SOBOLEV-POINCARÉ INEQUALITIES FOR DIFFERENTIAL FORMS AND CURRENTS IN $\mathbb{R}^n$
DISUGUAGLIANZE DI SOBOLEV E POINCARÉ PER FORME DIFFERENZIALI E CORRENTI IN $\mathbb{R}^n$

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Abstract. In this Note we collect some results in $\mathbb{R}^n$ about (p,q) Poincaré and Sobolev inequalities, with $1 \leq p < n$, for differential forms obtained in a joint research with Franchi and Pansu. In particular, we focus to the the case $p = 1$. From the geometric point of view, Poincaré and Sobolev inequalities for differential forms provide a quantitative formulation of the vanishing of the cohomology. As an application of the results obtained in the case $p = 1$ we obtain a Poincaré and Sobolev inequalities for Euclidean currents.

Sunto. In questa Nota presentiamo alcuni risultati ottenuti in una ricerca in collaborazione con Franchi e Pansu che riguardano disuguaglianze (p,q) di Poincaré e di Sobolev $(1 \leq p < n)$ per forme differenziali e correnti in $\mathbb{R}^n$. Ci soffermeremo in particolare sul caso dell' esponente $p = 1$. Dal punto di vista geometrico le disuguaglianze di Poincaré e Sobolev per forme differenziali sono una formulazione di tipo quantitativo in teoria della coomologia. Il caso $p = 1$ si presta inoltre ad essere generalizzato alla teoria delle correnti e in questa Nota otteniamo una disuguaglianz di Poincaré (e di Sobolev) per correnti.

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1. Introduction

In the Euclidean setting $\mathbb{R}^n$, when dealing with differential forms there is a well known topological problem, whether a given closed form is exact. Beside, for several applications to the cohomology theory for example, we can study also an analytical problem: whether a primitive ϕ of a given exact form ω can be upgraded to one which satisfies a (p, q) estimate of the type $\|\phi\|_q \leq c\|\omega\|_p$. More precisely, if $1 \leq p < n$, we ask whether, given a closed differential h-form ω in $L^p(\mathbb{R}^n)$, there exists an $(h-1)$-form ϕ in $L^q(\mathbb{R}^n)$ for some $q \geq p$ such that $d\phi = \omega$ and

$$\|\phi\|_q \leq C\, \|\omega\|_p, \tag{1}$$

for $C = C(n, p, q, h)$. We refer to the above inequality as to the (p, q)-Poincaré inequality for h-forms (notice that, by the scale invariance, we must have $\frac{1}{p} - \frac{1}{q} = \frac{1}{n}$). We notice that there is a connection between the above inequality and the classical Poincaré inequality. Recall that classical Poincaré inquality for functions says that, if $1 \leq p < n$, for any (say) Lipschitz continuous function u there exists a constant c_u such that

$$\|u - c_u\|_q \leq C(N, p)\, \|\nabla u\|_p \quad \text{provided} \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{n}.$$

Classical Poincaré inequality for functions (i.e. 0-forms) can be derived from Poincaré inequality for differential forms. Indeed, we notice that $du =: \omega$ is a closed form so that, if there exists ϕ in $L^q(\mathbb{R}^n)$ such that $d\phi = \omega$, then $u - \phi = c_u$ (since $u - \phi$ is closed) and then

$$\|u - c_u\|_q = \|\phi\|_q \leq C\, \|du\|_p \leq C\, \|\nabla u\|_p.$$

Sobolev inequality in $\mathbb{R}^n$ deals with compactly supported 0-forms, i.e. functions u on $\mathbb{R}^n$, and 1-forms, their differentials du. It states that

$$\|u\|_q \leq C_{p,q,n} \|du\|_p$$

whenever

$$1 \leq p, q < +\infty, \quad \frac{1}{p} - \frac{1}{q} = \frac{1}{n}.$$

In 1993 Iwaniec & Lutoborski (see [11], Corollary 4.2) proved the following remarkable Poincaré inequalities for differential forms, for $1 < p < \infty$. Let D be (say) a ball in

$\mathbb{R}^n$, and let ω be a h-form in $\mathbb{R}^n$ with distributional coefficients such that $d\omega$ has L^p coefficients, then there exists a closed h-form ω_D such that

$$\|\omega - \omega_D\|_{L^{pn/(n-p)}(D)} \le C\|d\omega\|_{L^p(D)}, \tag{2}$$

(a Sobolev inequality was proved by [14], Theorem 4.1 and equation (169)).

