THE NODAL SET OF SOLUTIONS TO ANOMALOUS EQUATIONS
L'INSIEME NODALE DI SOLUZIONI DI EQUAZIONI DEGENERI

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Abstract. This note focuses on the geometric-theoretic analysis of the nodal set of solutions to specific degenerate or singular equations.
As they belong to the Muckenhoupt class A_2, these operators appear in the seminal works of Fabes, Kenig, Jerison and Serapioni [FKS82, FKJ83, FJK82]. In particular, they have recently attracted a lot of attention in the last decade due to their link to the local realization of the fractional Laplacian. The goal is to get a glimpse of the complete theory of the nodal set of solutions of such equations in the spirit of the seminal works of Hardt, Simon, Han and Lin [HS89, Han94, Lin91].

Sunto. Queste note si concentrano sull'analisi geometrica dell'insieme nodale di soluzioni di specifiche equazioni degeneri o singolari.
Questa famiglia di operatori appartiene alla classe di Muckenhoupt A_2, ampiamente studiata nei lavori pionieristici di Fabes, Kenig, Jerison e Serapioni [FKS82, FKJ83, FJK82]. In particolare, tali operatori hanno ottenuto maggior attenzione negli ultimi decenni data il loro legame con la localizzazione del Laplaciano frazionario. L'obbiettivo è di riassumere i punti importanti della teoria degli insiemi nodali delle soluzioni di tali equazioni, nello spirito dei lavori influenti di Hardt, Simon, Han e Lin [HS89, Han94, Lin91].

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1. Introduction

In the last decades the study of the structure of the nodal set of solutions of elliptic equations was the center of the attention of the scientific community (see e.g.

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[DF88, Han94, HHL98, Lin91]), with a special focus on the measure theoretical features of its singular part, also in connection with the validity of a strong unique continuation principle, in order to ensure the existence of a finite vanishing order, as pointed out in [GL86, GL87, Lin91]. Recently, major progress has been done on the study of nodal sets of eigenfunctions (or critical sets of harmonic functions) by Logunov and Malinnikova [Log18b, Log18a, LM16] in connection with conjectures by Yau and Nadirashvili.

This is a note based on the work [STT18], written in collaboration with Y. Sire and S. Terracini. The aim is to give a complete overview of the structure of the nodal set in $\mathbb{R}^{n+1}$ of solutions of a class of degenerate-singular equations which has recently become very popular in connection with the study of fractional powers of the Laplacian, and firstly studied in the pioneering works [FKS82, FKJ83, FJK82]. Given $a \in (-1,1)$ and $X = (x,y) \in \mathbb{R}^n_x \times \mathbb{R}_y$ we consider a class of operators including

$$L_a = \mathrm{div}(|y|^a \nabla),$$

and their perturbations (here we denote by div and ∇ respectively the divergence and the gradient operator in $\mathbb{R}^{n+1}$). Our main purpose is to fully understand the local behaviour of L_a-harmonic functions near their nodal set and to develop a geometric analysis of its structure and regularity, in order to comprehend how the degenerate or singular character of the coefficients can affect the local picture of the nodal set itself. Thus, we introduce the notion of *characteristic manifold* Σ associated with the operator L_a, as the set of points where the coefficient either vanishes or blows up, and we study the properties of the nodal set $\Gamma(u)$ of solutions to equation

$$-L_a u = 0 \quad \text{in } B_1 \subset \mathbb{R}^{n+1}.$$

In particular, since the operator L_a is locally uniformly elliptic on $\mathbb{R}^{n+1} \setminus \Sigma$, we restrict our attention on the structure of the nodal set neighbouring the characteristic manifold Σ, trying to understand the structural difference between the whole nodal set $\Gamma(u) = \{x \in B_1,\ \ u(x) = 0\}$ and its restriction on Σ.

