LARGE SETS AT INFINITY AND MAXIMUM PRINCIPLE ON UNBOUNDED DOMAINS FOR A CLASS OF SUB-ELLIPTIC OPERATORS

INSIEMI LARGHI ALL'INFINITO E PRINCIPIO DEL MASSIMO SU DOMINI NON LIMITATI PER UNA CLASSE DI OPERATORI SUB-ELLITTICI

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Abstract. Maximum Principles on unbounded domains play a crucial rôle in several problems related to linear second-order PDEs of elliptic and parabolic type. In the present notes, based on a joint work with E. Lanconelli, we consider a class of sub-elliptic operators $\mathcal{L}$ in $\mathbb{R}^N$ and we establish some criteria for an unbounded open set to be a Maximum Principle set for $\mathcal{L}$. We extend some classical results related to the Laplacian (proved by Deny, Hayman and Kennedy) and to the sub-Laplacians on homogeneous Carnot groups (proved by Bonfiglioli and Lanconelli).

Sunto. I Principi del Massimo su domini non limitati rivestono un ruolo fondamentale in diversi problemi legati alle equazioni alle derivate parziali del secondo ordine, sia ellittiche sia paraboliche. In queste note, basate su un lavoro scritto in collaborazione con E. Lanconelli, consideriamo una classe di operatori sub-ellittici $\mathcal{L}$ in $\mathbb{R}^N$ e proviamo alcuni criteri che garantiscono la validità del Principio del Massimo per $\mathcal{L}$ su un aperto non limitato. In particulare, estendiamo alcuni classici risultati per l'operatore di Laplace (provati da Deny, Hayman e Kennedy) e per i sub-Laplaciani sui gruppi omogenei di Carnot (provati da Bonfiglioli e Lanconelli).

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1. Introduction

The present notes are based on the talk titled *Large sets at infinity and Maximum Principle on unbounded domains for a class of sub-elliptic operators*, given by the author in Bologna at the "Bruno Pini Mathematical Analysis Seminar". In its turn, this talk was based on the paper [5], which is a joint work with E. Lanconelli.

It is well known that, in Euclidean space $\mathbb{R}^N$, the classical Laplace operator Δ satisfies the following Weak Maximum Principle (WMP) on every *bounded* open set: if $\Omega \subseteq \mathbb{R}^N$ is a *bounded* open set and if $u \in C^2(\Omega, \mathbb{R})$, then

$$\text{(WMP)} \qquad \begin{cases} \Delta u \geq 0 & \text{in } \Omega, \\ \limsup\limits_{x \to \xi} u(x) \leq 0 & \text{for every } \xi \in \partial\Omega \end{cases} \implies u \leq 0 \text{ on } \Omega.$$

Moreover, it is not difficult to produce an example showing that the hypothesis that Ω be bounded cannot be dropped for the validity of (WMP).

Example 1.1. *Let $\Omega := \mathbb{R} \times (0, 2\pi) \subseteq \mathbb{R}^2$ and let $u : \Omega \to \mathbb{R}$ be defined as follows:*

$$u(x) := e^{x_1} \sin(x_2).$$

A direct computation shows that $\Delta u \equiv 0$ on Ω and

$$\limsup_{x \to \xi} u(x) = u(\xi) = 0, \qquad \textit{for every } \xi \in \partial\Omega = \mathbb{R} \times \{0, 2\pi\}.$$

On the other hand, $u \not\leq 0$ on Ω.

Even if Example 1.1 shows that (WMP) does not hold, in general, if Ω is not bounded, there are many meaningful situations where one needs a WMP on unbounded open sets, at least for bounded-above subharmonic functions. In this direction we have the following theorem, originally proved by Deny in 1947 (see [8]).

Theorem 1.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set such that $\mathbb{R}^N \setminus \Omega$ contains an infinite open cone. Consider a bounded-above function $u \in C^2(\Omega, \mathbb{R})$ satisfying*

$$\begin{cases} \Delta u \geq 0 & \textit{in } \Omega, \\ \limsup_{y \to x} u(y) \leq 0, & \textit{for every } x \in \partial\Omega, \end{cases}$$

Then $u(x) \leq 0$ *for every* $x \in \Omega$.

As a matter of fact, Theorem 1.1 was obtained by Deny as a corollary of the next result, which contains an interesting property concerning the behavior at infinity of any bounded-above subharmonic function.

Theorem 1.2 ([8, Theorem 3.1]). *Let* $u : \mathbb{R}^N \to \mathbb{R}$ *be a bounded-above subharmonic function on* $\mathbb{R}^N$. *Then, we have*

$$\lim_{|x|\to\infty} u(x) = \sup_{\mathbb{R}^N} u$$

along almost all fixed rays through the origin.

