty} \mathscr{F}_\varepsilon^{int}(u_\varepsilon,B_j).$$

Now, the lim inf of the functional $\mathscr{F}_\varepsilon^{int}$ can be explicitly estimated in case the domain is a ball. Indeed, we can use Proposition 4.3 in [38] that states[6] that

$$\liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon^{int}(u_\varepsilon,B_\rho) \geqslant \omega_{n-1}\rho^{n-1}\left(c_*-\eta(\alpha)\right), \tag{4.13}$$

with $\eta(\alpha)\to 0^+$ as $\alpha\to 0^+$ and c^* being the constant appearing in the definition of $\mathscr{F}$. Combining the above results we deduce that

$$\liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon^{int}(u_\varepsilon,\Omega) \geqslant \omega_{n-1}\left(c_*-\eta(\alpha)\right)\sum_{j=0}^{+\infty}\rho_j^{n-1} \geqslant \left(c_*-\eta(\alpha)\right)\left(\mathrm{Per}(E,\Omega)-\alpha\right),$$

and letting $\alpha\to 0^+$ we prove point (i).

The proof of point (ii) relies on the recovery sequence constructed in Proposition 4.6 of [38]. □

From Theorem 4.3 we easily observe that the two functionals $\mathscr{F}_\varepsilon^{int}$ and $\mathscr{F}_\varepsilon$ attain the same Γ-limit when $s\in[1/2,1)$. Indeed, since $\mathscr{F}_\varepsilon^{int}(u,\Omega)\leqslant\mathscr{F}_\varepsilon(u,\Omega)$ for every function u and domain Ω, from Theorem 4.3 we deduce that for any $u\in X$

(iii) for every u_ε that converges to u in X,

$$\liminf_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon,\Omega) \geqslant \mathscr{F}(u,\Omega);$$

(iv) there exists u_ε that converges to u in X and such that

$$\limsup_{\varepsilon\to 0^+} \mathscr{F}_\varepsilon^{int}(u_\varepsilon,\Omega) \leqslant \mathscr{F}(u,\Omega).$$

That is, combining (i), (ii), (iii), and (iv), both $\mathscr{F}_\varepsilon^{int}$ and $\mathscr{F}_\varepsilon$ converge to the Γ-limit $\mathscr{F}$, that for $s\in[1/2,1)$ is defined as

$$\mathscr{F}(u,\Omega) := \begin{cases} c_*\,\mathrm{Per}(E,\Omega) & \text{if } u_{|\Omega}=\chi_E-\chi_{\mathscr{C}E},\\ +\infty & \text{otherwise.}\end{cases}$$

[6] Observe that Proposition 4.3 in [38] is stated for $\mathscr{F}_\varepsilon^{int}$ as in (4.13), not for $\mathscr{F}_\varepsilon$.

Remark 4.4. The phenomena highlighted in [38] emphasizes a structural difference between the strongly nonlocal regime, i.e., when $s \in (0, 1/2)$, and the weakly nonlocal one in which $s \in [1/2, 1)$.

This difference also affects the different behavior of the interior and exterior contributions of the energy functional in the Γ-limit. Indeed, in the case $s \in (0, 1/2)$ Theorem 4.1 shows that both the interior and the exterior components of the fractional phase coexistence functional $\mathscr{F}_\varepsilon$ converge to two different and nontrivial Γ-limits, whose sum is the full fractional perimeter of a set E in a domain Ω.

On the other hand, when $s \in [1/2, 1)$, the nonlocal interactions on $\Omega \times \mathscr{C}\Omega$ in the functional $\mathscr{F}_\varepsilon$ disappear in the Γ-limit. As a matter of fact, since

$$0 \leqslant \mathscr{F}_\varepsilon^{ext}(u_\varepsilon, \Omega) = \mathscr{F}_\varepsilon(u_\varepsilon, \Omega) - \mathscr{F}_\varepsilon^{int}(u_\varepsilon, \Omega),$$

we have that

$$\begin{aligned} 0 \leqslant \limsup_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon^{ext}(u_\varepsilon, \Omega) &= \limsup_{\varepsilon \to 0^+} \Big(\mathscr{F}_\varepsilon(u_\varepsilon, \Omega) - \mathscr{F}_\varepsilon^{int}(u_\varepsilon, \Omega) \Big) \\ &\leqslant \limsup_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon(u_\varepsilon, \Omega) - \liminf_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon^{int}(u_\varepsilon, \Omega). \end{aligned}$$