As a further motivation, this analysis will be the starting point of the study of competition-diffusion systems of populations under an anomalous diffusion. More precisely, we can imagine that the characteristic manifold Σ is playing a major role in the diffusion phenomenon by penalizing or encouraging the diffusion across Σ, according with the value of $a \in (-1,1)$. More precisely the diffusion is

$a>0$	$a<0$
encouraged near Σ	penalized near Σ

Our intention is the study of nonlinear competition-diffusion systems of k components where the rules for the diffusion are influenced by the presence of a characteristic manifold

$$-|y|^{-a}\,\mathrm{div}(|y|^a\,\nabla u_i) = f_{i,\beta}(u_i) - \beta u_i \sum_{j\neq i} a_{ij} u_j^2, \quad i = 1,\cdots,k.$$

Inspired by [TVZ16, TVZ14, CL08, CTV05, NTTV10], in the case of strong competition, the limiting segregated configurations will satisfy a refection law which represents the only interaction between the different densities through the common free boundary. Thanks to this reflection property, the free boundary will be locally described as the nodal set of L_a-harmonic function.

As already mentioned, our operators belong to the class introduced in the 80's by Fabes, Jerison, Kenig and Serapioni in [FKS82, FKJ83, FJK82], where they established Hölder continuity of solutions within a general class of degenerate-singular elliptic operators $L = \mathrm{div}(A(X)\nabla\cdot)$ whose coefficient $A(X) = (a_{ij}(X))$ are defined starting from a symmetric matrix valued function such that

$$\lambda\omega(X)\,|\xi|^2 \leq (A(X)\xi,\xi) \leq \Lambda\omega(X)\,|\xi|^2\,, \quad \text{for some } \lambda,\Lambda > 0,$$

where the weight ω may either vanish, or be infinite, or both. In particular, the prototypes of weights considered in their analysis belong to the Muckenhoupt A_2-class, i.e. weights such that

$$\sup_{B\subset\mathbb{R}^{n+1}} \left(\frac{1}{|B|}\int_B \omega(X)\mathrm{d}X\right)\left(\frac{1}{|B|}\int_B \omega^{-1}(X)\mathrm{d}X\right) < \infty.$$

Our case corresponds to the choice $\omega(X) = |y|^a$, which is Muckenhoupt whenever $a \in (-1,1)$. Note however that this class of A_2-weights is not the optimal one to have

Hölder regularity as noticed in [FKS82]. However, for our purposes it provides a good model for applications.

Our approach is based upon the validity of Almgren and Weiss type monotonicity formulæ, the existence and uniqueness of non trivial tangent maps at every point of the nodal set, and on a complete classification of all possible homogenous configurations appearing in the blow-up limit. Nevertheless, the starting point of our analysis relies on the decomposition of L_a-harmonic functions with respect to the orthogonal direction to the characteristic manifold Σ. Indeed, denoting by $H^{1,\beta}(B_1)$ the Sobolev space with respect to the measure $|y|^\beta\, dy\, dx$, we have (see also [GZ03, CG11])

Proposition 1.1. *Given $a \in (-1,1)$ and u an L_a-harmonic function in B_1, there exist a unique couple of functions $u_e^a \in H^{1,a}(B_1), u_e^{2-a} \in H^{1,2-a}(B_1)$ symmetric with respect to Σ respectively L_a and L_{2-a} harmonic in B_1 and locally smooth, such that*

$$u(X) = u_e^a(X) + u_e^{2-a}(X) y\, |y|^{-a} \quad \text{in } B_1.$$

The previous proposition is deeply base on the results in [STV19], where the authors studied the regularity of solutions to degenerate or singular problem by introducing a method based upon blow-up and appropriate Liouville type theorems.

With this decomposition in mind, we can reduce the classification of the possible blow-up limits to the symmetric ones and eventually recover all the possible cases.

When one deal with the analysis of nodal sets of solutions of PDEs or free boundary problems, a key point is the possibility of performing blow-up analysis, which allow to better understand local behaviour of solutions near the geometric boundary. Indeed, this possibility is given by the validity of some monotonicity formulae, main tool of our analysis. More precisely, fixed $X_0 \in \Gamma(u) \cap \Sigma$ and $r \in (0,R)$, where $R > 0$ will be defined later, consider

$$E(X_0,u,r) = \frac{1}{r^{n+a-1}} \int_{B_r(X_0)} |y|^a\, |\nabla u|^2\, \mathrm{d}X, \quad H(X_0,u,r) = \frac{1}{r^{n+a}} \int_{\partial B_r(X_0)} |y|^a\, u^2 \mathrm{d}\sigma$$

and the Almgren quotient

$$N(X_0,u,r)=\frac{E(X_0,u,r)}{H(X_0,u,r)}=\frac{r\displaystyle\int_{B_r(X_0)}|y|^a\,|\nabla u|^2\,\mathrm{d}X}{\displaystyle\int_{\partial B_r(X_0)}|y|^a\,u^2\mathrm{d}\sigma}. \tag{1}$$