A straightforward computation shows that this statement can be formulated as in (1).

The statement above actually holds for bounded convex domains. However, for more general Euclidean domains, the validity of Poincaré inequality is sensitive to irregularities of boundaries. One way to eliminate such a dependence is to allow a loss on domain (this has been done, in a more general subriemannian setting in [2]). In fact, if we are for example interested in applications to the cohomology theory, a weaker form of Poincaré inequality (2) suffices. We can call it *interior* Poincaré inequality or Poincaré inequality with loss of domain (see [15]), and it reads as follows.

Let D, D' be two balls in $\mathbb{R}^n$, $D \subset\subset D'$ and let ω be a closed form in D' with L^p coefficients. Then ω admits a potential ϕ in $L^{pn/(n-p)}(D)$ and

$$\|\phi\|_{L^{pn/(n-p)}(D)} \le C\|\omega\|_{L^p(D')}.$$

Analogously, by *interior* Sobolev inequalities, we mean that, if ω is supported in D, then there exists ϕ supported in D' such that $d\phi = \alpha$ and

$$\|\phi\|_{L^{pn/(n-p)}(D')} \le C\,\|\omega\|_{L^p(D)}. \tag{3}$$

We stress that the interior Poincaré inequality, though apparently weaker than the Poincaré inequality without loss of domain, is, under other respects, more general since it is not affected by the geometry of the boundary of D (the word "interior" refers precisely to this feature). Moreover, the interior Sobolev and Poincaré inequalities that we derive from [11], [14] can be extended to the endpoint case $p = 1$ (the case $p = 1$ has been studied indeed in [3], see Theorem 2.1 below and in a subriemannian setting in [1]). The interior Sobolev and Poincaré inequalities can be extended also to the case $p = n$ (it is a straightforward consequence of [11] and [5] in $\mathbb{R}^n$, see also [4] for a non-Euclidean setting).

In this note, In Section 2 we expose the results and the scheme of the proof of Poincaré inequality for differential forms (1), in fact very shortly, since the details are already contained in the papers quoted above. On the other hand, interesting consequences of the results contained in [3] (see Theorem 2.1 below) are the following Poincaré and Sobolev type inequalities for Euclidean currents. The statement for currents reads as follows (see Section 3 below for precise definitions):

Theorem 1.1. *For $h = 1, \dots, n-1$, let $q = n/(n-1)$. Let $B \subset \mathbb{R}^n$ be a bounded open convex set, and let B' be an open set, $B \subset\subset B'$. Then there exists $C = C(n, B, B')$ with the following property:*

(1) *Interior Poincaré inequality. For every h-current T in B' with finite mass $M(T)$ and such that $\partial T = 0$, there exists an $(h-1)$-form $\phi \in L^q(B)$, such that, if we denote by T_ϕ the current associated with ϕ,*

$$\partial T_\phi = T, \qquad \text{and} \qquad \|\phi\|_{L^q(B)} \le C\, M(T).$$

(2) *Sobolev inequality. For every h-current supported in B with finite mass $M(T)$ and such that $\partial T = 0$, there exists an $(h-1)$-form $\phi \in L^q(B')$, with support in B', such that*

$$\partial T_\phi = T \qquad \text{and} \qquad \|\phi\|_{L^q(B')} \le C\, M(T).$$

2. Poincaré and Sobolev inequalities for differential forms in $\mathbb{R}^n$

Throughout the present note our setting will be the Euclidean space $\mathbb{R}^n$ with $n > 2$. If f is a real function defined in $\mathbb{R}^n$, we denote by ${}^{\text{v}}f$ the function defined by ${}^{\text{v}}f(p) := f(-p)$, and, if $T \in \mathcal{D}'(\mathbb{R}^n)$, then ${}^{\text{v}}T$ is the distribution defined by $\langle {}^{\text{v}}T|\phi\rangle := \langle T|{}^{\text{v}}\phi\rangle$ for any test function ϕ.

We remind also that the convolution $f * g$ is well defined when $f, g \in \mathcal{D}'(\mathbb{R}^n)$, provided at least one of them has compact support.

As customary, a basis of the tangent space $\bigwedge_1(\mathbb{R}^n) := \mathbb{R}^n$ is given by $(\partial_{x_1}, \dots, \partial_{x_n})$. We denote by $\langle \cdot, \cdot \rangle$ the scalar product making $(\partial_{x_1}, \dots, \partial_{x_n})$ orthonormal.