Thus, by Theorem 4.3 and assuming $s \in [1/2, 1)$, we have that for every $u \in X$ there exists a sequence u_ε converging to u in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ such that

$$\limsup_{\varepsilon \to 0^+} \mathscr{F}_\varepsilon^{ext}(u_\varepsilon, \Omega) = 0.$$

We recall that the convergence of the level sets of the minimizers described in (2.3) possesses a natural nonlocal counterpart, as established in [37, 39]. More precisely, the statement in (2.3) holds true for the rescaled minimizers of $\mathscr{F}_1$, defined as

$$\mathscr{F}_1(u, \Omega) := \mathscr{K}(u, \Omega) + \int_\Omega W(u(x))\, dx.$$

The only difference with the setting in (2.3) is that the limit set E is now a local minimizer for the classical perimeter when $s \in [1/2, 1)$, and a local minimizer for the nonlocal perimeter when $s \in (0, 1/2)$.

The geometric convergence proofs in [37, 39] also rely on energy and density estimates which can be seen as a nonlocal counterpart of the classical ones in (2.4) and (2.5). More precisely, while (2.5) holds the same in the nonlocal case (i.e., phases do not get lost in the measure theoretic sense), the nonlocal counterpart of (2.4) takes into account different scaling properties depending on the nonlocal exponent. Namely, if u is a minimizer of $\mathscr{F}_1$ in B_{R+1} with $R>2$, then

$$\mathscr{F}_1(u,B_R)\leqslant\begin{cases}CR^{n-2s} & \text{if } s\in(0,1/2),\\ CR^{n-1}\log R & \text{if } s=1/2,\\ CR^{n-1} & \text{if } s\in[1/2,1),\end{cases}\tag{4.14}$$

for some $C>0$ depending on n, s, and W.

That is, comparing (2.4) and (4.14), the energy of the nonlocal minimizers still behaves as if the interfaces were flat, but in this case the energy contribution in a large ball has a "faster" growth due to the strongly long-range interaction arising when $s\in(0,1/2]$. For further details on the one-dimensional case, see also [34].

We also mention that the results and the techniques in [38] have been used by the second and the third author in [30] to study the Γ-convergence of a nonlocal functional arising in a model for water waves (see also [22] for a detailed presentation of the physical models). The energy functional related to this problem depends on a parameter $s\in(0,1)$ and can be described as follows. One defines

$$S_s(\xi)=\frac{J_{1-s}(-i|\xi|)}{J_{s-1}(-i|\xi|)}\,|\xi|^{2s},$$

where J_k is the Bessel function of the first kind of order k. In this setting, S_s plays the role of a "Fourier multiplier", and it has an interesting algebraic property of interpolating between the Fourier symbol of $-\Delta$ for small frequencies and that of $(-\Delta)^s$ for high frequencies — see [30, Theorem 1.1] for details. Then, the energy functional considered in [30] on a compactly supported function u with values in $[0,1]$ takes the form

$$\mathscr{P}_\varepsilon(u):=\varepsilon^{2s}\int_{\mathbb{R}^n}S_s(\xi)\,|\widehat{u}(\xi)|^2\,d\xi+\int_{\mathbb{R}^n}W(u(x))\,dx,\tag{4.15}$$

where $\widehat{u}$ is the Fourier transform of u, and W is a nonnegative double-well function vanishing at 0 and 1. We observe that the scaling in (4.15) is reminiscent of the one in (4.4). Then, recalling the scaling factors in (4.5) and (4.6), one defines

$$\mathscr{Q}_\varepsilon(u) := \begin{cases} \varepsilon^{-2s}\,\mathscr{P}_\varepsilon(u) & \text{if } s \in (0,1/2), \\ |\varepsilon \log \varepsilon|^{-1}\ \mathscr{P}_\varepsilon(u) & \text{if } s = 1/2, \\ \varepsilon^{-1}\,\mathscr{P}_\varepsilon(u) & \text{if } s \in (1/2,1). \end{cases}$$

As proved in [30], when $s \in [1/2,1)$, the Γ-limit of the functional $\mathscr{Q}_\varepsilon$ turns out to be the classical perimeter (up to normalizing constants), in analogy with [38]. On the other hand, when $s \in (0,1/2)$, the Γ-limit of $\mathscr{Q}_\varepsilon$ is a new nonlocal energy functional, structurally different from the fractional Laplacian and from the ones that have been investigated in the literature, given by

$$\mathscr{Q}(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^n} S_s(\xi)\,|\widehat{u}(\xi)|^2\,d\xi & \text{if } u = \chi_E, \text{ for some } E \subset \mathbb{R}^n, \\ 0 & \text{otherwise.} \end{cases}$$

We refer to [30, Theorem 1.3] for a precise statement about the Γ-convergence of $\mathscr{Q}_\varepsilon$.