The following is the monotonicity result related to the Almgren quotient, which allows to define the vanishing order of a solution as the limit $N(X_0,u,0^+)=\lim_{r\to 0^+}N(X_0,u,r)$.

Proposition 1.2 ([CS07]). *Let $a\in(-1,1)$ and u be a L_a-harmonic function on B_1. Then, for every $X_0\in B_1\cap\Sigma$ we have that the map $r\mapsto N(X_0,u,r)$ is absolutely continuous and monotone nondecreasing on $(0,1-|X_0|)$.*
Hence, there always exists finite the limit

$$N(X_0,u,0^+)=\lim_{r\to 0^+}N(X_0,u,r)=\inf_{r>0}N(X_0,u,r).$$

to which we will refer as the Almgren frequency.

Motivated by Proposition 1.1, we classify the possible vanishing order of the solutions paying attention to the different behaviour of the solution across the characteristic manifold.

Corollary 1.1. *Let u be L_a-harmonic on B_1, then for every $X_0\in\Gamma(u)\cap\Sigma$ we have*

$$N(X_0,u,0^+)\geq\min\{1,1-a\}. \tag{2}$$

More precisely

- *if u is symmetric with respect to Σ, we have $N(X_0,u,0^+)\in 1+\mathbb{N}$,*
- *if u is antisymmetric with respect to Σ we have $N(X_0,u,0^+)\in 1-a+\mathbb{N}$.*

Thus, for $k\geq\min\{1,1-a\}$ we define

$$\Gamma_k(u)=\{X_0\in\Gamma(u)\colon N(X_0,u,0^+)=k\}.$$

and we prove the validity of the following local expansion near the nodal set.

Theorem 1.1. *For every $X_0 \in \Gamma_k(u) \cap \Sigma$ there exists a unique tangent map φ^{X_0} such that*

$$u(x,y) = \varphi^{X_0}(x - x_0, y) + o(|(x - x_0, y)|)^k. \tag{3}$$

Thus, the map $X_0 \mapsto \varphi^{X_0}$ is continuous.

The proof follows a standard approach, indeed we proved compactness of the blow-up sequence and uniqueness and non-degeneracy of the blow-up limit by using the following monotonicity type formulas:

- k-Weiss type monotonicity formula (see [STT18, Proposition 5.2.] for the proof of the monotonicity result)
$$r \mapsto W_k(X_0, u, r) = \frac{H(X_0, u, r)}{r^{2k}} \left(N(X_0, u, r) - k\right);$$
- Monneau type monotonicity formula (see [STT18, Proposition 5.3.] for the proof of the monotonicity)
$$r \mapsto \frac{H(X_0, u - \varphi_{X_0}, r)}{r^{2k}} = \frac{1}{r^{n+a+2k}} \int_{\partial B_r(X_0)} |y|^a \left(u - \varphi^{X_0}\right)^2 \mathrm{d}\sigma,$$
with $\varphi_{X_0} \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ a k-homogeneous L_a-harmonic polynomial.

Equivalently, we can define as tangent map the unique nonzero homogeneous map $\varphi^{X_0} \in H^{1,a}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ such that

$$u_{X_0,r}(X) = \frac{u(X_0 + rX)}{r^k} \longrightarrow \varphi^{X_0}(X),$$

with k the vanishing order of u at X_0. It is straightforward to notice that the main weakness of the concept of tangent map, in this setting, is that it takes care either of the symmetric part of u or of the even one since they do not share the same optimal regularity and the same possible vanishing orders. More precisely, for every $X_0 \in \Gamma_k(u)$

$$\begin{aligned} u_{X_0,r}(X) &= \frac{u_e(X_0 + rX)}{r^k} + \frac{u_o(X_0 + rX)}{r^k} \\ &= \frac{u_e^a(X_0 + rX)}{r^k} + \frac{u_e^{2-a}(X_0 + rX)}{r^{k-1+a}} y\,|y|^{-a}, \end{aligned}$$

where both u_e^a and u_e^{2-a} are symmetric with respect to Σ. Then, by the classification of the vanishing orders we deduce that fixed $k > 0$, just one of the two terms in the previous equality *survives* as $r \to 0^+$.

