MAXIMUM PRINCIPLES FOR VISCOSITY SOLUTIONS OF WEAKLY ELLIPTIC EQUATIONS
PRINCIPI DI MASSIMO PER SOLUZIONI DI VISCOSITÀ DI EQUAZIONI DEBOLMENTE ELLITTICHE

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Abstract. Maximum principles play an important role in the theory of elliptic equations. In the last decades there have been many contributions related to the development of fully nonlinear equations and viscosity solutions. Here we consider degenerate elliptic equations, where the main term is a partial trace of the Hessian matrix of the solution. We establish maximum principles in domains that are unbounded in some directions, contained in slabs, and extended maximum principles, which lead to removable singularity results.

Sunto. I principi di massimo rivestono un ruolo importante nella teoria delle equazioni ellittiche. Negli ultimi decenni vi sono stati molti contributi correlati allo sviluppo delle equazioni completamente non lineari e alle soluzioni di viscosità. In questo lavoro si considerano equazioni ellittiche degeneri, dove il termine principale è la traccia parziale della matrice hessiana della soluzione. Si stabiliscono principi di massimo in domini illimitati in qualche direzione, contenuti in lastre infinite, ed estensioni del principio di massimo che consentono di ottenere risultati sulle singolarità eliminabili.

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1. Introduction

We consider two classes of elliptic operators, which are in different ways partial traces of $n \times n$ real symmetric matrices X. Namely:

$$\mathcal{P}_{D^{\sigma_k}}(X) = X_{i_1 i_1} + \cdots + X_{i_k i_k} \tag{1}$$

and

$$\mathcal{P}_{\lambda_{\sigma_k}}(X) = \lambda_{i_1}(X) + \cdots + \lambda_{i_k}(X), \tag{2}$$

where $\sigma_k = (i_1, \ldots, i_k)$ is a choice of k positive integers $i_1 < \cdots < i_k$ between 1 and n. We have denoted by X_{ij} the coefficient in the i-th row and j-th column, and by $\lambda_i(X)$ the i-th largest eigenvalues of X.

If $k = n$, both the operators coincide with the full trace of X. If $k < n$, we get therefore different partial traces, which continue to be degenerate elliptic, but no more uniformly elliptic, according to the definition of Section 2.

There are other perspectives which link the two kinds of operators, as it will be clear throughout the paper.

Operators like $\mathcal{P}_{D^{\sigma_k}}$ arise in directional diffusion problems and have been investigated in [12, 13, 14] as well operators like $\mathcal{P}_{\lambda_{\sigma_k}}$ arise in geometric problems of mean partial curvature, see [49, 42, 43], or stochastic differential games, see [5, 6], and have been investigated in [44, 29, 12, 13, 14, 30, 31, 1, 24, 25, 48, 22].

Here, we investigate the validity of the maximum principle (**MP**) for second-order differential elliptic operators having as principal part $\mathcal{F}[u] = \mathcal{F}(D^2u)$, where $\mathcal{F}$ could be $\mathcal{P}_{D^{\sigma_k}}$ or $\mathcal{P}_{\lambda_{\sigma_k}}$, and their generalizations. We denote by D^2u the Hessian matrix of u, when u is C^2, as well as D will stand for the gradient of u. For the meaning of (**MP**) we refer to the next section.

We will see, generally speaking, that these operators are degenerate, non-uniformly elliptic, when $k < n$, and have a different amount of ellipticity: the directional ellipticity for $\mathcal{P}_{D^{\sigma_k}}$'s and the weaker non-totally degenerate ellipticity for $\mathcal{P}_{\lambda_{\sigma_k}}$'s, see Section 2.

Different maximum principles are obtained in the two different cases , if $k < n$. When the principal part is $\mathcal{P}_{D^{\sigma_k}}$, (**MP**) holds with bounded first-order coefficients of any magnitude. On the other hand, when $\mathcal{P}_{\lambda_{\sigma_k}}$ is the principal part, (**MP**) holds provide the coefficients of the gradient term are suitably small, depending on the ellipticity constant and the diameter of the domain. See Section 2 below.

In particular, for operators of type $\mathcal{P}_{D^{\sigma_k}}$, we address the issue of maximum principles in unbounded domains, where generally a control on the growth of the solution is required as $|x| \to \infty$ (Phragmén-Lindelöf principles). We refer to [9, 10, 45, 46, 47, 15, 16] for the uniformly elliptic case.

The first result below is concerned with degenerate elliptic equations in domains which are bounded in some directions, contained in slabs. We recall that existence, uniqueness and maximum principles for uniformly elliptic equations in such domains have been considered since B. Pini [37, 38] and D. Gilbarg [26]. For a general look to maximum principles for second-order partial differential equations we refer to [28] and [39].

The following theorem generalizes the results of [17, 18] to cases in which ellipticity constants and the solution are possibly unbounded, up to a subquadratic growth. It is obtained assuming condition (**SC1**), which is defined in the next section. Such condition is based in turn on the standard condition (**SC0**), which essentially means that $\mathcal{F}$ is a proper, degenerate elliptic operator with Lipschitz-continuous dependence on the gradient; see (6) , (14), (15) and (16). To obtain (**SC1**), it is needed in addition: the coercivity with respect to matrix variations (strict ellipticity) along a bounded direction (11); the Lipschitz-continuity with respect to matrix variations along the bounded directions (27).

Theorem 1.1. *Let Ω be an unbounded domain of $\mathbb{R}^n = \mathbb{R}^h \times \mathbb{R}^k$ such that $\Omega \subset R_h \times \mathbb{R}^k$, where R_h is an open bounded interval in $\mathbb{R}^h$. Let $\mathcal{F}$ satisfy* (**SC1**) *with strict ellipticity* (11) *with respect to a direction $\nu \in \mathbb{R}^h \times \{0\}^k$.*

Let $u^+(x) = o(|x|^\alpha)$ as $x \to \infty$, with $\alpha \in [0, 2]$. Suppose that the following conditions hold for the coefficients b, λ and Λ, involved in (14), (11) *and* (27)*:*

$$\frac{b(x)}{\lambda(x)} = O(1), \quad \frac{\Lambda(x)}{\lambda(x)} = O(|x|^{2-\alpha}). \tag{3}$$

Then (**MP**) *holds.*

This results fills a gap between the condition $\Lambda(x) = O(|x|)$, assumed in [18] for the validity of (**MP**), and the condition $\Lambda(x) >> |x|^2$ of the counterexample to MP of [17, Example 1.9], as discussed in the sequel, after the proof of Theorem 1.1.

We also remark that, in the case that the ellipticity constants $\lambda(x)$, $\Lambda(x)$ and the first-order coefficients $b(x)$ appearing in condition (**SC1**) are constant, then a Phragmén-Lindelöf result holds in slabs, assuming a suitable exponential growth for u. See for instance [17, 18].

Concerning the partial trace operators (1), the above maximum principle follows from the uniformly elliptic case, if the partial sum contains $\lambda_1(X)$. We cannot hope instead to have a weak maximum principle in domains unbounded in some directions for partial trace operators like (1) when the partial sum does not contain $\lambda_1(X)$. See Section 3.

On the other hand, for operators like $\mathcal{P}_{\lambda_{\sigma_k}}$'s, we obtain an extended maximum principle in bounded domains, where the subsolution is given except on a singular set. The result below generalizes those ones of [1, 25] for isolated singularities. Here we use conditon (**SC2**), which will be defined in the next section. It consists of condition (**SC0**) combined with the non-totally degenerate ellipticity (10) and the Lipschitz-continuity (28) with respect to non-negative matrix variations.

Theorem 1.2. *Let Ω be a bounded domain of $\mathbb{R}^n$. Let $\mathcal{F}$ satisfy* (**SC2**) *with* $\sup_\Omega \Lambda(x)/\lambda(x) < \infty$*. There exists $\alpha > 0$, only depending on the parameters of condition* (**SC2**)*, such that, for all $x_0 \in \Omega$ the following extended* (**MP**) *holds.*

Suppose that $u(x) = o(|x - x_0|^{-\alpha})$, as $x \to x_0$, is a viscosity subsolution of the equation $\mathcal{F}[u] = 0$ in $\Omega \backslash \{x_0\}$, and $u \le 0$ on $\partial\Omega$. Then $u \le 0$ in Ω.