In the context of nonlocal models for the phase separations of a fluid in a container, we also mention the articles [1, 2], in which the authors study the Γ-convergence of an interaction energy with a summable kernel. In this case, the functional has a singularity which is weaker than the one in [38], and other techniques, different from the ones in [38], are used.

We also mention that an analogue of Conjecture 2.1 for the fractional Allen-Cahn equation $(-\Delta)^s u(x) = W'(u(x))$ opens an interesting line of research. For this, we refer to the recent surveys [15, 22, 23].

5. Limits in the fractional parameter s

Till now, our main focus in this note was on the limit behavior of phase transition energy functionals for the rescaling parameter ε going to zero and for a fixed nonlocal exponent s. However, it is also possible to consider limits in the fractional parameter s. The first result that we present in this setting is a "pointwise" limit, for $s \to (1/2)^-$, of the interior and the exterior contributions in the fractional perimeter of a set E inside the container Ω, that converge, respectively, to the perimeter of E inside Ω, and to the perimeter of E on the boundary of Ω. Recalling the notation in (4.7) and (4.8), we state it in the following theorem.

Theorem 5.1 (Lombardini [27], Maggi and Valdinoci [29]). *Let $s \in (0,1/2)$, $\Omega' \subset \mathbb{R}^n$ be an open set, and $E \subset \mathbb{R}^n$ with locally finite perimeter in Ω'.*

Then, for every open set Ω compactly contained in Ω' and with Lipschitz boundary, it holds that

$$\lim_{s\to(1/2)^-}\left(\frac{1}{2}-s\right)\operatorname{Per}_s^{int}(E,\Omega) = \omega_{n-1}\operatorname{Per}(E,\Omega),$$
$$\lim_{s\to(1/2)^-}\left(\frac{1}{2}-s\right)\operatorname{Per}_s^{ext}(E,\Omega) = \omega_{n-1}\mathscr{H}^{n-1}\left(\partial^* E\cap\partial\Omega\right),$$

where ω_{n-1} is the measure of the $(n-1)$-dimensional unit ball and $\partial^* E$ is the reduced boundary of E.

The study of the pointwise limit of the s-perimeter addressed in Theorem 5.1 has its foundations in the results about the limit as $s \to (1/2)^-$ of the $W^{2s,1}$-seminorm of a function. This study was initiated by Bourgain, Brezis, and Mironescu [7] (see also [17] for optimal assumptions), establishing that the $W^{2s,1}$-seminorm

$$|u|_{W^{2s,1}(\Omega)} := \iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|}{|x-y|^{n+2s}}\,dx\,dy$$

rescaled by $(1/2-s)$ converges as $s \to (1/2)^-$ to the L^1-norm of ∇u. Some further results in this direction are obtained in [35], also establishing the Γ-convergence of the $W^{2s,1}$-seminorm to its pointwise limit. We point out that in [7,17,35] the authors consider exponents $1 \leqslant p < +\infty$ and more general kernels than $|x-y|^{-n-2s}$, studying integrals of the type

$$\iint_{\Omega\times\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^p}\rho_i(x-y)\,dx\,dy,$$

where ρ_i is a sequence of radial mollifiers and the limit is taken for $i \to \infty$.

We also mention the recent contributions [9–11] carrying out the study of both pointwise and Γ-limits as $\delta \to 0$ of a family of nonlocal and nonconvex functionals of the type

$$\delta^p \iint_{\Omega\times\Omega}\frac{\varphi\left(|u(x)-u(y)|/\delta\right)}{|x-y|^{n+p}}\,dx\,dy,$$

where φ is a non-decreasing function satisfying some boundedness and growth assumption.