Remark 1.1. *It is easy to see that neither the assumption of the upper-boundedness of* u *nor the assumption on the "size" of* $\mathbb{R}^N \setminus \Omega$ *in Theorem 1.1 can be dropped.*

(i) *Let* $\Omega := \{x \in \mathbb{R}^2 : x_2 > 0\}$ *and let*

$$u : \Omega \longrightarrow \mathbb{R}, \qquad u(x) := x_2.$$

Clearly, $\mathbb{R}^2 \setminus \Omega$ *contains an infinite open cone; moreover,* $\Delta u \equiv 0$ *on* Ω *and*

$$\limsup_{x\to\xi} u(x) = u(\xi) = 0, \qquad \text{for every } \xi \in \partial\Omega = \{x \in \mathbb{R}^2 : x_2 = 0\}.$$

On the other hand, $u > 0$ *on* Ω.

(ii) *Let* $\Omega := \mathbb{R}^3 \setminus \{0\}$ *and let* $u : \Omega \to \mathbb{R}$ *be defined as follows:*

$$u(x) := 1 - \frac{1}{\|x\|}.$$

A direct computation shows that

- u *is subharmonic and bounded-above on* Ω*;*
- $\limsup_{x\to 0} u(x) = -\infty$.

On the other hand, $u \not\leq 0$ *on* Ω *(note that* $\mathbb{R}^3 \setminus \Omega = \{0\}$*).*

Motivated by the phenomena just described, in these notes we consider a class of subelliptic operators $\mathcal{L}$ in $\mathbb{R}^N$ and we try to establish some criteria ensuring the validity of the WMP (for $\mathcal{L}$) on unbounded open sets.

Remark 1.2. *It is interesting to point out that, in the particular case of* $\mathbb{R}^2$, *it is not possible to construct an analog of the example in Remark 1.1-(ii): in fact, Theorem 1.1 holds without the need of assuming that* $\mathbb{R}^2 \setminus \Omega$ *contains an infinite open cone.*

The proof of this assertion is essentially based on the fact that there do not exist a Green function for the whole space $\mathbb{R}^2$*; if, instead,* $N \geq 3$, *the Green function for* $\mathbb{R}^N$ *exists and it coincides with the fundamental solution of* Δ.

2. Main assumptions and notations

On Euclidean space $\mathbb{R}^N$, we consider linear second-order differential operators $\mathcal{L}$ (PDOs, in the sequel) of the following quasi-divergence form

$$\mathcal{L} = \frac{1}{V(x)} \sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \Big(V(x)\, a_{i,j}(x)\, \frac{\partial}{\partial x_j} \Big), \tag{1}$$

and satisfying the *structural assumptions* listed below:

- **(S):** $V, a_{i,j} \in C^\infty(\mathbb{R}^N, \mathbb{R})$ and $V > 0$ on $\mathbb{R}^N$;
- **(DE):** $A(x) = \Big(a_{i,j}(x) \Big) \geq 0$ for every $x \in \mathbb{R}^N$;
- **(NTD):** $\mathrm{trace}(A(x)) > 0$ for every $x \in \mathbb{R}^N$;
- **(HY):** there exists a real $\varepsilon > 0$ such that both $\mathcal{L}$ and $\mathcal{L}_\varepsilon := \mathcal{L} - \varepsilon$ are C^∞-hypoelliptic in every open subset of $\mathbb{R}^N$.

The class of operators of the form (1) and satisfying the structural assumptions (S)-to-(HY) is quite large, as the next Example 2.1 shows.

Example 2.1. (a) *Let* $\mathbb{G} = (\mathbb{R}^N, *)$ *be a Lie group and let* μ *be a Haar measure on* $\mathbb{G}$. *If* $\mathcal{Z} = \{Z_1, \ldots, Z_m\}$ *is a system of Lie-generators for* $\mathrm{Lie}(\mathbb{G})$, *then*

$$\Delta_{\mathbb{G}} := -\sum_{j=1}^{m} Z_j^{*,\mu} Z_j,$$

takes the form (1) *and satisfies assumptions* (S)*-to-* (HY). *Here,* $Z_j^{*,\mu}$ *denotes the* (*formal*) *adjoint of* Z_j *in the space* $L^2(\mathbb{R}^N, \mathrm{d}\mu)$ (*for* $j = 1, \ldots, m$).

(b) *More generally, if* $X_1, \ldots, X_m$ *are smooth vector fields on* $\mathbb{R}^N$ *satisfying the Hörmander Rank Condition at every point of* $\mathbb{R}^N$, *then*

$$\mathcal{L}_X := -\sum_{j=1}^{m} X_j^* X_j,$$

takes the form (1) *and satisfies assumptions* (S)-*to*-(HY).

(c) *Let $X_1, \dots, X_m$ be smooth vector fields on $\mathbb{R}^N$ satisfying the Hörmander Rank Condition at every point of $\mathbb{R}^N$. If $\mathrm{div}(X_i) \equiv 0$ for every $i = 1, \dots, m$, then the PDO*

$$\mathcal{L}_X := X_1^2 + \cdots + X_m^2,$$

takes the form (1) *and satisfies assumptions* (S)-*to*-(HY).