It is worthwhile introducing a new notion of *tangent field* Φ^{X_0} of u at a nodal point, which takes care of the different behaviour of both the symmetric and antisymmetric part of an L_a-harmonic function, which will be of crucial use in our results.

Definition 1.1. *Let $a \in (-1,1), u$ be an L_a-harmonic function in B_1 and $X_0 \in \Gamma_k(u) \cap \Sigma$, for some $k \geq \min\{1, 1-a\}$. We define as* tangent field *of u at X_0 the unique nontrivial vector field $\Phi^{X_0} \in (H^{1,a}_{loc}(\mathbb{R}^{n+1}))^2$ such that*

$$\Phi^{X_0} = (\varphi_e^{X_0}, \varphi_o^{X_0}),$$

where $\varphi_e^{X_0}$ and $\varphi_o^{X_0}$ are respectively the tangent map of the symmetric part u_e of u and of the antisymmetric one u_o.

First, the notion of the tangent field allows us to describe the topology of the nodal set by proving a *vectorial* counterpart of the classic result of upper semi-continuity of the vanishing order. In order to define properly the relevant subsets, we define

$$\partial_y^a u = \begin{cases} |y|^a \, \partial_y u & \text{if } X \notin \Sigma \\ \lim_{y \to 0} |y|^a \, \partial_y u(x,y) & \text{if } X \in \Sigma \end{cases}.$$

This quantity, as observed already in previous works, is the nontrivial one to be considered as far as the derivative in y is concerned.

In the light of this observation, it is natural to define the regular part $\mathcal{R}(u)$ and the singular part $\mathcal{S}(u)$ of the nodal set as follows:

$$\begin{aligned} \mathcal{R}(u) &= \left\{ X \in \Gamma(u) \,\middle|\, \begin{array}{cc} N(X_0, u, 0^+) = 1 & \text{if } X_0 \notin \Sigma \\ N(X_0, u_e, 0^+) = 1 \text{ or } N(X_0, u_0, 0^+) = 1 - a & \text{if } X_0 \in \Sigma \end{array} \right\} \\ &= \left\{ X \in \Gamma(u) \,\middle|\, |\nabla_x u(X)|^2 + \left|\partial_y^a u(X)\right|^2 \neq 0 \right\} \end{aligned}$$

$$\begin{aligned} \mathcal{S}(u) &= \left\{ X \in \Gamma(u) \,\middle|\, \begin{array}{cc} N(X_0, u, 0^+) \geq 2 & \text{if } X_0 \notin \Sigma \\ N(X_0, u_e, 0^+) \geq 2 \text{ and } N(X_0, u_0, 0^+) \geq 2 - a & \text{if } X_0 \in \Sigma \end{array} \right\}, \\ &= \left\{ X \in \Gamma(u) \,\middle|\, |\nabla_x u(X)|^2 + \left|\partial_y^a u(X)\right|^2 = 0 \right\}. \end{aligned}$$

The previous sets are natural generalisation of their counterpart for the case of uniformly elliptic operator. The main difficulty in the degenerate case is to understand how the vanishing order and the generalised gradient $|\nabla_x u(X)|^2 + \left|\partial_y^a u(X)\right|^2$ change across the characteristic manifold. This feature is fundamental in order to draw a complete picture of the topology of the nodal set in the whole $\mathbb{R}^{n+1}$.