The dual space of $\bigwedge_1(\mathbb{R}^n)$ is denoted by $\bigwedge^1(\mathbb{R}^n) =: (\mathbb{R}^n)^*$. The basis of $\bigwedge^1(\mathbb{R}^n)$, dual to the basis $(\partial_{x_1}, \dots, \partial_{x_n})$, is the family of covectors $(dx_1, \dots, dx_n)$ and we again indicate as $\langle \cdot, \cdot \rangle$ the inner product in $(\mathbb{R}^n)^*$ that makes $(dx_1, \dots, dx_n)$ an orthonormal basis.

We put $\bigwedge_0(\mathbb{R}^n) = \bigwedge^0(\mathbb{R}^n) := \mathbb{R}$ and, for $1 \le h \le n$,

$$\bigwedge_h(\mathbb{R}^n) := \operatorname{span}\{\partial_{x_{i_1}} \wedge \cdots \wedge \partial_{x_{i_h}} : 1 \le i_1 < \cdots < i_h \le n\}$$

and

$$\bigwedge^h(\mathbb{R}^n) := \operatorname{span}\{dx_{i_1} \wedge \cdots \wedge dx_{i_h} : 1 \le i_1 < \cdots < i_h \le n\}\,.$$

If $I := (i_1, \dots, i_h)$ with $1 \le i_1 < \cdots < i_h \le n$, we set $|I| := h$ and

$$dx^I := dx_{i_1} \wedge \cdots \wedge dx_{i_h}.$$

The elements of $\bigwedge_h(\mathbb{R}^n)$ and $\bigwedge^h(\mathbb{R}^n)$ are called *h-vectors* and *h-covectors* respectively. The scalar products in the spaces of 1-vectors and 1-covectors can be canonically extended to $\bigwedge_h(\mathbb{R}^n)$ and $\bigwedge^h(\mathbb{R}^n)$ respectively.

The *Hodge star operator* is a linear operator

$$* : \bigwedge^h(\mathbb{R}^n) \to \bigwedge^{n-h}(\mathbb{R}^n)$$

defined by $\xi \wedge \eta = \langle \xi, *\eta \rangle$ for any $\eta \in \bigwedge^{n-h}(\mathbb{R}^n)$.

If $v \in \bigwedge_h(\mathbb{R}^n)$ and $\xi \in \bigwedge^h(\mathbb{R}^n)$, $|v|$ and $|\xi|$ denote as costumary their Euclidean norm. We recall now the definition of the comass norm of a covector (see [9], Chapter 2, Section 2.1).

Definition 2.1. *We denote by* $\|\xi\|$ *the comass norm of a covector* $\xi \in \bigwedge^h(\mathbb{R}^n)$ *by*

$$\|\xi\| = \sup\left\{ \langle \xi | v \rangle \;\middle|\; v \in \bigwedge_h(\mathbb{R}^n),\ |v| \le 1,\ v \text{ simple} \right\}.$$

By formula (13) of [10], Chapter 1, Section 2.2, there exists a geometric constant $c_1 > 0$ such that

$$c_1^{-1}|\xi| \le \|\xi\| \le |\xi| \qquad \text{for all } \xi \in \bigwedge^h(\mathbb{R}^n). \tag{4}$$

By translation, $\bigwedge^h(\mathbb{R}^n)$ defines a fibre bundle over $\mathbb{R}^n$, still denoted by $\bigwedge^h(\mathbb{R}^n)$. A differential form on $\mathbb{R}^n$ is a section of this fibre bundle.

Through this Note, if $0 \le h \le n$ and $\mathcal{U} \subset \mathbb{R}^n$ is an open set, we denote by $\Omega^h(\mathcal{U})$ the space of differential h-forms on $\mathcal{U}$, and by $d : \Omega^h(\mathcal{U}) \to \Omega^{h+1}(\mathcal{U})$ the exterior differential.

Thus $(\Omega^\bullet(\mathcal{U}), d)$ is the de Rham complex in $\mathcal{U}$ and any $u \in \Omega^h$ can be written as $u = \sum_{|I|=h} u_I dx^I$.

If $\mathcal{U}$ is an open set in $\mathbb{R}^n$ we write that a h-form $\omega \in D(\mathcal{U})$ if their components with respect to a fixed basis belong to $D(\mathcal{U})$. Analogously, a h form $\omega \in L^p(\mathcal{U})$ if their components with respect to a fixed basis are in $L^p(\mathcal{U})$, endowed with its natural norm. Clearly, these definition are independent of the choice of the basis itself.