(d) *The following semi-elliptic* non-Hörmander operators

$$\mathcal{L}_1 := \frac{\partial^2}{\partial x_1^2} + \Big(\exp(-1/|x_1|)\,\frac{\partial}{\partial x_2}\Big)^2 + \Big(\exp(-1/|x_1|)\,\frac{\partial}{\partial x_3}\Big)^2,$$

$$\mathcal{L}_2 := \frac{\partial^2}{\partial x_1^2} + \Big(\exp(-1/\sqrt{|x_1|})\,\frac{\partial}{\partial x_2}\Big)^2 + \frac{\partial^2}{\partial x_3^2},$$

$$\mathcal{L}_3 := \Big(x_2\,\frac{\partial}{\partial x_1}\Big)^2 + \frac{\partial^2}{\partial x_2^2} + \Big(\exp(-1/\sqrt[3]{|x_1|})\,\frac{\partial}{\partial x_3}\Big)^2 + \frac{\partial^2}{\partial x_4^2},$$

take the form (1) *and satisfy assumptions* (S)-*to*-(HY) (*the validity of (HY) for the $\mathcal{L}_i$'s was proved, respectively, by Christ* [7], *Kusuoka and Stroock* [9] *and Morimoto* [11]).

Remark 2.1. *The C^∞-hypoellipticity of $\mathcal{L}$ implies that of $\mathcal{L} - \varepsilon$ (for every real $\varepsilon > 0$) in the following meaningful cases* (*see* [12] *for a discussion on this problem*)*:*

- *for Hörmander operators, and, more generally, for second order sub-elliptic operators (in the usual sense of fulfilling a subelliptic estimate);*
- *for operators with* real-analytic *coefficients.*

Under assumptions (S)-to-(HY), a satisfactory Potential Theory for $\mathcal{L}$ can be constructed (see, e.g., [1, 2]). In this theory, the 'harmonic' functions are the *$\mathcal{L}$-harmonic functions*, that is, the (smooth) solutions to

$$\mathcal{L}u = 0$$

on some open subset of $\mathbb{R}^N$. The corresponding *$\mathcal{L}$-subharmonic functions* are the upper semi-continuous (u.s.c., for short) functions $u : \Omega \to [-\infty, \infty)$ (where Ω is an open subset of $\mathbb{R}^N$) satisfying the next two properties:

(i) $\{x \in \Omega : u(x) > -\infty\}$ is dense in Ω;

(ii) for every bounded open set $V \subseteq \overline{V} \subseteq \Omega$ and for every function h $\mathcal{L}$-harmonic in V and continuous up to ∂V such that $u_{|\partial V} \le h_{|\partial V}$, one has $u \le h$ in V.

Accordingly, a function $u : \Omega \to (-\infty, \infty]$ (where $\Omega \subseteq \mathbb{R}^N$ is open) is $\mathcal{L}$-*superharmonic* in Ω if $-u$ is $\mathcal{L}$-subharmonic in Ω.

By the results in [2] (and by the fact that $h \equiv 1$ is $\mathcal{L}$-harmonic), the following Maximum Principle for $\mathcal{L}$-subharmonic functions holds true (see [5, Theorem A.2]):

Let $\Omega \subseteq \mathbb{R}^N$ be open and *bounded* and let $u \in \underline{\mathcal{L}}(\Omega)$. Then

$$\limsup_{x \to \xi} u(x) \le 0 \text{ for every } \xi \in \partial\Omega \quad \Longrightarrow \quad u \le 0 \text{ in } \Omega. \tag{2}$$

Throughout the sequel, if $\Omega \subseteq \mathbb{R}^N$ is open, we shall use the subsequent notations:

- $\underline{\mathcal{L}}(\Omega)$ denotes the cone of the $\mathcal{L}$-subharmonic functions in Ω;
- $\underline{\mathcal{L}}_b(\Omega)$ denotes the cone of the bounded-above $\mathcal{L}$-subharmonic functions in Ω;
- $\mathcal{L}(\Omega)$ denotes linear space of the $\mathcal{L}$-harmonic functions in Ω;
- $\overline{\mathcal{L}}(\Omega)$ denotes the cone of the $\mathcal{L}$-superharmonic functions in Ω.

We explicitly observe that all the above cones are actually *convex cones.*

We now fix the following (crucial) definition.

Definition 2.1 (MP set for $\mathcal{L}$). *Let $\Omega \subseteq \mathbb{R}^N$ be open. We say that Ω is a* maximum principle set (*MP set, in short*) *for $\mathcal{L}$ if it satisfies the following property:*

$$\begin{cases} u \in \underline{\mathcal{L}}_b(\Omega) \\ \limsup\limits_{x \to \xi} u(x) \le 0 \quad \textit{for every } \xi \in \partial\Omega \end{cases} \quad \Longrightarrow \quad u \le 0 \quad \textit{in } \Omega.$$

Remark 2.2. *A couple of remarks are in order.*

(i) *If $\Omega \subseteq \mathbb{R}^N$ is open and bounded, then Ω is a maximum principle set for $\mathcal{L}$ (and one can replace the cone $\underline{\mathcal{L}}_b(\Omega)$ with $\underline{\mathcal{L}}(\Omega)$).*

(ii) *Since $\mathcal{L}(1) = 0$, if $\Omega \subseteq \mathbb{R}^N$ is an MP set for $\mathcal{L}$ we have*

$$\begin{cases} u \in \underline{\mathcal{L}}_b(\Omega) \\ \limsup\limits_{x \to \xi} u(x) \le M \quad \textit{for every } \xi \in \partial\Omega \end{cases} \quad \Longrightarrow \quad u \le M \quad \textit{in } \Omega$$

(*whatever the chosen $M \in \mathbb{R}$*).