This is a basic tool in the study of removable singularities of the equation $\mathcal{F}[u] = 0$, which goes back to [27, 40, 41] and has been diffusely investigated for instance in [33, 35, 36, 30, 31, 1, 25]. Generally, a control on the growth of the solution near the singular set is required. For unconditional result, we refer for instance to [8, 7, 34, 48].

Indeed, a continuation $\tilde{u}$ across x_0 of a solution u of the equation $\mathcal{F}[u] = 0$ in $\Omega\backslash\{x_0\}$ can be obtained solving the Dirichlet problem

$$\begin{cases} \mathcal{F}[\tilde{u}] = 0 & \text{in } \Omega \\ \tilde{u} = u & \text{on } \partial\Omega\,, \end{cases} \tag{4}$$

when it is possible, so that x_0 is a removable singularity.

In the case of the partial trace operators as (2) or the weighted version $\mathcal{F}[u] = a_1\lambda_1(D^2u) + \cdots + a_n\lambda_n(D^2u)$, existence results are established in [5, 22]. A proof of the removability result is contained in [30, 1].

Here we prove a similar result for operators as (1) and more generally for the weighted version $\mathcal{F}[u] = a_1\frac{\partial^2 u}{\partial x_1^2} + \cdots + a_n\frac{\partial^2 u}{\partial x_n^2}$, when $a_i \geq 0$ for all $i = 1, \ldots, n$ and $a_j > 0$ for some $j \in \{1, \ldots, n\}$.

This is obtained combining the extended (**MP**) of Theorem 1.2 for directional elliptic operators (1) and the existence result for the Dirichlet problem (4) shown in Lemma 5.1. The following removability result follows.

Theorem 1.3. *Let Ω be a bounded domain of $\mathbb{R}^n$, and $x_0 \in \Omega$. There exists $\alpha > 0$ such that the following removable singularity result holds.*

Suppose that $u(x)$ is a viscosity solution of the equation

$$a_1\frac{\partial^2 u}{\partial x_1^2} + \cdots + a_n\frac{\partial^2 u}{\partial x_n^2} = f(x) \quad \text{in } \Omega\backslash\{x_0\} \tag{5}$$

where the a_i's are non-negative coefficients such that $\overline{a} = \max_i a_i > 0$ and f is continuous and bounded in Ω.

There exists $\alpha > 0$ such that, if $u(x) = o(|x - x_0|^{-\alpha})$, then u can be continued to a viscosity solution $\tilde{u}$ of equation (5) in Ω. In particular, we can take $\alpha = n/\overline{a} - 2$.

This returns the classical removability result for the Laplace operator in dimension $n \geq 3$. In this case $a_1 = \cdots = a_n = 1$, an harmonic function u in the punctured space $\mathbb{R}^n\backslash\{0\}$ such that $u(x) = o(1/|x|^{-(n-2)})$ as $x \to 0$ can be continued to an harmonic function in the whole space $\mathbb{R}^n$.
The exponent $\alpha = n - 2$ is therefore optimal, since the fundamental solution $u(x) = 1/|x|^{-(n-2)}$ has a singularity at $x = 0$ which cannot be removed.

The paper is organized as follows: in Section 2 we introduce the different notions of ellipticity and the viscosity solutions as well maximum principles; in Section 3 we prove the maximum principle in unbounded domains of Theorem 1.1; in Section 4 we show the extended maximum principle of Theorem 1.2; in Section 5 we prove an existence and uniqueness result as well as Theorem 1.3 on removable isolated singularities.

2. Notations and preliminary results

We consider a fully nonlinear operator $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ of the variables $(x, t, \xi, X) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n$. Here Ω is a domain (open connected set) of $\mathbb{R}^n$ and $\mathcal{S}^n$ is the space of $n \times n$ real symmetric matrices.

The full nonlinearity consists in a possibly nonlinear dependence with respect to X, which implies a nonlinear dependence on the higher derivatives of the corresponding second-order differential operator

$$\mathcal{F}[u] = \mathcal{F}(x, u, Du, D^2u).$$

We recall that Du and D^2u stand for the gradient and the Hessian matrix of u, when $u \in C^2(\Omega)$, but partial differential equations $\mathcal{F}[u] = 0$ will have a (weak) sense also for $u \in C(\Omega)$, as it will be seen below.

We recall the definitions of degenerate and uniform ellipticity, introducing in $\mathcal{S}^n$ the following partial order relationship: $X \leq Y$ if and only if $Y - X$ is semidefinite positive.

Degenerate ellipticity. $\mathcal{F}$ is degenerate elliptic in Ω if it is non-decreasing in X:

$$X \leq Y \quad \Rightarrow \quad \mathcal{F}(x, t, \xi, X) \leq \mathcal{F}(x, t, \xi, Y) \tag{6}$$

for all $(x, t, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$.

Uniform ellipticity. $\mathcal{F}$ is uniformly elliptic in Ω if there exist positive constants λ and $\Lambda \geq \lambda$ (ellipticity constants) such that:

$$X \leq Y \quad \Rightarrow \quad \lambda \mathrm{Tr}(Y - X) \leq \mathcal{F}(x, t, \xi, Y) - \mathcal{F}(x, t, \xi, X) \leq \Lambda \mathrm{Tr}(Y - X) \tag{7}$$

for all $(x, t, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$. We have denoted by $\mathrm{Tr}(X)$ the trace of $X \in \mathbb{S}^n$.

It is plain that the uniform ellipticity implies the degenerate ellipticity, but the converse is not true. For instance, the trace operator $\mathrm{Tr}(X)$ is uniform elliptic with ellipticity constants $\lambda = 1 = \Lambda$, but the partial trace operators $\mathcal{P}_{D^{\sigma_k}}$ or $\mathcal{P}_{\lambda_{\sigma_k}}$ are both non-uniformly elliptic when $\sigma_k = (i_1, \dots, i_k)$ with $k < n$.

In fact, let $\{e_i\}_{i=1,\dots,n}$ the canonical basis of $\mathbb{R}^n$: $e_1 = (1, \dots, 0), \dots,\ e_n = (0, \dots, 1)$. For $\xi, \eta \in \mathbb{R}^n$, we denote by $\xi \otimes \eta$ the $n \times n$ the matrix of entries $(\xi \otimes \eta)_{ij} = \xi_i \eta_j$. Let also I be the $n \times n$ identity matrix $(I)_{ij} = \delta_{ij}$, the Kronecker symbol, such that $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

For $n \geq 2$, the operator $\mathcal{P}_{D_1}(X) = X_{11}$ is degenerate elliptic, since $X \leq Y$ implies $X_{11} \leq Y_{11}$. On the other hand, let $X = e_1 \otimes e_1$ and $Y = I$. Hence $X \leq Y$. Then we have $X_{11} = 1 = Y_{11}$, so that $\mathcal{P}_{D_1}(Y) - \mathcal{P}_{D_1}(X) = 0 < \lambda = \lambda \mathrm{Tr}(Y - X)$ for all $\lambda > 0$. Therefore $\mathcal{P}_{D_1}$ is not uniformly elliptic.