In the following two subsections we sketch the principal steps used to prove the Poincaré inequality (1) respectively for the case $p > 1$ and $p = 1$. The details of the proofs, as remarked above, can be found in the references already quoted [11] and [3] (and if one is interested to a non-Euclidean setting analougus results are proved respectively in [2] and [1] for the Heisenberg groups).

2.1. **Scheme of proof of** (1) **for** $p > 1$**: homotopy operators.** The most efficient way to prove a Poincaré inequality in the whole $\mathbb{R}^n$ is to find a homotopy between identity and 0 on the complex of differential forms, i.e. a linear operator K that raises the degree by 1 and satisfies

$$1 = dK + Kd.$$

The Laplacian provides us with such a homotopy. Write $\Delta = dd^* + d^*d$ where d^* is the formal L^2-adjoint of d. Denote by Δ^{-1} the operator of convolution with the fundamental solution of the Laplacian. Then Δ^{-1} commutes with d and its adjoint d^*, hence $K := d^*\Delta^{-1}$ satisfies $1 = dK + Kd$ on globally defined L^p differential forms. The operator K is given by convolution with a homogeneous kernel of type 1 in the terminology of [6], hence it is bounded from L^p to L^q if $p > 1$. This proves the (p, q)-Poincaré inequality in $\mathbb{R}^n$.

To pass to bounded sets, we recall that Poincaré's Lemma asserts that every closed form on a ball is exact. We need a quantitative version of this statement. The standard proof of Poincaré's Lemma relies on a homotopy operator which depends on the choice of an origin. Iwaniec and Lutoborski, [11] observed that averaging it over origins yields a bounded operator $K : L^p \to L^q$. Hence we get the Euclidean Poincaré inequality for convex domains. A support preserving variant $J : L^p \to L^q$ appears in Mitrea-Mitrea-Monniaux, [14] and this proves the (p,q)-Sobolev inequality for bounded convex Euclidean domains with $p > 1$.

2.2. **Scheme of the proof of** (1) **for** $p = 1$**.** As noticed above, the case $p > 1$ has been fully understood on bounded convex sets by Iwaniec & Lutoborsky ([11]) and in the full space $\mathbb{R}^n$ an easy proof consists in putting $\phi = d^*\Delta^{-1}\omega$. Unfortunately, this argument does not suffice for $p = 1$ since, by the Hardy-Littlewood-Sobolev inequality (see [16] Theorem 1 pag.119), $d^*\Delta^{-1}$ maps L^1 only into the weak Marcinkiewicz space $L^{n/(n-1),\infty}$. Upgrading from $L^{n/(n-1),\infty}$ to $L^{n/(n-1)}$ is possible for functions (see [13], [7], [8]). but the trick used for functions does not seem to generalize to differential forms.

We explain here, very roughtly the idea contained in [3] to prove Poincaré inequality (1) in the case $p = 1$.

Set $q = n/(n-1)$. The core of the proof consists of two points. The first one is the following result due to Lanzani-Stein [12], that says that for smooth compactly supported differential forms ϕ of degrees $< n-1$ with $d^*\phi = 0$, we have

$$\|\phi\|_{n/(n-1)} \le C\,\|d\phi\|_1 \,. \tag{5}$$

In other words, the classical Gagliardo-Nirenberg inequality is the first link of a chain of analogous inequalities for compactly supported smooth differential forms. Starting from a closed differential forms ω in L^1, to prove Poincaré inequality, micking the proof for $p > 1$ one could be tempted to take again $\phi = d^*\Delta^{-1}\omega$ and replace the usual $L^p - L^q$ boundedness of singular integrals of potential type by using Lanzani-Stein inequality (5). Indeed, it is easy to check $d^*\phi = 0$ and the desired estimate $\|\phi\|_{n/n-1} \le \|\omega\|_1$ almost follows from Lanzani-Stein inequality, except that ϕ is not compactly supported so cannot be directly plugged in (5). The trick used in [3] then is to show instead (up to a regularization argument) that

$$\|\phi_R\|_{n/n-1} \le c\|\omega\|_1 + o(1)$$

where $\phi_R := d^*(\chi_R\Delta^{-1})$ is a suitable smooth localization of ϕ to a (large) ball of radius R, and $o(1) \to 0$ as $R \to +\infty$.