3. A first characterization of MP sets: $\mathcal{L}$-largeness

To present our first result, we fix the following definition.

Definition 3.1 ($\mathcal{L}$-largeness). *We say that a subset F of $\mathbb{R}^N$ is* $\mathcal{L}$-large at infinity *if*

$$\limsup_{\substack{x\to\infty\\ x\in F}} u(x) = \limsup_{\substack{x\to\infty\\ x\in\mathbb{R}^N}} u(x) \quad \text{for every } u\in\underline{\mathcal{L}}_b(\mathbb{R}^N). \tag{3}$$

If $F\subseteq\mathbb{R}^N$ is not *$\mathcal{L}$-large at infinity, we shall say that F is* $\mathcal{L}$-thin at infinity. *More explicitly, F is $\mathcal{L}$-thin at infinity if and only if there exists $u\in\underline{\mathcal{L}}_b(\mathbb{R}^N)$ such that*

$$\limsup_{\substack{x\to\infty\\ x\in F}} u(x) < \limsup_{\substack{x\to\infty\\ x\in\mathbb{R}^N}} u(x).$$

Remark 3.1. *A set $F\subseteq\mathbb{R}^N$ is $\mathcal{L}$-large at infinity if and only if*

$$(\star) \qquad \sup_F u = \sup_{\mathbb{R}^N} u, \qquad \text{for every } u\in\underline{\mathcal{L}}_b(\mathbb{R}^N).$$

In fact, let $u\in\underline{\mathcal{L}}_b(\mathbb{R}^N)$ be arbitrarily fixed. By using the Strong Maximum Principle for $\mathcal{L}$-subharmonic functions in [5, Theorem A.2], *together with the fact that u is upper semi-continuous on $\mathbb{R}^N$, it can be proved that*

$$\sup_{\mathbb{R}^N} u = \limsup_{\substack{x\to\infty\\ x\in\mathbb{R}^N}} u(x).$$

Thus, if Ω is $\mathcal{L}$-large at infinity (so that (3) *holds), we obtain*

$$\limsup_{\substack{x\to\infty\\ x\in F}} u(x) \le \sup_F u \le \sup_{\mathbb{R}^N} u = \limsup_{\substack{x\to\infty\\ x\in\mathbb{R}^N}} u(x) = \limsup_{\substack{x\to\infty\\ x\in F}} u(x),$$

and this proves that $(\star)$ is satisfied.

Conversely, if we assume that $(\star)$ holds, it is not difficult to deduce that

$$\sup_{F\cap B(0,r)} u < \sup_{F\setminus B(0,r)} u \qquad \text{for every } r>0;$$

from this, by letting $r\to\infty$ we derive

$$\sup_F u \le \limsup_{\substack{x\to\infty\\ x\in F}} u(x) \le \limsup_{\substack{x\to\infty\\ x\in\mathbb{R}^N}} u(x) \le \sup_{\mathbb{R}^N} u = \sup_F u,$$

and this proves that (3) *is satisfied.*

Here is our first basic result.

Theorem 3.1 (MP sets and $\mathcal{L}$-largeness)**.** *An open set $\Omega \subseteq \mathbb{R}^N$ is a maximum principle set for $\mathcal{L}$ if and only if its complement $\mathbb{R}^N \setminus \Omega$ is $\mathcal{L}$-large at infinity.*

Proof. Let us assume that Ω is a maximum principle set for $\mathcal{L}$, and let $u \in \underline{\mathcal{L}}_b(\mathbb{R}^N)$. We define $u_0 := \sup_{\mathbb{R}^N\setminus\Omega} u$. Since u is u.s.c. on $\mathbb{R}^N$, we have

$$\limsup_{x\to\xi} u(x) \leq u(\xi) \leq u_0 \qquad (\xi \in \partial\Omega);$$

as a consequence, Ω being an MP set for $\mathcal{L}$, we get

$$u \leq u_0 = \sup_{\mathbb{R}^N\setminus\Omega} u \text{ on } \Omega, \qquad \text{whence } \sup_{\mathbb{R}^N\setminus\Omega} u = \sup_{\mathbb{R}^N} u.$$

This proves that $\mathbb{R}^N \setminus \Omega$ is $\mathcal{L}$-large at infinity.