Indeed, we prove a quasi upper semi-continuity of the Almgren frequency formula: given $(X_i)_i \in \Gamma_k(u) \setminus \Sigma$ with $k \in 1+\mathbb{N}$ such that $X_i \to X_0 \in \Gamma(u) \cap \Sigma$, then

$$N(X_i, u, 0^+) \leq \begin{cases} N(X_0, u_e, 0^+), \\ N(X_0, u_o, 0^+) + a. \end{cases}$$

This result deeply used the existence and uniqueness of the blow-up limit in $\mathbb{R}^{n+1} \setminus \Sigma$ and Σ and the smoothness of symmetric L_a-harmonic function.

The next step is to develop a blow-up analysis in order to fully understand the structure of $\Gamma(u)$ in $\mathbb{R}^{n+1}$ and its restriction on Σ. The following is a summary of our main result describing the stratified structure of both the regular and singular parts of the nodal set.

Theorem 1.2. *Let $a \in (-1,1), a \neq 0$ and u be an L_a-harmonic function in B_1. Then the regular set $\mathcal{R}(u)$ is locally a $C^{k,r}$ hypersurface on $\mathbb{R}^{n+1}$ in the variable $(x, y\,|y|^{-a})$ with*

$$k = \left\lfloor \frac{2}{1-a} \right\rfloor \quad \text{and} \quad r = \frac{2}{1-a} - \left\lfloor \frac{2}{1-a} \right\rfloor.$$

On the other hand, there holds

$$\mathcal{S}(u) \cap \Sigma = \mathcal{S}^*(u) \cup \mathcal{S}^a(u)$$

where $\mathcal{S}^*(u)$ is contained in a countable union of $(n-2)$-dimensional C^1 manifolds and $\mathcal{S}^a(u)$ is contained in a countable union of $(n-1)$-dimensional C^1 manifolds. Moreover

$$\mathcal{S}^*(u) = \bigcup_{j=0}^{n-2} \mathcal{S}^*_j(u) \quad \text{and} \quad \mathcal{S}^a(u) = \bigcup_{j=0}^{n-1} \mathcal{S}^a_j(u),$$

where both $\mathcal{S}^*_j(u)$ and $\mathcal{S}^a_j(u)$ are contained in a countable union of j-dimensional C^1 manifolds.

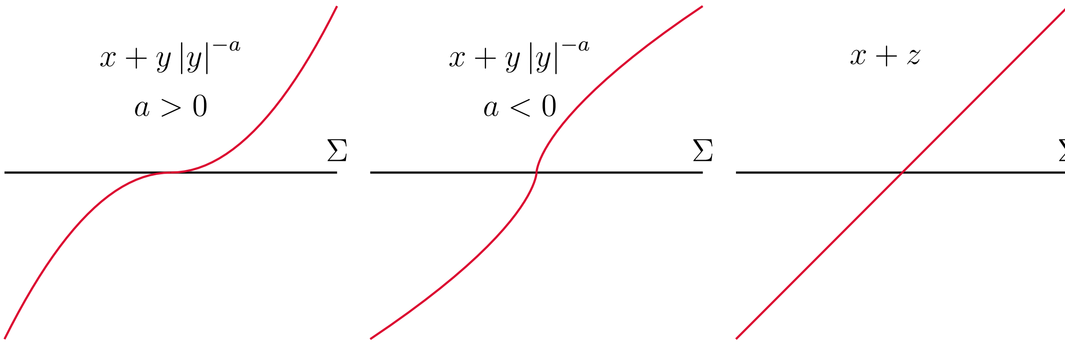


FIGURE 1. The change of variable allows to regularize the problem in the y-direction and it emphasizes the role of the vectorial tangent map.

The proof is deeply based on a suitable change of variable $\Phi\colon \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that

$$\Phi\colon (x,z) \mapsto \left(x, (1-a)z\,|z|^{\frac{a}{1-a}}\right),$$
$$\Phi^{-1}\colon (x,y) \mapsto \left(x, \frac{y\,|y|^{-a}}{(1-a)^{1-a}}\right),$$

with Jacobian $|J_{\Phi^{-1}}(x,y)| = (1-a)^a\,|y|^{-a}$ and $\Phi(X_0) = X_0$, for every $X_0 \in \Sigma$. Given $\tilde{u} = u \cdot \Phi$, we get

$$|\nabla_x u(X_0)|^2 + \left|\partial^a_y u(X_0)\right|^2 \neq 0 \longleftrightarrow |\nabla_x \tilde{u}(X_0)|^2 + |\partial_z \tilde{u}(X_0)|^2 \neq 0,$$

which allows to translate in our setting the classic Implicit function theorem.