We also recall the two Pucci extremal operators, for given ellipticity constants λ and Λ, namely:

$$\mathcal{M}^+_{\lambda,\Lambda}(X) = \Lambda \sum_{i=1}^n \lambda_i^+(X) - \lambda \sum_{i=1}^n \lambda_i^-(X)$$

and

$$\mathcal{M}^-_{\lambda,\Lambda}(X) = \lambda \sum_{i=1}^n \lambda_i^+(X) - \Lambda \sum_{i=1}^n \lambda_i^-(X),$$

where $\lambda^+ = \max(\lambda, 0)$ and $\lambda^- = -\min(\lambda, 0)$. The maximality of $\mathcal{M}^+_{\lambda,\Lambda}$ and the minimality of $\mathcal{M}^-_{\lambda,\Lambda}$ derive from the following inequalities:

$$\mathcal{M}^-_{\lambda,\Lambda}(Y - X) \leq \mathcal{F}(x, t, \xi, Y) - \mathcal{F}(x, t, \xi, X) \leq \mathcal{M}^+_{\lambda,\Lambda}(Y - X). \tag{8}$$

This is equivalent to the uniform ellipticity of $\mathcal{F}$. If in addition $\mathcal{F}(x, t, \xi, O) = 0$, then (8) implies

$$\mathcal{M}^-_{\lambda,\Lambda}(X) \leq \mathcal{F}(x, t, \xi, X) \leq \mathcal{M}^+_{\lambda,\Lambda}(X).$$

Note that $\mathcal{M}^-_{1,1}(D^2u) = \Delta u = \mathcal{M}^+_{1,1}(D^2u)$. We also list some useful properties of the Pucci extremal operators:

$$\begin{aligned} &\mathcal{M}^+_{\lambda,\lambda}(-X) = -\mathcal{M}^-_{\lambda,\lambda}(X); \\ &\mathcal{M}^+_{\lambda,\lambda}(X + Y) \leq \mathcal{M}^+_{\lambda,\lambda}(X) + \mathcal{M}^+_{\lambda,\lambda}(Y). \end{aligned} \tag{9}$$

Here we will consider two notions of ellipticity which, even combined with (6), are both weaker than the uniform ellipticity: the non-totally degenerate ellipticity, see [4], and the directional ellipticity, see [17, 18]. In all cases, the operator $\mathcal{F} : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ will be assumed continuous throughout this paper.

Non-totally degenerate ellipticity. $\mathcal{F}$ is non-totally degenerate elliptic in Ω if and only if $\mathcal{F}$ is degenerate elliptic and

$$\mathcal{F}(x,t,\xi,X+\tau I) - \mathcal{F}(x,t,\xi,X) \geq \lambda(x)\tau, \quad \tau > 0, \tag{10}$$

for all $(x,t,\xi,X) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{S}^n$, where λ is a continuous function such that $\lambda(x) > 0$ in Ω.

Directional ellipticity. $\mathcal{F}$ is strictly elliptic in Ω with respect to the direction $\nu \in S^{n-1} \equiv \{\xi \in \mathbb{R}^n : |\xi| = 1\}$ if and only if $\mathcal{F}$ is degenerate elliptic and

$$\mathcal{F}(x,t,\xi,X+\tau\,\nu\otimes\nu) - \mathcal{F}(x,t,\xi,X) \geq \lambda(x)\tau, \quad \tau > 0, \tag{11}$$

for all $(x,t,\xi,X) \in \Omega \times \mathbb{R}^+ \times \mathbb{R}^n \times \mathcal{S}^n$, where λ is a continuous function such that $\lambda(x) > 0$ in Ω.

It is immediate to see that the uniform ellipticity implies the directional ellipticity. We also remark that the directional ellipticity implies the non-totally degenerate ellipticity. In fact, assuming (11) and observing that $I = \nu\otimes\nu + (I - \nu\otimes\nu)$, we have:

$$\begin{aligned} &\mathcal{F}(x,t,\xi,X+\tau I) - \mathcal{F}(x,t,\xi,X) \\ \geq\, &\mathcal{F}(x,t,\xi,X+\tau\,\nu\otimes\nu) - \mathcal{F}(x,t,\xi,X) \geq\ \lambda(x)\tau, \quad \tau > 0, \end{aligned}$$

where in the first inequality we have used the degenerate ellipticity (6) and the positive definiteness of the matrix $(I - \nu\otimes\nu)$.

As we will see in the sequel, the partial trace operators $\mathcal{P}_{D^{\sigma_k}}(X) = X_{i_1 i_1} + \cdots + X_{i_k i_k}$, introduced at the beginning (1), correspond to the differential operators

$$\mathcal{P}_{D^{\sigma_k}}[u] = \frac{\partial^2 u}{\partial x_{i_1}^2} + \cdots + \frac{\partial^2 u}{\partial x_{i_k}^2}, \tag{12}$$

which are strictly elliptic with respect to the directions $e_{i_1}, \ldots, e_{i_k}$, non-uniformly elliptic if $k < n$ directional ellipticity.

On the other hand, the partial trace operators $\mathcal{P}_{\lambda_{\sigma_k}}(X) = \lambda_{i_1}(X) + \cdots + \lambda_{i_k}(X)$, defined in (2), correspond to the differential operators

$$\mathcal{P}_{\lambda_{\sigma_k}}[u] = \lambda_{i_1}(D^2u) + \cdots + \lambda_{i_k}(D^2u) \tag{13}$$

which are non-totally degenerate elliptic operators, less than directional elliptic if $k < n$.

In fact, for all $\nu \in S^{n-1}$, with $n \geq 2$, there exists a matrix X such that $\lambda_n(X + \nu \otimes \nu) = 1 = \lambda_n(X)$, so that λ_n cannot be strictly elliptic with respect to any direction: it is sufficient to take $X = \mu \otimes \mu$, where $\mu \in S^{n-1}$ is orthogonal to ν.

Concerning the first order terms, we will assume that $\mathcal{F}$ is Lipschitz continuous with respect to ξ, precisely

$$|\mathcal{F}(x, t, \eta, X) - \mathcal{F}(x, t, \xi, X)| \leq b(x)|\eta - \xi|, \quad \xi, \eta \in \mathbb{R}^n, \tag{14}$$

for all $(x, t, X) \in \Omega \times \mathbb{R} \times \mathcal{S}^n$, where b is a continuous function, bounded on bounded domains, such that $b(x) \geq 0$ in Ω.

With respect to zero-order terms, we will consider the monotonicity assumption

$$\mathcal{F}(x, s, \xi, X) \leq \mathcal{F}(x, t, \xi, X), \quad s < t, \tag{15}$$

for all $(x, \xi, X) \in \Omega \times \mathbb{R}^n \times \mathcal{S}^n$.

We are concerned with solutions u of the equation

$$\mathcal{F}[u] \equiv \mathcal{F}(x, u, Du, D^2u) = f(x),$$

and $f(x)$ will be assumed continuous in Ω, assuming that

$$\mathcal{F}(x, 0, 0, O) = 0. \tag{16}$$

There is no loss of generality assuming (16). In fact, condition (16) is in general satisfied with the operator $\mathcal{G}(x, t, \xi, X) = \mathcal{F}(x, t, \xi, X) - \mathcal{F}(x, 0, 0, O)$ instead of $\mathcal{F}(x, t, \xi, X)$. So we can apply the results based on (16) to the equation $\mathcal{G}[u] = g(x)$, where $g(x) = f(x) - \mathcal{F}(x, 0, 0, O)$.

(**SC0**) We say that $\mathcal{F}$ satisfies the structural condition (**SC0**) if (6), (14), (15) and (16) hold.

If $u \in C^2(\Omega)$, the equation for $\mathcal{F}[u] = f$ is intended in the classical sense:

$$\mathcal{F}(x, u(x), Du(x), D^2u(x)) = f(x) \text{ for all } x \in \Omega.$$

However, we can consider solutions which are only continuous, in the viscosity sense defined here below.