Conversely, let us assume that $\mathbb{R}^N \setminus \Omega$ is $\mathcal{L}$-large at infinity, and let $u \in \underline{\mathcal{L}}_b(\Omega)$ satisfy

$$\limsup_{x\to\xi} u(x) \leq 0, \qquad \text{for every } \xi \in \partial\Omega. \tag{4}$$

We then consider the function $v : \mathbb{R}^N \to \mathbb{R}$ defined as follows:

$$v(x) := \begin{cases} \max\{u(x), 0\}, & \text{if } x \in \Omega, \\ 0, & \text{if } x \notin \Omega. \end{cases}$$

Taking into account (4), it can be proved that $v \in \underline{\mathcal{L}}_b(\mathbb{R}^N)$; as a consequence, since $\mathbb{R}^N \setminus \Omega$ is $\mathcal{L}$-large at infinity we get (see Remark 3.1)

$$\sup_{\mathbb{R}^N} v = \sup_{\mathbb{R}^N\setminus\Omega} v = 0, \qquad \text{whence } u \leq 0 \text{ on } \Omega.$$

This proves that Ω is an MP set for $\mathcal{L}$. □

4. A sufficient condition for $\mathcal{L}$-largeness: $p_{\mathcal{L}}$-unboundedness

In view of the results in Section 3 (see, in particular, Theorem 3.1), it is natural to look for some simple criteria ensuring the $\mathcal{L}$-largeness at infinity of a set $F \subseteq \mathbb{R}^N$.

In order to do this, we require the following additional assumptions on $\mathcal{L}$.

(FS) $\mathcal{L}$ is endowed with a *"well-behaved" global fundamental solution*, that is, it is possible to find a function

$$\Gamma : \mathcal{O} := \big\{(x,y) \in \mathbb{R}^N \times \mathbb{R}^N : x \neq y\big\} \longrightarrow \mathbb{R}$$

satisfying the following properties:

(a) $\Gamma \in C^\infty(\mathcal{O}, \mathbb{R})$ and $\Gamma(x,y) > 0$ for every $x, y \in \mathcal{O}$;

(b) Γ is symmetric, that is, $\Gamma(x,y) = \Gamma(y,x)$ for every $(x,y) \in \mathcal{O}$;

(c) for every $x \in \mathbb{R}^N$, we have $\Gamma(x,\cdot) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and

$$\int_{\mathbb{R}^N} \Gamma(x,y)\, \mathcal{L}\varphi(y)\, V(y)\, \mathrm{d}y = -\varphi(x), \quad \forall\, \varphi \in C_0^\infty(\mathbb{R}^N, \mathbb{R});$$

(d) for every $x \in \mathbb{R}^N$, $\Gamma(x,\cdot)$ has a pole at x and it vanishes at infinity, i.e,

$$\lim_{y \to x} \Gamma(x,y) = \infty \qquad \text{and} \qquad \lim_{\|y\| \to \infty} \Gamma(x,y) = 0.$$

For the sake of brevity, given $x \in \mathbb{R}^N$, in the sequel we set:

$$\Gamma_x : \mathbb{R}^N \setminus \{x\} \longrightarrow \mathbb{R}, \quad \Gamma_x(y) := \Gamma(x,y).$$

(G) Defining the *open* Γ-*ball* of centre x and radius r as

$$\Omega(x,r) := \{y \in \mathbb{R}^N \setminus \{x\} : \Gamma_x(y) > 1/r\} \cup \{x\},$$

there exists a constant $\theta \in (0,1)$ such that

$$x \notin \Omega(y,r) \qquad \Longrightarrow \qquad \Omega(x,\theta r) \cap \Omega(y,\theta r) = \emptyset$$

for every $x, y \in \mathbb{R}^N$ and every $r > 0$.

(L) If $u \in \mathcal{L}(\mathbb{R}^N)$ is any $\mathcal{L}$-harmonic function which is bounded from above (or from below) on $\mathbb{R}^N$, then u is constant throughout $\mathbb{R}^N$.

Remark 4.1. *It is easy to see that assumption* (G) *is equivalent to requiring that* $\gamma := 1/\Gamma$ *is a* quasi-distance *in* $\mathbb{R}^N$*, that is, there exists a real* $\mathbf{c} > 1$ *such that*

$$\gamma(x,z) \le \mathbf{c}\big(\gamma(x,y) + \gamma(y,z)\big), \quad \textit{for any } x, y, z \in \mathbb{R}^N.$$

Remark 4.2. *Assumptions* (FS)*-to-*(L) *are satisfied by any sub-Laplacian on a homogeneous Carnot group* (*see, e.g.,* [6, Chap. 5])*; moreover, we shall see that these assumptions are fulfilled also by any homogeneous Hörmander operator in* $\mathbb{R}^N$*.*

Under assumptions (FS)-to-(L), we obtain a geometrical criterion for $\mathcal{L}$-largeness at infinity which is based on the notion of $p_{\mathcal{L}}$-unboundedness introduced below.