Finally, we can provide applications of our results in the context of nonlocal elliptic equations by using the local realisation of fractional operator. Inspired by [CS07], we exploit the local realisation of the fractional Laplacian, defined by

$$(-\Delta)^s u(x) = C(n,s)\ \text{P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} \mathrm{d}y\ ,$$

for $s \in (0,1)$ and

$$C(n,s) = \frac{2^{2s} s \Gamma(\frac{n}{2}+s)}{\pi^{n/2}\Gamma(1-s)} \in \left(0, 4\Gamma\left(\frac{n}{2}+1\right)\right]\ ,$$

as the Dirichlet-to-Neumann map for a variable v depending on one more space dimension. More precisely, the extended solution v is defined as

$$\begin{cases} \operatorname{div}(y^a \nabla v) = 0 & \text{in } \mathbb{R}^{n+1}_+, \\ v(x,0) = u(x) & \text{in } \Sigma\ , \end{cases}$$

with $a = 1 - 2s \in (-1, 1)$. Such an extension exists unique and is given by the formula

$$v(x,y) = \gamma(n,s) \int_{\mathbb{R}^n} \frac{y^{2s} u(x)}{(|x-\eta|^2 + y^2)^{n/2+s}} \mathrm{d}\eta \qquad \text{where } \gamma(n,s)^{-1} =: \int_{\mathbb{R}^n} \frac{1}{(|\eta|^2+1)^{n/2+s}} \mathrm{d}\eta\,,$$

where the nonlocal operator $(-\Delta)^s$ translates into the Dirichlet-to-Neumann opeartor type

$$(-\Delta)^s \colon H^s(\mathbb{R}^n) \to H^{-s}(\mathbb{R}^n), \quad u \longmapsto -\frac{C(n,s)}{\gamma(n,s)} \lim_{y\to 0^+} y^{1-2s} \partial_y v(x,y).$$

More generally, thanks to the generalisation in [CG11, ST10], we can consider the case of fractional powers of divergence form operators L with Lipschitz leading coefficient, in order to study the structure and the regularity of the nodal set of $(-L)^s$-harmonic functions, for $s \in (0,1)$. More precisely, we combine the extension technique with a geometric reduction introduced in [AKS62] and exploited in the seminal papers [GL86, GL87]. This will allow us to extend our analysis to:

1. fractional powers $(-L)^s$ of divergence form operators with Lipschitz leading coefficients, i.e.
$$Lu = \mathrm{div}\,(A(x)\nabla u) = \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j} u\right);$$
2. fractional powers $(-\Delta_M)^s$ of the Laplace-Beltrami operator on a Riemannian manifold M with Lipschitz metric;
3. given $V \in W^{1,q}(B_1)$, for some $q \geq n/2s$, our analysis holds true for nontrivial solutions of the equation
$$(-\Delta)^s u = V(x) u \quad \text{in } B_1.$$

Our techniques are quite robust and, we believe, can apply to a wider class of operators on manifolds like the conformally covariant ones of fractional order formulated on conformally compact Einstein manifolds and asymptotically hyperbolic manifold (see [CG11] for more details in this direction).

Our results show some genuinely nonlocal features in the Taylor expansion of $(-L)^s$-harmonic functions near their zero set and their deep impact on the structure of the nodal set itself. We prove that the first term of the Taylor expansion of an $(-L)^s$-harmonic function is either an homogeneous harmonic polynomial or any possible homogeneous polynomial. In particular, this implies

Theorem 1.3. *Given L, a divergence form operator with Lipschitz leading coefficients, and $s \in (0,1)$, let u be $(-L)^s$-harmonic in B_1. Then there holds*