Let u be an upper (resp. lower) semicontinuous function in Ω, for short $u \in \mathrm{usc}(\Omega)$ (resp. $u \in \mathrm{lsc}(\Omega)$) . We say that u is a viscosity subsolution (resp. supersolution) of the equation $\mathcal{F}[u] = f(x)$ in Ω, or also a viscosity solution of the differential inequality $\mathcal{F}[u] \geq f(x)$ (resp. $\mathcal{F}[u] \leq f(x)$), if and only if:
for all $x_0 \in \Omega$ and all C^2 (test) functions $\varphi(x)$ touching u from above at x_0, namely $\varphi(x_0) = u(x_0)$ and $\varphi(x) \geq u(x)$ in a neighbourhood of x_0, we have

$$F(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq f(x_0)\,, \tag{17}$$

resp. for all $x_0 \in \Omega$ and all C^2 (test) functions $\varphi(x)$ touching u from below at x_0, namely $\varphi(x_0) = u(x_0)$ and $\varphi(x) \leq u(x)$ in a neighbourhood of x_0, we have

$$F(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \leq f(x_0)\,. \tag{18}$$

A viscosity solution of the equation $\mathcal{F}[u] = f(x)$ is a function $u \in C(\Omega)$ which is a viscosity subsolution and supersolution of the given equation.

We will use in the sequel the following property of viscosity solutions with respect to the pointwise sup and inf operations. Let $\{u_j\}$ be a finite family of viscosity subsolutions (resp. supersolutions) of the equation $\mathcal{F}[u] = f(x)$. Then $v = \sup_j u_j$ (resp. $\inf_j u_j$) is in turn a viscosity subsolution of the equation $\mathcal{F}[v] = -f^-$ (resp. a supersolution of the equation $\mathcal{F}[v] = f^+$). See [11].

For more properties of viscosity solutions we refer to [20], [11], [19], [32].

Suppose that $\mathcal{F}$ satisfies condition (**SC0**) in a bounded domain Ω. Assuming that $\mathcal{F}$ uniformly elliptic, we have for $u \in \mathrm{usc}(\overline{\Omega})$ the following maximum principle:

(**MP**) $\mathcal{F}[u] \geq 0$ in Ω, $u \leq 0$ on $\partial\Omega \;\Rightarrow\; u \leq 0$ on Ω.

See for instance [9, 10, 45, 46], which have been established for more general domains, satisfying the measure-geometric property introduced by Cabré [9], and are based on the weak Harnack inequality, which is not available for directional elliptic operators as (1).

However, the full uniform ellipticity is not necessary: (**MP**) continues to hold if $\mathcal{F}$ is strictly elliptic in at least a direction or $\mathcal{F}$ is non-totally degenerate elliptic and (14) holds with sufficiently small first-order coefficient $b(x)$, as it is shown by the result below. For other conditions on the first-order term see [2, 23].

Theorem 2.1. *Let Ω be a bounded domain of $\mathbb{R}^n$, and $\mathcal{F}$ satisfy condition (***SC0***). Then* (**MP**) *holds if in addition:*

(i) $\mathcal{F}$ is strictly elliptic as in (11), with respect to some direction $\nu \in S^{n-1}$, and

$$\inf_{\Omega} \lambda(x) > 0 \tag{19}$$

or

(ii) $\mathcal{F}$ is non-totally degenerate elliptic as in (10) with

$$\inf_{\Omega} (\lambda(x) - b(x)|x - x_0|) > 0 \tag{20}$$

for some $x_0 \in \mathbb{R}^n$.

Proof. Suppose $u \in \mathrm{usc}(\overline{\Omega})$, $\mathcal{F}[u] \geq 0$ in Ω and $u \leq 0$ on $\partial\Omega$. To prove (**MP**) we have to show that $u \leq 0$ in Ω.

Strict subsolutions

Suppose in addition $\mathcal{F}[u] > 0$ in Ω. Arguing by contradiction, suppose there exists a positive maximum of u in Ω, say $u = M > 0$. Then the function $\varphi \equiv M$ touches from above the subsolution u, and therefore $\mathcal{F}(x, 0, 0, O) \geq \mathcal{F}(x, M, 0, O) > 0$, contradicting (16). Then $u \leq 0$ in Ω, as we needed to prove.

All subsolutions

Next, we only assume the non-strict differential inequality $\mathcal{F}[u] \geq 0$ in Ω, so that $\mathcal{F}[u^+] \geq 0$ in Ω in the viscosity sense, since $u^+(x) = \sup(u(x), 0)$ and $\mathcal{F}[0] = 0$.

Case (i)

We may assume that the direction ν in (11) is $e_1 = (1, 0, \dots, 0)$, and $0 \le x_1 \le d$. For $\varepsilon > 0$ we consider the function $v(x) := u^+(x) - \varepsilon(e^{\beta d} - e^{\beta x_1})$, with $\beta > 0$ to be suitably chosen.

Then $v \le u^+$ in $\overline{\Omega}$, and by (15), (11), (14):

$$\begin{aligned} \mathcal{F}[v] &\ge \mathcal{F}(x, u^+, Du^+ + \varepsilon\beta e^{\beta x_1}, D^2u^+ + \varepsilon\beta^2 e^{\beta x_1} e_1 \otimes e_1) \\ &\ge \varepsilon\beta\left(\beta\lambda(x) - b(x)\right) e^{\beta x_1} \end{aligned} \tag{21}$$

in the viscosity sense. Therefore, choosing β large enough, we get $\mathcal{F}[v] > 0$ in Ω.

Moreover, since $v(x) \le u^+(x) = 0$ on $\partial\Omega$, the case of *strict subsolutions* implies that $v \le 0$ in Ω, namely:

$$u^+(x) \le v(x) + \varepsilon e^{\beta d} \le \varepsilon e^{\beta d},$$

from which, letting $\varepsilon \to 0^+$, we get $u^+ = 0$, as we wanted to prove.

Case (ii)

In this case, let $R > 0$ be the radius of a ball $B_R(x_0)$ centered at x_0 such that $\Omega \subset B_R(x_0)$. For $\varepsilon > 0$ we define the function $v(x) := u^+(x) - \frac{1}{2}\varepsilon\left(R^2 - |x - x_0|^2\right)$. As before $v \le u^+$, and by (15), (11), (14):

$$\begin{aligned} \mathcal{F}[v] &\ge \mathcal{F}(x, u^+, Du^+ + \varepsilon C|x - x_0|, D^2u^+ + \varepsilon\, C\, I) \\ &\ge \varepsilon C\left(\lambda(x) - b(x)|x - x_0|\right) \end{aligned} \tag{22}$$

in the viscosity sense. Therefore, assuming (20), we get $\mathcal{F}[v] > 0$ in Ω. Since $v \le u^+ \le 0$ on $\partial\Omega$, by the case of *strict subsolutions*, we deduce as in (i) that $v \le 0$ in Ω, namely

$$u^+(x) \le v(x) + \frac{1}{2}\varepsilon R^2 \le \frac{1}{2}\varepsilon R^\cdot$$

Also in this case, letting $\varepsilon \to 0^+$ yields $u^+ = 0$. This concludes the proof. □

Remark 2.1. Setting

$$\overline{b} = \sup_{x\in\Omega} b(x), \quad \underline{\lambda} = \inf_{x\in\Omega} \lambda(x), \tag{23}$$

condition (20) is satisfied assuming

$$\overline{b}\, d < \underline{\lambda}. \tag{24}$$

To see this, it is sufficient to take any $x_0 \in \Omega$ in (20).

In [25] the authors showed the optimality of this condition, giving a counterexample when Ω is a ball: if $bd > \lambda$, there exists a continuous viscosity subsolution u of the equation $\lambda_n(D^2u) = 0$ such that $u = 0$ on $\partial\Omega$, but u has positive values somewhere in Ω. $\square$

The maximum principles of Theorem 2.1 can be generalized to obtain the following above estimates.

Corollary 2.1. *Let Ω be a bounded domain of $\mathbb{R}^n$, and $\mathcal{F}$ satisfy condition* (**SC0**). *Suppose that $u \in usc(\overline{\Omega})$ is a subsolution of the equation $\mathcal{F}[u] = f(x)$.*

(i) If $\mathcal{F}$ is strictly elliptic (11), with respect to a direction $\nu \in S^{n-1}$, and satisfies (19), *let*

$$\sup_{x\in\Omega} \frac{b(x)}{\lambda(x)} < B < \infty, \quad \sup_{x\in\Omega} \frac{f^-(x)}{\lambda(x)} = K < \infty.$$

We have

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + CK, \tag{25}$$

where C is a positive constant depending on Bd and $d > 0$ is the thickness of any slab $S_{h,k,\nu} = \{x \in \mathbb{R}^n : h \le \langle x, \nu\rangle \le k\} \supset \Omega$.