In expression above we have $o(1)$ since the use of an argument of truncation produces commutation terms. They have to be handled carefully:

$$d\phi_R = [dd^*, \chi_R]\Delta^{-1}\omega + \chi_R dd^*\Delta^{-1}\omega = [dd^*, \chi_R]\Delta^{-1}\omega + \chi_R\omega.$$

Using (5) (possible because now ϕ_R is compactly supported),

$$\begin{aligned} \|\phi_R\|_{L^{n/(n-1)}} &\le C\|d\phi_R\|_{L^1(\mathbb{R}^n)} = C\|dd^*(\chi_R\Delta^{-1}\omega)\|_{L^1(\mathbb{R}^n)} \\ &\le C\big\{\|[dd^*, \chi_R](\Delta^{-1}\omega)\|_{L^1(\mathbb{R}^n)} + \|\chi_R(dd^*\Delta^{-1}\omega)\|_{L^1(\mathbb{R}^n)}\big\}. \end{aligned} \tag{6}$$

The second term of the left hand side can be handled by a duality argument. In fact, we can show that

$$\begin{aligned} \omega &= \Delta\Delta^{-1}\omega = dd^*\Delta^{-1}\omega + d^*d\Delta^{-1}\omega \\ &= dd^*\Delta^{-1}\omega + d^*\Delta^{-1}d\omega = dd^*\Delta^{-1}\omega. \end{aligned}$$

The point is to estimate the garbage term $\|[dd^*, \chi_R]\Delta^{-1}\omega\|_1$.

Notice that $[dd^*, \chi_R]$ is a first order differential operator, of the form $[dd^*, \chi_R] = P_0 + P_1$ where P_0 has order 0 and depends on second derivatives $\nabla^2\chi_R$ and P_1 has order 1 and depends on first derivatives $\nabla\chi_R$ only. Both $P_0\Delta^{-1}$ and $P_1\Delta^{-1}$ have homogeneous kernels.

Now we can use the key trick used in [3] (Lemma 2.6), that is if P is the operator of convolution with a kernel of type $\mu > 0$, and $\omega \in L^1$, then the L^1 norm of $P\omega$ on shells $B(0, 2R) \setminus B(0, R)$ is $o(R^\mu)$.

Therefore, if we take χ_R such that $d\chi_R$ is supported in the shell $B(0, 2R) \setminus B(0, R)$, $|\nabla\chi_R| \le \frac{1}{R}$ and $|\nabla^2\chi_R| \le \frac{1}{R^2}$, then $\|P_0\Delta^{-1}\omega\|_1$ and $\|P_1\Delta^{-1}\omega\|_1$ tend to 0 as $R \to \infty$. Then, letting $R \to \infty$ in (6), $\|\phi\|_q$ stays uniformly bounded, yielding eventually that $d^*\Delta^{-1}\omega \in L^q$, thanks to Fatou's theorem. The Poincaré inequality (1) is proved also for p=1.

2.2.1. *Interior results for $p = 1$.* We recall now to the following result proved in [3] (see Corollary 1.2 and the proof therein), that concerns interior Poincaré and interior Sobolev inequality in the case $p = 1$, where the world "interior" is meant to stress the loss of domain from B' to B.

Theorem 2.1. *For $h = 1, \dots, n-1$, let $q = n/(n-1)$. Let $B \subset \mathbb{R}^n$ be a bounded open convex set, and let B' be an open set, $B \subset\subset B'$. Then there exists $C = C(n, B, B')$ with the following property:*

(1) *Interior Poincaré inequality. For every closed h-form ω in $L^1(B')$, there exists an $(h-1)$-form $\phi \in L^q(B)$, such that*

$$d\phi = \omega_{|B}, \qquad \textit{and} \qquad \|\phi\|_{L^q(B)} \le C\,\|\omega\|_{L^1(B')}.$$

(2) *Sobolev inequality. For every closed h-form $\omega \in L^1$ with support in B, there exists an $(h-1)$-form $\phi \in L^q$, with support in B', such that*

$$d\phi = \omega \qquad \textit{and} \qquad \|\phi\|_{L^q(B')} \le C\,\|\omega\|_{L^1(B)}.$$

Sketch of the proof: The proof of this local Poincaré inequality is based on Iwaniec-Lutoborsky's homotopy, [11]. The core of Iwaniec & Lutoborski's argument relies on the construction of an homotopy operator $T_{IL} : L^p(B) \to W^{1,p}(B)$ which is defined by a kernel k that can be estimated by a singular integral of potential type that is homogeneous of degree $1-N$.