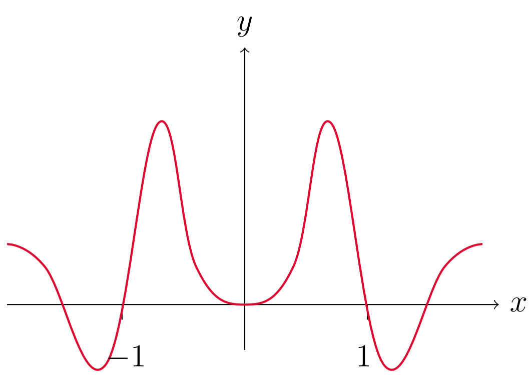
$$\mathcal{S}(u) = \mathcal{S}^*(u) \cup \mathcal{S}^s(u)$$

where $\mathcal{S}^*(u)$ is contained in a countable union of $(n-2)$-dimensional C^1 manifolds and $\mathcal{S}^s(u)$ is contained in a countable union of $(n-1)$-dimensional C^1 manifolds. Moreover

$$\mathcal{S}^*(u) = \bigcup_{j=0}^{n-2} \mathcal{S}^*_j(u) \quad \textit{and} \quad \mathcal{S}^s(u) = \bigcup_{j=0}^{n-1} \mathcal{S}^s_j(u),$$

where both $\mathcal{S}^*_j(u)$ and $\mathcal{S}^s_j(u)$ are contained in a countable union of j-dimensional C^1 manifolds.

We underline that the result on the existence of $(n-1)$-dimensional singular set is optimal in the sense that for any vanishing order $k \geq 2$ there exists an s-harmonic function in B_1 which vanishes of order k and such that $\Gamma(u) = \mathcal{S}^s_{n-1}(u)$.



$$\begin{cases} (-\Delta)^s u = 0 & \text{in } (-1,1) \\ u = g & \text{in } \mathbb{R} \setminus (-1,1) \end{cases}$$

$$g(x) = \frac{2 + x^s(s-2)}{(s-2)x^2}(x^2-1)^s$$

FIGURE 2. Explicit example of s-harmonic function in $(-1,1) \subset \mathbb{R}$ with Dirichlet condition on $\mathbb{R}\setminus(-1,1)$ which vanishes at the origin with vanishing order $\mathcal{O}(u,0) = 2$.

Finally, we are able to state and prove in our context, the nonlocal counterpart of a conjecture proposed by Lin in [Lin91]. Following his strategy, we give an explicit estimate on the $(n-1)$-Hausdorff measure of the nodal set $\Gamma(u)$ of s-harmonic functions in terms of the Almgren frequency previously introduced. We have

Theorem 1.4. *Given $s \in (0,1)$, let u be an s-harmonic function in B_1 and $0 \in \Gamma(u)$. Then*

$$\mathcal{H}^{n-1}\left(\Gamma(u) \cap B_{\frac{1}{2}}\right) \leq C(n,s)N,$$

where v is the L_a-harmonic extension of u in B_1^+ and $N = N(0,v,1)$ is the frequency defined by

$$N = \frac{\displaystyle\int_{B_1^+} |y|^a \, |\nabla v|^2 \, \mathrm{d}X}{\displaystyle\int_{\partial B_1^+} |y|^a \, v^2 \mathrm{d}\sigma}.$$

Recently, in [BET17] the authors studied the geometry of sets that admit arbitrarily good local approximations by zero sets of harmonic polynomials. In the light of the previous Theorems, it would be interesting to adapt their strategies to our degenerate-singular framework.

References

[AKS62] N. Aronszajn, A. Krzywicki, and J. Szarski. A unique continuation theorem for exterior differential forms on Riemannian manifolds. *Ark. Mat.*, 4:417–453 (1962), 1962.

[BET17] M. Badger, M. Engelstein, and T. Toro. Structure of sets which are well approximated by zero sets of harmonic polynomials. *Anal. PDE*, 10(6):1455–1495, 2017.

[CG11] S.-Y. A. Chang and M. del Mar González. Fractional Laplacian in conformal geometry. *Adv. Math.*, 226(2):1410–1432, 2011.

[CL08] L. Caffarelli and F.-H. Lin. Singularly perturbed elliptic systems and multi-valued harmonic functions with free boundaries. *J. Amer. Math. Soc.*, 21(3):847–862, 2008.

[CS07] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[CTV05] M. Conti, S. Terracini, and G. Verzini. Asymptotic estimates for the spatial segregation of competitive systems. *Adv. Math.*, 195(2):524–560, 2005.