(ii) If $\mathcal{F}$ is non-totally degenerate elliptic (10) and satisfies (20), let

$$\sup_{x\in\Omega} \frac{f^-(x)}{\lambda(x) - b(x)|x - x_0|} = K_b < \infty.$$

We have

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+ + \frac{1}{2} K_b R^2, \tag{26}$$

where R is the radius of any ball $B_R(x_0) \supset \Omega$.

Proof. Recall that $\mathcal{F}[u] \ge f(x)$ implies $\mathcal{F}[u^+] \ge -f^-(x)$, and let $K = \sup_\Omega f^-$.

Case (i)

We suppose $\nu = e_1$ and $\Omega \subset S_{0,d,e_1}$. Let us consider the function $v(x) = u^+(x) - \sup_{\partial\Omega} u^+ - CK\left(e^{\beta d} - e^{\beta x_1}\right)$.

Choosing $\beta = 2B$ and $C \geq 1/2B^2$, by direct computation we have:

$$\begin{aligned}\mathcal{F}[v] &\geq \mathcal{F}(x, u^+, Du^+ + CK\beta\, e^{\beta x_1}, D^2u^+ + C\beta^2 e^{\beta x_1}\, I)\\ &\geq \mathcal{F}(x, u^+, Du^+, D^2u^+) + CK\beta(-b(x) + \beta\,\lambda(x))\\ &\geq -f^-(x) + CK\lambda(x)\beta(-B + \beta)\\ &\geq -f^-(x) + 2B^2Cf^-(x) \geq 0.\end{aligned}$$

Since $v(x) \leq u^+(x) - \sup_{\partial\Omega} u^+ = 0$ on $\partial\Omega$, Theorem 2.1 yields $v \leq 0$ in Ω, and therefore

$$u^+(x) \leq \sup_{\partial\Omega} u^+ + CK(e^{2Bd} - 1),$$

from which (25).

Case (ii)

We consider the function $v(x) = u^+(x) - \sup_{\partial\Omega} u^+ - \frac{1}{2}\, K_b\, (R^2 - |x - x_0|^2)$. Then

$$\begin{aligned}\mathcal{F}[v] &\geq \mathcal{F}(x, u^+, Du^+ + K_b|x - x_0|, D^2u^+ + K_b\, I)\\ &\geq \mathcal{F}(x, u^+, Du^+, D^2u^+) + K_b\,(-b(x)|x - x_0| + \lambda(x))\\ &\geq -f^-(x) + f^-(x) = 0.\end{aligned}$$

Since $v(x) \leq u^+(x) - \sup_{\partial\Omega} u^+ = 0$ on $\partial\Omega$, Theorem 2.1 yields $v \leq 0$ in Ω, and therefore

$$u^+(x) \leq \sup_{\partial\Omega} u^+ + \tfrac{1}{2} K_b R^2,$$

from which (26). □

Note that no assumption has been made to control the variation of $\mathcal{F}$ appearing in (10) or (11) from above, up to now.

In the case that Ω is unbounded in some direction, we need instead such a control with respect to the unbounded directions.

To avoid inessential complications, suppose that $\Omega \subset R_k \times \mathbb{R}^k$, where R_h is an open bounded interval of $\mathbb{R}^h$ and $h + k = n$. In the case of directional elliptic operators with respect to some direction $\nu \in \mathcal{R}^h \times \{0\}^k$, we will assume the following condition:

$$\begin{aligned}F(x, t, \xi, X + Q) - F(x, t, \xi, X) \leq{}& \Lambda(x)\,\lambda_n(Q)\\ &\text{for all } Q \geq O \ \text{ s.t. } \langle Q\nu, \nu\rangle = 0,\end{aligned} \tag{27}$$

for some positive continuous function $\Lambda(x)$.

In order to deal with (**MP**) in unbounded domains, we need to strengthen condition (**SC0**).

(**SC1**) We say that $\mathcal{F}$ satisfies the structural condition (**SC1**) if (11) and (27) hold, in addition to (**SC0**).

Similar conditions can be used for the issue of removable singularities, which we consider here in the case of non-totally degenerate elliptic operators. To this end we need to assume the above condition on the variation of $\mathcal{F}$ for all possible non-negative increments on $X \in \mathcal{S}^n$, namely:

$$F(x,t,\xi,X+Q) - F(x,t,\xi,X) \leq \Lambda(x)\,\mathrm{Tr}(Q) \quad \text{for all } Q \geq O. \tag{28}$$

In this case, we need a different condition strengthening (**SC0**).

(**SC2**) We say that $\mathcal{F}$ satisfies the structural condition (**SC2**) if (10), (20) and (28) hold, in addition to (**SC0**).

3. Maximum principles in unbounded domains

When dealing with unbounded domains, we need in general assume an a-priori control on the growth of the subsolutions at infinity, even in the uniformly elliptic case. Recall for instance that the function $u(x) = e^{x_2} \sin x_1$ is a smooth harmonic function in the plane. Nevertheless, u is positive in the strip $S_0 = \{0 < x_1 < \pi\}$, even though $u = 0$ on $\partial S = \{x_1 = 0\} \cup \{x_1 = \pi\}$.

A typical assumption is that u is bounded above, see [9], [10], [45], [15], [17], [18].

Considering $\Omega \subset R_h \times \mathbb{R}^k$, where R_h is an open bounded interval in $\mathbb{R}^h$, the coordinates of a point of Ω will be splitted as $x = (y,z)$ with $y = (y_1,\ldots,y_h)$ and $z = (z_1,\ldots,z_k)$.

We will assume that $u^+(x) = o(\psi(x))$ in Ω as $|x| \to \infty$ in a weak sense, namely

$$\liminf_{\substack{|x|\to\infty \\ x\in\Omega}} \frac{u^+(x)}{\psi(x)} = 0,$$

where ψ is a smooth function in Ω, continuous up to the boundary, such that $\psi(x) > 0$ in Ω.

According to the directional ellipticity (i) of Theorem 2.1, we can consider $\psi(x) = \psi(y,z) = \varphi(z) = \phi(|z|)$, which depends only on $z = (z_1, \dots, z_k) \in \mathbb{R}^k$.

Lemma 3.1. *Let Ω be an unbounded domain of $\mathbb{R}^n$ such that $\Omega \subset R_h \times \mathbb{R}^k$, where R_h is an open bounded interval in $\mathbb{R}^h$. Let $\mathcal{F}$ satisfy condition* (**SC1**) *with direction of strict ellipticity $\nu \in \mathbb{R}^h \times \{0\}^k$, see* (11).

Let $\phi : \mathbb{R}^+ \to \mathbb{R}$ be a positive C^2 function such that

$$\phi'(t) \geq 0, \quad \phi''(t) \leq \frac{\varphi'(t)}{t},. \tag{29}$$

$$\sup_{x\in\Omega} \frac{b(x)}{\lambda(x)} < \infty, \quad \sup_{x\in\Omega} \left(\frac{b(x)}{\lambda(x)} \phi'(|z|) + \frac{\Lambda(x)}{\lambda(x)} \frac{\phi'(|z|)}{|z|} \right) < \infty, \tag{30}$$

where $x = (y,z) = (y_1, \dots, y_h, z_1, \dots, z_k)$.

Then (**MP**) *holds in Ω for subsolutions of the equation $\mathcal{F}[u] = 0$ such that $u^+(x) = o(\phi(|z|))$ as $|z| \to \infty$ in the following weak sense:*

$$\liminf_{\substack{|z|\to\infty \\ x\in\Omega}} \sup_{y\in R_h} \frac{u^+(y,z)}{\phi(|z|)} = 0. \tag{31}$$

Proof. By assumption, easily passing to $u^+ = \max(u, 0)$ in the viscosity setting, from condition (16) we have $\mathcal{F}(x, u^+, Du^+, D^2u^+) \geq 0$ in $\Omega \subset R_h \times \mathbb{R}^k$.