A Γ-convergence counterpart of Theorem 5.1 is provided by a result in [6], which establishes the Γ-convergence of the fractional perimeter to the classical perimeter, as the fractional parameter s converges to $1/2$:

Theorem 5.2 (Ambrosio, De Philippis, and Martinazzi [6]). *Let $E \subset \mathbb{R}^n$ be a measurable set, and Ω compactly contained in $\mathbb{R}^n$ with Lipschitz boundary. Then,*

(i) *for every sequences $s_i \to (1/2)^-$ and E_i of measurable sets with $\chi_{E_i} \to \chi_E$ in $L^1_{\rm loc}(\mathbb{R}^n)$, we have*

$$\liminf_{i\to\infty} \left(\frac{1}{2} - s_i\right) \mathrm{Per}^{int}_{s_i}(E_i, \Omega) \geqslant \omega_{n-1}\mathrm{Per}(E, \Omega);$$

(ii) *for every sequence $s_i \to (1/2)^-$ there exists a sequence E_i with $\chi_{E_i} \to \chi_E$ in $L^1_{\rm loc}(\mathbb{R}^n)$, such that*

$$\limsup_{i\to\infty} \left(\frac{1}{2} - s_i\right) \mathrm{Per}_{s_i}(E_i, \Omega) \leqslant \omega_{n-1}\mathrm{Per}(E, \Omega).$$

We observe that the role played by interior and exterior contributions in Theorem 5.2 is similar in some aspects to the one in Theorem 4.3. Indeed, from Theorem 5.2 and the fact that $\mathrm{Per}_s(E, \Omega) \geqslant \mathrm{Per}^{int}_s(E, \Omega)$, we immediately deduce that the interior contributions in the fractional perimeter Per^{int}_s and the full s-perimeter Per_s attain the same Γ-limit as s converges to $1/2$. In this sense, the exterior contributions in the fractional perimeter, which are given by the term Per^{ext}_s, do not contribute[7] to the Γ-limit.

We also stress that the boundary contributions in the limit present significant differences when the Γ-limit is replaced by the pointwise one, as a close comparison between Theorems 5.1 and 5.2 clearly shows. Indeed, if one considers the pointwise convergence for $s \to (1/2)^-$, as done in Theorem 5.1, then the interior and the exterior contributions of the fractional perimeter converge, respectively, to the classical perimeter of the set inside the container and to the measure of the part of the boundary of the set E that coincides with the boundary of the container Ω.

[7] The counterpart of this fact for long-range phase transition models was discussed in Remark 4.4.

More specifically, from Theorem 5.2 it follows that, in the sense of Definition 1.1,

$$\Gamma - \lim_{s\to(1/2)^-} \left(\frac{1}{2} - s\right) \operatorname{Per}_s^{ext}(E,\Omega) = 0, \tag{5.1}$$

but from Theorem 5.1 it holds that, for a given set E with Lipschitz boundary,

$$\lim_{s\to(1/2)^-} \left(\frac{1}{2} - s\right) \operatorname{Per}_s^{ext}(E,\Omega) = \omega_{n-1} \operatorname{Per}(E,\partial\Omega). \tag{5.2}$$

Even if at a first glance the "mismatch" between (5.1) and (5.2) can be surprising, or a bit disturbing, several arguments suggest important differences between the Γ-limit in (5.1) and the "pointwise" limit in (5.2). First of all, the Γ-convergence dealt with in our setting relies on the L^1-topology, which is "weak" enough to allow the approximation of every set E with a sequence of sets E_k such that $(\partial E_k) \cap (\partial\Omega) = \varnothing$. This fact makes it possible to "optimize" the recovery sequence in the lim sup inequality of the Γ-convergence setting (recall in particular point (ii) in Definition 1.1) in such a way to "avoid additional boundary contributions".

Another reason for the discrepancy between the limits in (5.1) and (5.2) lies in the "variational nature" of Γ-convergence with a fixed boundary datum. For this, the allowed variations for the related minimization problem are taken with compact support inside the domain Ω. In this sense, the Γ-limit is typically not naturally endowed with additional boundary contributions, which would be not compatible with the notion of local minimizers of the limit problem.

Acknowledgments

S.D. and E.V. are supported by the Australian Research Council Discovery Project DP170104880 "N.E.W. Nonlocal Equations at Work".

S.D. is supported by the DECRA Project DE180100957 "PDEs, free boundaries and applications".

P.M. is supported by the MINECO grant MTM2017-84214-C2-1-P and is part of the Catalan research group 2017SGR1392.

The authors are members of INdAM-GNAMPA. Part of this work was carried out on the occasion of a very pleasant visit of the second author to the University of Western Australia, which we thank for the warm hospitality. This visit was partially supported by a Ferran Sunyer i Balaguer scholarship of the Institut d'Estudis Catalans, granted to the second author.

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