[DF88] H. Donnelly and C. Fefferman. Nodal sets of eigenfunctions on Riemannian manifolds. *Invent. Math.*, 93(1):161–183, 1988.

[FJK82] E. Fabes, D. Jerison, and C. Kenig. The Wiener test for degenerate elliptic equations. *Ann. Inst. Fourier (Grenoble)*, 32(3):vi, 151–182, 1982.

[FKJ83] E. B. Fabes, C. E. Kenig, and D. Jerison. Boundary behavior of solutions to degenerate elliptic equations. In *Conference on harmonic analysis in honor of Antoni Zygmund, Vol. I, II (Chicago, Ill., 1981)*, Wadsworth Math. Ser., pages 577–589. Wadsworth, Belmont, CA, 1983.

[FKS82] E. Fabes, C. Kenig, and R. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. Partial Differential Equations*, 7(1):77–116, 1982.

[GL86] N. Garofalo and F.-H. Lin. Monotonicity properties of variational integrals, A_p weights and unique continuation. *Indiana Univ. Math. J.*, 35(2):245–268, 1986.

[GL87] N. Garofalo and F.-H. Lin. Unique continuation for elliptic operators: a geometric-variational approach. *Comm. Pure Appl. Math.*, 40(3):347–366, 1987.

[GZ03] C. Robin Graham and Maciej Zworski. Scattering matrix in conformal geometry. *Invent. Math.*, 152(1):89–118, 2003.

[Han94] Q. Han. Singular sets of solutions to elliptic equations. *Indiana Univ. Math. J.*, 43(3):983–1002, 1994.

[HHL98] Q. Han, R. Hardt, and F.-H. Lin. Geometric measure of singular sets of elliptic equations. *Comm. Pure Appl. Math.*, 51(11-12):1425–1443, 1998.

[HS89] R. Hardt and L. Simon. Nodal sets for solutions of elliptic equations. *J. Differential Geom.*, 30(2):505–522, 1989.

[Lin91] F.-H. Lin. Nodal sets of solutions of elliptic and parabolic equations. *Comm. Pure Appl. Math.*, 44(3):287–308, 1991.

[LM16] Alexander Logunov and Eugenia Malinnikova. Ratios of harmonic functions with the same zero set. *Geom. Funct. Anal.*, 26(3):909–925, 2016.

[Log18a] Alexander Logunov. Nodal sets of Laplace eigenfunctions: polynomial upper estimates of the Hausdorff measure. *Ann. of Math. (2)*, 187(1):221–239, 2018.

[Log18b] Alexander Logunov. Nodal sets of Laplace eigenfunctions: proof of Nadirashvili's conjecture and of the lower bound in Yau's conjecture. *Ann. of Math. (2)*, 187(1):241–262, 2018.

[NTTV10] B. Noris, H. Tavares, S. Terracini, and G. Verzini. Uniform Hölder bounds for nonlinear Schrödinger systems with strong competition. *Comm. Pure Appl. Math.*, 63(3):267–302, 2010.

[ST10] P. R. Stinga and J. L. Torrea. Extension problem and Harnack's inequality for some fractional operators. *Comm. Partial Differential Equations*, 35(11):2092–2122, 2010.

[STT18] Y. Sire, S. Terracini, and G. Tortone. On the nodal set of solutions to degenerate or singular elliptic equations with an application to $s-$harmonic functions. *ArXiv e-prints*, August 2018.

[STV19] Yannick Sire, Susanna Terracini, and Stefano Vita. Liouville type theorems and regularity of solutions to degenerate or singular problems part I: even solutions. *arXiv e-prints*, page arXiv:1904.02143, Apr 2019.

[TVZ14] S. Terracini, G. Verzini, and A. Zilio. Uniform Hölder regularity with small exponent in competition-fractional diffusion systems. *Discrete and Continuous Dynamical Systems- Series A*, 34(6):2669–2691, 2014.

[TVZ16] S. Terracini, G. Verzini, and A. Zilio. Uniform Hölder bounds for strongly competing systems involving the square root of the laplacian. *Journal of the European Mathematical Society*, 18(12):2865–2924, 2016.

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