Definition 4.1 ($p_{\mathcal{L}}$-unboundedness)**.** *Let* $F \subseteq \mathbb{R}^N$ *and let* $p \in (1,\infty)$*. We say that* F *is* $p_{\mathcal{L}}$-bounded *if there exists a family* $\mathcal{F} = \big\{\Omega(x_n, r_n)\big\}_{n \in J}$*, with* $J \subseteq \mathbb{N}$*, such that*

(a) $F \subseteq \bigcup_{n\in J} \Omega(x_n, r_n)$;

(b) $\displaystyle\sum_{n\in J} \big(\Gamma(0; x_n)\, r_n\big)^p < \infty.$

*If $F \subseteq \mathbb{R}^N$ is not $p_{\mathcal{L}}$-bounded, we shall say that F is $p_{\mathcal{L}}$-*unbounded.

We then have the following result (see [5, Theorem 1.5]).

Theorem 4.1 (Criterion for $\mathcal{L}$-largeness)**.** *Let $F \subseteq \mathbb{R}^N$ be any non-void set. If there exists some $p \in (1, \infty)$ such that F is $p_{\mathcal{L}}$-unbounded, then F is $\mathcal{L}$-large at infinity.*

The proof of Theorem 4.1 rests on the following result, which is resemblant to the classical result demonstrated by Deny in the case $\mathcal{L} = \Delta$ (see Theorem 1.2).

Theorem 4.2 (of Deny-type)**.** *Let $u \in \underline{\mathcal{L}}_b(\mathbb{R}^N)$ and let $p \in (1, \infty)$ be arbitrarily fixed. Then, it is possible to construct a $p_{\mathcal{L}}$-bounded set $F \subseteq \mathbb{R}^N$ such that*

$$\lim_{\substack{x\to\infty \\ x\notin F}} u(x) = \sup_{\mathbb{R}^N} u.$$

For a proof of Theorem 4.2, see [5, Theorem 1.6].

5. A sufficient condition for $p_{\mathcal{L}}$-unboudedness: Γ-cones

In view of the criterion for $\mathcal{L}$-largeness contained in Theorem 4.1, it is natural to look for some geometrical conditions ensuring the $p_{\mathcal{L}}$-unboundedness of a set $F \subseteq \mathbb{R}^N$.

To this end, we require another additional assumption on $\mathcal{L}$.

(D) There exist two constants $\alpha', \alpha'' > 2$, with $\alpha' < \alpha''$, such that

$$\alpha' \left|\Omega(x, r)\right| \leq \left|\Omega(x, 2r)\right| \leq \alpha'' \left|\Omega(x, r)\right|$$

for every $x \in \mathbb{R}^N$ and every $r > 0$ (here, $|A|$ indicate the standard N-dimensional Lebesgue measure in $\mathbb{R}^N$ of a Borel set $A \subseteq \mathbb{R}^N$).

Remark 5.1. *As for the case of assumptions* (FS)*-to-*(L)*, also assumption* (D) *is satisfied by any sub-Laplacian on a Carnot group; moreover, as we shall see, this assumption is satisfied also by any homogeneous Hörmander operator in $\mathbb{R}^N$.*

Under assumption (D), the sufficient condition for $p_{\mathcal{L}}$-unboundedness we obtain is related with the definition of Γ-cone given below.

Definition 5.1 (Γ-cone)**.** *Let $K \subseteq \mathbb{R}^N$ be any set. We say that K is a* Γ-cone *if it contains a countable family $\mathcal{F} = \{\Omega(z_j, R_j)\}_{j\in\mathbb{N}}$ of Γ-balls such that*

(i) $\|z_j\| \to \infty$ *as* $j \to \infty$*;*

(ii) $\liminf_{j\to\infty} \big(\Gamma_0(z_j)\, R_j\big) > 0$.

Then, the following theorem holds true (see [5, Theorem 1.8]).

Theorem 5.1 (Criterion for $p_{\mathcal{L}}$-unboundedness)**.** *Let $F \subseteq \mathbb{R}^N$ and let us assume that there exists a Γ-cone $K \subseteq F$. Then, there exists $p > 1$ such that F is $p_{\mathcal{L}}$-unbounded.*

Gathering together Theorems 3.1, 4.1 and 5.1 we obtain the following result, in which all the hypotheses (H1)-to-(H3), (FS), (G), (L) and (D) are assumed.

Theorem 5.2. *The open set $\Omega \subseteq \mathbb{R}^N$ is a maximum principle set for $\mathcal{L}$ if one of the following* (*sufficient*) *conditions is satisfied:*

(i) $\mathbb{R}^N \setminus \Omega$ *is $\mathcal{L}$-large at infinity* (*this condition is also necessary*)*;*

(ii) $\mathbb{R}^N \setminus \Omega$ *is $p_{\mathcal{L}}$-unbounded* (*for a suitable $p > 1$*)*;*

(iii) $\mathbb{R}^N \setminus \Omega$ *contains a Γ-cone.*

Proof. (i) This is precisely the statement of Theorem 3.1.