Let $\psi(x) = \psi(y,z) = \varphi(z) = \phi(|z|)$. By condition (31), we can take sequences of numbers $\varepsilon > 0$ and $R_\varepsilon > 0$ such that $R_\varepsilon \to \infty$ as $\varepsilon \to 0^+$ and $u^+(x) - \varepsilon\psi(x) \leq 0$ as $|z| \geq R_\varepsilon$.

Correspondingly, we define the function $v_\varepsilon(x) = u^+(x) - \varepsilon\psi(x)$, noting that $v_\varepsilon \leq 0$ for $|y| \geq R_\varepsilon$. Then $S_\varepsilon = \{x \in \Omega : |y| < R_\varepsilon\}$ is an open bounded set such that

$$v_\varepsilon \leq 0 \quad \text{on } \partial S_\varepsilon. \tag{32}$$

On the other hand, by (15):

$$\mathcal{F}[v_\varepsilon] \geq F(x, u^+, Du^+ - \varepsilon D\psi, D^2u^+ - \varepsilon D^2\psi). \tag{33}$$

Next, we compute $D\psi$ and $D^2\psi$ (for $z \neq 0$):

$$D\psi(x) = \phi'(|z|)\,\frac{z}{|z|}$$

(34)

$$D^2\psi(x) = \left(\phi''(|z|) - \frac{\phi'(|z|)}{|z|}\right)\,\frac{z}{|z|} \otimes \frac{z}{|z|} + \frac{\phi'(|z|)}{|z|}\,I_z\,,$$

where

$$I_z = \begin{pmatrix} 0 & 0 \\ 0 & I_k \end{pmatrix}$$

and I_k is the $k \times k$ identity matrix.

By assumption (29), we have $D^2\psi(x) \leq \frac{\phi'(|z|)}{|z|}\,I_z$. Then, inserting in (33) and using (6), (14), (27), we get in S_ε:

$$\begin{aligned} \mathcal{F}[v_\varepsilon] &\geq F(x, u^+, Du^+ - \varepsilon\phi'(|z|)\,\tfrac{z}{|z|}, D^2u^+ - \varepsilon\tfrac{\phi'(|z|)}{|z|}\,I_z) \\ &\geq F(x, u^+, Du^+, D^2u^+) - \varepsilon\left(b(x)\phi'(|z|) + k\,\Lambda(x)\,\tfrac{\phi'(|z|)}{|z|}\right) \\ &\geq -\,\varepsilon\left(b(x)\phi'(|z|) + k\,\Lambda(x)\,\tfrac{\phi'(|z|)}{|z|}\right). \end{aligned} \tag{35}$$

From Corollary 2.1 we deduce:

$$v_\varepsilon(x) \leq \varepsilon\,C\sup_{x\in\Omega}\left(1 + \frac{b(x)}{\lambda(x)}\,\phi'(|z|) + \frac{\Lambda(x)}{\lambda(x)}\,\frac{\phi'(|z|)}{|z|}\right).$$

It follows, for $x \in \Omega$ such that $|y| < R_\varepsilon$:

$$u^+(x) \leq \varepsilon\phi(x) + \varepsilon\,C\sup_{x\in\Omega}\left(1 + \frac{b(x)}{\lambda(x)}\,\phi'(|z|) + \frac{\Lambda(x)}{\lambda(x)}\,\frac{\phi'(|z|)}{|z|}\right). \tag{36}$$

Let us fix $x = (y_1, \ldots, y_h, z_1, \ldots, z_k) \in \Omega$. Then $|z| < R_\varepsilon$ for sufficiently small $\varepsilon > 0$. So, letting $\varepsilon \to 0^+$ in (36), we get $u^+(x) = 0$, as we needed to prove. $\square$

Theorem 1.1 follows from this lemma by a suitable choice of the function ϕ.

Proof of Theorem 1.1. Let $\alpha \in [0,2]$. We choose a function $\phi : (0,\infty) \to \mathbb{R}_+$ such that, in addition to (29) satisfies, for some $K \in \mathbb{R}_+$:

$$\phi'(t) \leq K(1+t)^{\alpha-1}, \quad t \geq 0, \tag{37}$$

For instance, in the case $\alpha = 1$, we can choose

$$\phi(t) = \begin{cases} 1 - e^{-t^2} & \text{if } 0 \leq t \leq 1/\sqrt{2} \\ \sqrt{2}\,e^{-1/2}t + 1 - 2e^{-1/2} & \text{if } t > 1/\sqrt{2}\,. \end{cases} \tag{38}$$

Then, by the assumptions on $b(x)$ and $\Lambda(x)$, Lemma 3.1 yields the result. $\square$

Consider for instance a viscosity subsolution u, in the strip $\Omega = (-1,1)\times\mathbb{R}$, of the following fully nonlinear equation of Bellman type:

$$\sup_{0\le t\le 1}\left\{\frac{\partial^2 u}{\partial x_1^2} + t|x|^{2-\alpha}\frac{\partial^2 u}{\partial x_2^2} + (1-t)\left(\frac{\partial u}{\partial x_1}+\frac{\partial u}{\partial x_2}\right)\right\} = 0. \tag{39}$$

By (3) of Theorem 1.1, if $u^+(x) = o(|x|^\alpha)$ as $|x|\to\infty|$, with $0\le\alpha\le 2$, then $u\le 0$ on $\partial\Omega$ implies $u\le 0$ in Ω. In particular, if u is bounded above, then (**MP**) holds with $\Lambda(x) = O(|x|^{2-\alpha})$ for arbitrarily small $\alpha>0$.

For the partial trace operators (1) we have different results. If the partial sum contains $\lambda_1(X)$, a maximum principle in domains which are bounded in some direction follows from the uniformly elliptic case, since

$$\lambda_{i_1}(X)+\cdots+\lambda_{i_k}(X) \le \frac{1}{n-1}\sum_{i=1}^{n-1}\lambda_i(X) + (k-1)\lambda_n(X),$$

which is a uniformly elliptic operator with ellipticity constants $\lambda = \frac{1}{n-1}$ and $\Lambda = k-1$, see [22]. Therefore $\lambda_{i_1}(D^2u)+\cdots+\lambda_{i_k}(D^2u)\ge 0$ implies $\mathcal{M}^+_{\lambda,\Lambda}(D^2u)\ge 0$.

If instead the partial sum does not contain $\lambda_1(X)$, we cannot hope to have a similar maximum principle, as the following counterexample shows. Let $u(x) = \sin x_1$ in $\mathbb{R}^n$, then $u(x)$ is a subsolution, bounded above, of the equation

$$\mathcal{P}_k^+(D^2u) := \lambda_{n-k+1}(D^2u)+\cdots+\lambda_n(D^2u) = 0$$

in $\Omega = (0,\pi)\times\mathbb{R}^{n-1}$, if $k\le n-1$. Note also that $u(x)=0$ on $\partial\Omega$. However, $u(x)>0$ in Ω and the maximum principle fails to hold.

4. Extended maximum principles

Let Ω be an bounded open set, and x_0 be a point of Ω. Suppose to have the subsolution $u\in\mathrm{usc}(\overline{\Omega}\backslash\{x_0\})$ of an elliptic equation $\mathcal{F}[u]=0$ in $\dot\Omega := \Omega\backslash\{x_0\}$.

We call x_0 a singular point or isolated singularity. We ask for conditions in order that the solution u can be continued to a subsolution $\tilde u$ in all Ω. In affirmative case, we call x_0 a removable singularity.