(ii) If $\mathbb{R}^N \setminus \Omega$ is $p_{\mathcal{L}}$-unbounded (for some $p > 1$), we know from Theorem 4.1 that $\mathbb{R}^N \setminus \Omega$ is $\mathcal{L}$-large at infinity; thus, by (i), Ω is an MP set for $\mathcal{L}$.

(iii) If $\mathbb{R}^N \setminus \Omega$ contains a Γ-cone K, we know from Theorem 5.1 that there exists a real $p > 1$ such that $\mathbb{R}^N \setminus \Omega$ is $p_{\mathcal{L}}$-unbounded; thus, by (ii), we conclude that Ω is a maximum principle set for $\mathcal{L}$. This ends the proof. $\square$

As a consequence of Theorem 5.2 we easily obtain the following result.

Corollary 5.1. *Let $\Omega \subseteq \mathbb{R}^N$ be an open set satisfying one of conditions* (i)-to-(iii) *in Theorem 5.2. Moreover, let $f : \Omega \times \mathbb{R} \to \mathbb{R}$ be such that*

$$f(x,z) \leq 0 \qquad \text{for every } x \in \Omega \text{ and } z \geq 0. \tag{5}$$

If $u \in C^2(\Omega, \mathbb{R})$ is bounded above and satisfies

$$\begin{cases} \mathcal{L}u + f(x,u) \geq 0 & \text{in } \Omega, \\ \limsup\limits_{x\to y} u(x) \leq 0 & \text{for every } y \in \partial\Omega, \end{cases} \tag{6}$$

then $u \leq 0$ *throughout* Ω.

Proof. We argue by contradiction and we assume the existence of some point $x_0 \in \Omega$ such that $u(x_0) > 0$. We then consider the following set

$$\Omega^+ := \{x \in \Omega : u(x) > 0\} \neq \emptyset. \tag{7}$$

By combining (5) with (6) we infer that, on Ω^+, we have $\mathcal{L}u \geq -f(x,u) \geq 0$; as a consequence, $u \in \underline{\mathcal{L}}(\Omega^+)$. On the other hand, by the boundary condition in (6) and the fact that $u \equiv 0$ on $\partial\Omega^+ \cap \Omega$, it is readily seen that

$$\limsup_{x \to y} u(x) \leq 0 \quad \text{for every } y \in \partial\Omega^+.$$

From this, it can be proved that (see [5, Lemma 2.1])

$$v : \Omega \longrightarrow \mathbb{R}, \qquad v(x) = \max\{u(x), 0\}$$

is $\mathcal{L}$-subharmonic in Ω; furthermore, since u is bounded above in Ω, the same is true of v. Taking into account that, by assumption, Ω is an MP-set for $\mathcal{L}$, we conclude that $v \leq 0$, whence $u \leq 0$, but this is in contradiction with (7). $\square$

6. A class of examples: homogeneous Hörmander operators

Let $\mathcal{X} = \{X_1, \ldots, X_m\}$ be a family of linearly independent smooth vector fields on Euclidean space $\mathbb{R}^N$, with $N \geq 3$, satisfying the following properties:

(H.1) $X_1, \ldots, X_m$ are δ_λ-homogeneous of degree 1 with respect to a family of non-isotropic dilations $\{\delta_\lambda\}_{\lambda>0}$ of the following type

$$\delta_\lambda : \mathbb{R}^N \to \mathbb{R}^N, \qquad \delta_\lambda(x) = \big(\lambda^{\sigma_1} x_1, \ldots, \lambda^{\sigma_N} x_N\big),$$

where $1 = \sigma_1 \leq \ldots \leq \sigma_N$ are positive integers;

(H.2) $X_1, \ldots, X_m$ satisfy the Hörmander rank condition, i.e.,

$$\dim\Big\{X(x) : X \in \mathrm{Lie}\big\{X_1, \ldots, X_m\big\}\Big\} = N \quad \text{for every } x \in \mathbb{R}^N.$$

Then, the second-order linear operator $\mathcal{L}$ defined by

$$\mathcal{L} := \sum_{j=1}^m X_j^2,$$

will be called a *homogeneous Hörmander operator*.

Example 6.1. *Some examples of homogeneous Hörmander operators are in order.*

(i) *The Bony-type vector fields on* $\mathbb{R}^N$*:*

$$X_1 = \partial_{x_1}, \quad X_2 = x_1\,\partial_{x_2} + \cdots + x_1^{N-1}\,\partial_{x_N},$$

which are homogeneous of degree 1 *w.r.t.* $\delta_\lambda(x) = (\lambda x_1, \lambda^2 x_2, \ldots, \lambda^N x_N)$.

(ii) *The vector fields on* $\mathbb{R}^3$*:*

$$X_1 = \partial_{x_1}, \quad X_2 = x_1\,\partial_{x_2}, \quad X_3 = x_1\,\partial_{x_3},$$

which are homogeneous of degree 1 *w.r.t.* $\delta_\lambda(x) = (\lambda x_1, \lambda^2 x_2, \lambda^2\, x_3)$.