In this case, the maximum principle would imply $\tilde{u}(x) \le \sup_{\partial\Omega} \tilde{u}^+ = \sup_{\partial\Omega} u^+$, and therefore, since $\tilde{u}(x) = u(x)$ for $x \neq x_0$:

$$u(x) \le \sup_{\partial\Omega} u^+ \quad \text{for all } x \in \dot{\Omega} \tag{40}$$

whereas instead $\partial\dot{\Omega} = \partial\Omega \cup \{x_0\}$.

We will refer to inequalities like (40) as to an extended maximum principle.

The above discussion shows that inequality (40), namely the extended (**MP**), is a necessary condition for removable isolated singularities. A stronger extended (**MP**) holds when inequality (40) is satisfied with a subset $S \subset \Omega$ instead of $\{x_0\}$. It is a necessary condition for non-isolated removable singularities.

For uniformly elliptic operators, singular sets with a suitable vanishing Riesz or logarithmic capacity are removable as well as for the upper partial trace operators $\mathcal{P}_k^+[u] := \lambda_{n-k+1}(D^2u) + \cdots + \lambda_n(D^2u)$ singular sets with Riesz or logarithmic capacity $C_{k-2}(S) = 0$ are removable. See for instance [30, 1]. In the case of the Laplace operator ($k = n$) condition $C_{n-2}(S) = 0$ completely characterizes removable singularities.

In this Section we see how the structure condition (**SC2**), including lower order terms, leads to the extended (**MP**) of Theorem 1.2 for isolated singularities.

To do this we introduce the fundamental supersolutions $\psi(x) = |x - x_0|^{-\alpha}$ with $\alpha > 0$, in analogy with the fundamental solutions of the Laplace operator, which are smooth in $\mathbb{R}^n \backslash \{x_0\}$ and tend to infinity as $x \to x_0$. See for instance [21, 3]. Let $\phi(t) = t^{-\alpha}$ with $\alpha > 0$, then $\psi(x) = \phi(|x - x_0|)$.

Reasoning as for (34) we have

$$\begin{aligned} D\psi(x) &= -\,\alpha|x - x_0|^{-\alpha-1}\tfrac{x-x_0}{|x-x_0|} \\ D^2\psi(x) &= \alpha(\alpha+2)\,|x - x_0|^{-\alpha-2}\frac{x - x_0}{|x - x_0|} \otimes \frac{x - x_0}{|x - x_0|} - \alpha|x - x_0|^{-\alpha-2}\, I \end{aligned} \tag{41}$$

Proof of Theorem 1.2. Let $u \in \mathrm{usc}(\overline{\Omega}\backslash\{x_0\})$ be a viscosity subsolution of the equation $\mathcal{F}[u] = 0$ in $\Omega\backslash\{x_0\}$, such that $u \le 0$ on $\partial\Omega$, for a non-totally degenerate elliptic operator satisfying condition (**SC2**).

For $\varepsilon > 0$ we define the function $u_\varepsilon(x) = u^+(x) - \varepsilon\psi(x)$. Then, using (41) and condition (**SC2**), and recalling that u^+ is in turn a subsolution, we get:

$$
\begin{aligned}
\mathcal{F}(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) &\geq \mathcal{F}(x, u^+, Du^+ - \varepsilon D\psi, D^2u^+ - \varepsilon D^2\psi) \\
&\geq \mathcal{F}(x, u^+, Du^+, D^2u^+) \\
&+ \varepsilon\alpha\left(\lambda(x) - \alpha\Lambda(x) - b(x)|x - x_0|\right)|x - x_0|^{-\alpha-2} \\
&\geq \ \varepsilon\alpha\left(\lambda(x) - \alpha\Lambda(x) - b(x)|x - x_0|\right)|x - x_0|^{-\alpha-2}
\end{aligned}
\tag{42}
$$

Since $\sup_\Omega \frac{\Lambda(x)}{\lambda(x)} < \infty$, by condition (20) we can find $\alpha > 0$ such that

$$
\mathcal{F}(x, u_\varepsilon, Du_\varepsilon, D^2u_\varepsilon) \geq 0. \tag{43}
$$

Moreover, supposing $u^+(x) = o(|x - x_0|^{-\alpha})$ as $x \to x_0$, then $u_\varepsilon(x) \to -\infty$ as $x \to x_0$. From (**MP**) it follows that $u_\varepsilon(x) \leq \sup_{\partial\Omega} u_\varepsilon^+$. From this:

$$
u^+(x) = u_\varepsilon(x) + \varepsilon|x - x_0|^{-\alpha} \leq \sup_{\partial\Omega} u^+(x) + \varepsilon d_0^{-\alpha} = \varepsilon d_0^{-\alpha}, \tag{44}
$$

where $d_0 = \mathrm{dist}(x_0, \partial\Omega)$.

Letting $\varepsilon \to 0^+$, we get therefore $u^+ = 0$, as we wanted to prove. □

We observe that Theorem 1.2 holds for directional elliptic operators (1).

5. Removable singularities for directional elliptic operators

The extended (**MP**) stated in the previous section can be used to get at once a removable singularity result, at least in the case of smooth solutions, for operators (1) and more generally for the operators

$$
\mathcal{F}[u] = a_1\frac{\partial^2 u}{\partial x_1^2} + \cdots + a_n\frac{\partial^2 u}{\partial x_n^2}, \tag{45}
$$

with non-negative coefficients a_i such that $\overline{a} = \max_i a_i > 0$.

For the general case of viscosity solutions, we will establish below an extended comparison principle. See the proof of Theorem 1.3.

A complimentary tool is the following lemma, which yields an existence and uniqueness result for the Dirichlet problem.

We pick from [5] the following geometric property for the bounded domain Ω: for every $r > 0$ there exists $\delta > 0$ such that, for every $x \in B_\delta(y)$ and direction $v \in \mathbb{R}^n$,

$$(x + \mathbb{R}\, v) \cap B_r(y) \cap \partial\Omega \neq \emptyset. \tag{G_1}$$

Lemma 5.1. *Suppose $a_i \geq 0$, and there exists at least one $j \in \{1, \dots, n\}$ such that $a_j > 0$. Let Ω be a bounded domain, endowed with the geometric condition (G_1), where f is continuous and bounded. Let also g be a continuous function on $\partial\Omega$. Then the Dirichlet problem*

$$\begin{cases} a_1 \frac{\partial^2 \tilde{u}}{\partial x_1^2} + \cdots + a_n \frac{\partial^2 \tilde{u}}{\partial x_n^2} = f(x) & \text{in } \Omega \\ \tilde{u} = g & \text{on } \partial\Omega \end{cases} \tag{46}$$

has a unique solution.

Proof. We use the Perron method. Therefore existence and uniqueness are proved once we have proved the comparison principle and the existence of subsolutions and supersolutions in Ω, which are equal to g on $\partial\Omega$.

Comparison principle

Let $\mathcal{F}$ as in (45). Suppose $\mathcal{F}[u] \geq f(x)$ and $\mathcal{F}[v] \leq f(x)$ in Ω such that $u \leq v$ on $\partial\Omega$. We claim that $u \leq v$ in Ω.

(i) Strict subsolutions. As for (**MP**), we firstly make the stronger assumption $F[u] \geq f(x) + \varepsilon \geq F[v] + \varepsilon$ in Ω.

By contradiction, suppose that $u - v$ has a positive maximum in Ω. Following the proof of [20, Theorem 3.3], we find two sequences of points $x_k, y_k \in \Omega$ and matrices $X_k, Y_k \in \mathcal{S}^n$ such that:

$$\mathcal{F}(X_k) \geq f(x_k) + \varepsilon, \quad \mathcal{F}(Y_k) \leq f(y_k), \tag{47}$$

and

$$\lim_{k \to \infty} k|x_k - y_k|^2 = 0, \quad X_k \leq Y_k. \tag{48}$$

Then, using the degenerate ellipticity (6), we get

$$\begin{aligned}\mathcal{F}(X_k) &\geq f(x_k) + \varepsilon \geq f(y_k) + f(x_k) - f(y_k) + \varepsilon \\ &\geq \mathcal{F}(Y_k) + f(x_k) - f(y_k) + \varepsilon \\ &\geq F(X_k) + f(x_k) - f(y_k) + \varepsilon.\end{aligned}$$

Letting $k \to \infty$, by the continuity of f this would yield the contradiction $\varepsilon < 0$.