(iii) *The vector fields on* $\mathbb{R}^3$*:*

$$X_1 = \partial_{x_1}, \quad X_2 = \partial_{x_2}, \quad X_3 = x_1 x_2\,\partial_{x_3},$$

which are homogeneous of degree 1 *w.r.t.* $\delta_\lambda(x) = (\lambda x_1, \lambda x_2, \lambda^3 x_3)$.

Let now $\mathcal{L} = \sum_{j=1}^m X_j^2$ be a homogeneous Hörmander operator in $\mathbb{R}^N$, with $N \geq 3$. By exploiting several known results, we see that $\mathcal{L}$ satisfies most of the assumptions introduced in the previous sections; in fact, we have

(a) $\mathcal{L}$ satisfies assumptions (S)-to-(HY) (see Example 2.1-(c));
(b) $\mathcal{L}$ satisfies assumption (FS) (by the results in [3]);
(c) $\mathcal{L}$ satisfies assumption (L) (by the results in [10]).

Furthermore, the following theorem holds true.

Theorem 6.1. *If* $\mathcal{L} = \sum_{j=1}^m X_j^2$ *is a homogeneous Hörmander operator in* $\mathbb{R}^N$ (*with* $N \geq 3$), *then* $\mathcal{L}$ *satisfies assumptions* (G) *and* (D).

The proof of Theorem 6.1 is quite long, and we refer to [5, Section 5] for all the details. Here, we limit ourselves to point out that the key ingredients for proving Theorem 6.1 are suitable global estimates for two objects associated with $\mathcal{L}$: its global fundamental solution Γ and the measure of the balls in the Carnot-Carathéodory metric associated with $X_1, \ldots, X_m$ (see the very recent paper [4]).

Gathering together assertions (a)-to-(c), Theorem 6.1 and Theorem 5.2, we obtain the following result (which is just a restatement of Theorem 5.2 in the present setting).

Corollary 6.1. *Let $\mathcal{L}$ be a homogeneous Hörmander operator on $\mathbb{R}^N$ (with $N \geq 3$).*

An open subset Ω of $\mathbb{R}^N$ is a maximum principle set for $\mathcal{L}$ if one of the following (sufficient) conditions is satisfied:

(i) *$\mathbb{R}^N \setminus \Omega$ is $\mathcal{L}$-large at infinity (this condition is also necessary);*

(ii) *$\mathbb{R}^N \setminus \Omega$ is $p_{\mathcal{L}}$-unbounded (for a suitable $p > 1$);*

(iii) *$\mathbb{R}^N \setminus \Omega$ contains a Γ-cone.*

On the other hand, by using the family of dilations $\{\delta_\lambda\}_{\lambda>0}$, we can obtain a "homogeneous version" of the cone criterion (iii), which is based on the definition of δ_λ-cone.

Definition 6.1 (δ_λ-cone)**.** *Let $C \subseteq \mathbb{R}^N$ be any set. We say that C is a* non-degenerate δ_λ-cone *if it satisfies the following properties:*

(i) $\mathrm{int}(F) \neq \emptyset$*;*

(ii) *there exists $\lambda_0 > 0$ such that $\delta_\lambda(C) \subseteq C$ for every $\lambda \geq \lambda_0$.*

Then, we have the following crucial result (see [5, Prop. 1.12] for a proof).

Proposition 6.1. *If $F \subseteq \mathbb{R}^N$ contains a non-degenerate δ_λ-cone, then there exists $p > 1$ such that F is $p_{\mathcal{L}}$-unbounded. (in the sense of Definition 4.1).*

It can be easily proved that every half-space of $\mathbb{R}^N$ contains a non-degenerate δ_λ-cone (see [5, Remark 5.11]); as a consequence, by combining Proposition 6.1 with Theorem 5.2, we readily obtain the announced "homogeneous version" of Corollary 6.1.

Theorem 6.2. *Let $\mathcal{L}$ be a homogeneous Hörmander operator in $\mathbb{R}^N$ (with $N \geq 3$) and let $\Omega \subseteq \mathbb{R}^N$ be an open set satisfying one of the following conditions:*

(i) *$\mathbb{R}^N \setminus \Omega$ contains a non-degenerate δ_λ-cone;*

(ii) *Ω is contained in a half-space.*

Then Ω is a maximum principle for $\mathcal{L}$.

Proof. (i) If $\mathbb{R}^N \setminus \Omega$ contains a non-degenerate δ_λ-cone, it follows from Proposition 6.1 that $\mathbb{R}^N \setminus \Omega$ is $p_{\mathcal{L}}$-unbounded (for some $p > 1$); as a consequence, Corollary 6.1-(ii) allows us to conclude that Ω is a maximum principle set for $\mathcal{L}$.

(ii) If Ω is contained in a half-space H, then $\mathbb{R}^N \setminus \Omega$ contains the half-space $H' = \mathbb{R}^N \setminus H$; since H' contains a non-degenerate δ_λ-cone (see [5, Remark 5.11]), we conclude from (i) that Ω is a maximum principle set for $\mathcal{L}$. $\square$

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