(ii) General case. Given a subsolution u, namely $\mathcal{F}[u] \geq f(x)$, we construct, for $\varepsilon > 0$, the strict subsolution $u_\varepsilon = u + \frac{\varepsilon}{2|a|}\,|x - x_0|^2$, where $x_0 \in \Omega$ and $|a| = a_1 + \cdots + a_n > 0$: in fact

$$\mathcal{F}[u_\varepsilon] = a_1 \frac{\partial^2 u_\varepsilon}{\partial x_1^2} + \cdots + a_n \frac{\partial^2 u_\varepsilon}{\partial x_n^2} \geq \mathcal{F}[u] + \varepsilon \geq f(x) + \varepsilon.$$

By (i), $u_\varepsilon(x) - v(x) \leq \sup_{\partial\Omega}(u_\varepsilon - v)$ in Ω, and therefore

$$u(x) - v(x) \leq u_\varepsilon(x) - v(x) \leq \sup_{\partial\Omega} \left(u(x) + \frac{\varepsilon}{2|a|}\,|x - x_0|^2 - v(x) \right) \leq \frac{\varepsilon}{2|a|}\,d^2.$$

Letting $\varepsilon \to 0^+$, we obtain $u \leq v$ in Ω, as claimed.

Subsolutions and supersolutions

From [5], see also [22], in a domain Ω satisfying condition (G_1), we can solve the Dirichlet problem

$$\begin{cases} \lambda_j(D^2 u_j) = f(x) & \text{in } \Omega \\ u_j = |a| g & \text{on } \partial\Omega \end{cases} \tag{49}$$

for each $j = 1, \ldots, n$. Therefore, since

$$\lambda_1(D^2 u_j) \leq \frac{\partial^2 u_j}{\partial x_i^2} \leq \lambda_n(D^2 u_j)$$

for all $i, j = 1, \ldots, n$, it turns out that

$$\frac{a_1}{|a|}\frac{\partial^2 u_1}{\partial x_1^2} + \cdots + \frac{a_n}{|a|}\frac{\partial^2 u_1}{\partial x_n^2} \geq \lambda_1(D^2 u_1) = f(x)$$

and

$$\frac{a_1}{|a|}\frac{\partial^2 u_n}{\partial x_1^2} + \cdots + \frac{a_n}{|a|}\frac{\partial^2 u_n}{\partial x_n^2} \leq \lambda_n(D^2 u_n) = f(x).$$

Then $\underline{u} = u_1/|a|$ and $\overline{u} = u_n/|a|$ are respectively a subsolution and a supersolution of the equation considered in (46) satisfying the boundary condition $\underline{u} = g = \overline{u}$.

By [20, Theorem 4.1], this concludes the proof. □

As discussed in the Introduction, the extended (**MP**) and the existence result lead to the removability result.

Proof of Theorem 1.3. Let u be a continuous viscosity solution of the equation

$$a_1 \frac{\partial^2 u}{\partial x_1^2} + \cdots + a_n \frac{\partial^2 u}{\partial x_n^2} = f(x) \quad \text{in } \Omega \backslash \{x_0\}. \tag{50}$$

Let B be a ball centered at x_0 such that $\overline{B} \subset \Omega$. Using Lemma 5.1, we find a unique viscosity solution U of the Dirichlet problem

$$\begin{cases} a_1 \frac{\partial^2 U}{\partial x_1^2} + \cdots + a_n \frac{\partial^2 U}{\partial x_n^2} = f(x) & \text{in } B \\ U = u & \text{on } \partial B \end{cases} \tag{51}$$

(i) *Case of smooth functions*

If U or u are C^2 functions, we proceed observing that by linearity the function $v = U - u$ is a viscosity solution of the following Dirichlet problem:

$$\begin{cases} a_1 \frac{\partial^2 v}{\partial x_1^2} + \cdots + a_n \frac{\partial^2 v}{\partial x_n^2} = 0 & \text{in } B \backslash \{x_0\} \\ v = 0 & \text{on } \partial B. \end{cases} \tag{52}$$

Since the operator $\mathcal{F}(X) = a_1 X_{11} + \cdots + a_n X_{nn}$ satisfies in particular condition (**SC2**), then the extended (**MP**) of Theorem 1.2 holds, for a suitable $\alpha > 0$, yields $U = u$ in $B \backslash \{x_0\}$.

This is enough: the function $U(x)$, defined in Ω by

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \neq x_0 \\ U(x_0) & \text{if } x = x_0 \end{cases} \tag{53}$$

is a continuation of the solution u across x_0 in Ω.

(ii) *General case*

Suppose now that U and u are continuous functions, respectively in B and $B \backslash \{x_0\}$. Supposing $x_0 = 0$, as we may, we proceed as in the proof of Lemma 5.1 (Comparison principle), comparing in $B \backslash \{0\}$ the functions $U(x)$ and

$$u_\varepsilon(x) = u(x) - \varepsilon \psi(x),$$

where $\psi(x) = \phi(|x|)$ and $\phi(t) = t^{-\alpha}$, as in the proof of Theorem 1.2.

Taking $\alpha > 0$ eventually smaller, in order that $\alpha \le n/\overline{a} - 2$, , where $\overline{a} = \max_i a_i$, from (41) we get

$$
\begin{aligned}
a_1 \frac{\partial^2 \psi}{\partial x_1^2} + \cdots + a_n \frac{\partial^2 \psi}{\partial x_n^2} &= \left(\alpha(\alpha+2) \left(a_1 \tfrac{x_1^2}{|x|^2} + \cdots + a_1 \tfrac{x_1^2}{|x|^2} \right) - \alpha n \right) |x|^{-\alpha-2} \\
&\le \alpha \left((\alpha+2)\, \overline{a} - n \right) |x|^{-\alpha-2} \le 0.
\end{aligned} \tag{54}
$$

Therefore

$$
\begin{aligned}
a_1 \frac{\partial^2 u_\varepsilon}{\partial x_1^2} + \cdots + a_n \frac{\partial^2 u_\varepsilon}{\partial x_n^2} = \; & a_1 \frac{\partial^2 u}{\partial x_1^2} + \cdots + a_n \frac{\partial^2 u}{\partial x_n^2} \\
& - \varepsilon \left(a_1 \frac{\partial^2 \psi}{\partial x_1^2} + \cdots + a_n \frac{\partial^2 \psi}{\partial x_n^2} \right) \ge 0,
\end{aligned}
$$

so that u_ε is a viscosity subsolution of equation (50) in $B \backslash \{0\}$.

Since $u(x) = o(|x|^{-\alpha})$ as $x \to 0$, then $u_\varepsilon(x) \to -\infty$ as $x \to 0$.

Hence, considering that U is a supersolution and applying the comparison principle as in the proof of Lemma 5.1, we obtain in $B \backslash \{0\}$:

$$
\begin{aligned}
u(x) = & \, u_\varepsilon(x) + \varepsilon \psi(x) \le U(x) + \sup_{\partial B} \left(u(x) - \varepsilon \psi(x) - U(x) \right) + \varepsilon \psi(x) \\
\le & \, U(x) + \varepsilon \psi(x)
\end{aligned}
$$

Letting $\varepsilon \to 0^+$, we get $u(x) \le U(x)$ in $B \backslash \{0\}$.

Using alternatively $U(x)$ as a subsolution and $u_\varepsilon(x) = u(x) + \varepsilon |x|^\alpha$ as a supersolution, we also get the reverse inequality $U(x) \le u(x)$ in $B \backslash \{0\}$. So we have $u(x) = \tilde{u}(x)$ in $B \backslash \{0\}$. Then U is the continuous extension of u across $x = 0$, and we conclude as in Case (i). □

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