We start from the homotopy operator $d^*\Delta^{-1}$ (that is associated with an homogeneous kernel), through successive localizations obtained by means of a family of cut-off functions, we obtain an approximate homotopy formula for L^1-forms α on B' such that $d\alpha \in L^1(B')$:

$$\alpha = dT\alpha + Td\alpha + S\alpha \qquad \text{on } B$$

(here is the loss of the domain). Here T a bounded operator

$$T : L^1(B') \cap d^{-1}(L^1(B')) \to L^q(B)$$

with if $q = n/(n-1)$ (in other words, T has the good continuity properties), and S is a smoothing operator

$$S : L^1(B') \to W^{s,q}(B).$$

Take now $\alpha = \omega$ that is a closed form. Thus $S\omega = \omega - dT\omega$ is closed and belongs to $L^q(B)$, with norm controlled by the L^1-norm of ω in B'. Thus we can apply Iwaniec & Lutoborski's homotopy T_{IL} to the smoothed form $S\omega$ to obtain

$$S\omega = dT_{IL}S\omega =: d\gamma$$

on B with the norm of γ in $W^{1,q}(B)$ controlled by the L^q-norm of $S\omega$ in B, and therefore by the L^1-norm of ω. Set $\phi := T\omega + \gamma$. Clearly

$$d\phi = dT\omega + d\gamma = dT\omega + S\omega = \omega.$$

In addition

$$\|\phi\|_{L^q(B)} \le C\big(\|\omega\|_{L^1(B)} + \|S\omega\|_{L^q(B)}\big) \le C\|\omega\|_{L^1(B)},$$

and we are done.

□

3. Poincaré and Sobolev inequalities for currents

To keep the paper self contained we recall briefly some definitions and results concerning Euclidean currents. We refer to e.g. [9] for a detailed presentation.

Definition 3.1. *If $\mathcal{U} \subset \mathbb{R}^n$ is an open set and $0 \le h \le n$, we say that T is a h-current on $\mathcal{U}$ if T is a continuous linear functional on smooth compactly differential h-forms endowed with the usual topology. The space of h-dimensional currents in $\mathcal{U}$ is denoted by $D_h(\mathcal{U})$. If u is a h differential form in $L^1_{\mathrm{loc}}(\mathcal{U})$, then u can be identified canonically with a h-current T_u through the formula*

$$\langle T_u|\varphi\rangle := \int_{\mathcal{U}} *u \wedge \varphi = \int_{\mathcal{U}} \langle u, \varphi\rangle\, dx$$

for any h form φ smooth compactly supported on $\mathcal{U}$.

From now on, if there is no way to misunderstandings, and u is differential forms with coefficients belonging to $L^1_{\mathrm{loc}}(\mathcal{U})$, we could write also u instead of T_u.

Suppose now u be a h form sufficiently smooth (take for instance $u \in C^\infty(\mathbb{R}^n)$. If $\phi \in \mathcal{D}(\mathbb{R}^n)$ is $n-h+1$ form, then, by Stokes formula,

$$\int_{\mathbb{R}^n} du \wedge \phi\, dx = (-1)^h \int_{\mathbb{R}^n} u \wedge d\phi\, dx.$$

Thus, if $T \in D_h(\mathbb{R}^n)$ it is natural to set

$$\langle \partial T|\phi\rangle = \langle T|d\phi\rangle$$

for any $(h-1)$ form $\phi \in \mathcal{D}(\mathbb{R}^n)$ and we call the $h-1$ current ∂T the *boundary of* T.

Definition 3.2. *Let $\mathcal{U}$ be open set. Let $T, T_j \in D_h(\mathcal{U})$, we say that the sequence $\{T_j\}$ converges in the sense of currents to T, and we write $T_j \to T$ in the sense of currents, if $\langle T_j|\alpha\rangle \to \langle T|\alpha\rangle$ for any h form $\alpha \in \mathcal{D}(\mathcal{U})$.*

As for distributions, the *support of a current* $T \in D_h(\mathcal{U})$ is defined as $\operatorname{spt} T = \bigcap\{K \subset \mathcal{U} \mid \text{ K relatively closed in } \mathcal{U},\ \langle T|\alpha\rangle = 0 \text{ for all } h \text{ form } \alpha \in D(\mathcal{U}) \text{ with } \operatorname{supp}\alpha \subset \mathcal{U}\setminus K\}$.

Following [9] Section 2.3 and keeping in mind Definition 2.1, we introduce also the notion of mass of a current.

Definition 3.3. *Let $\mathcal{U}, \mathcal{V}$ be open sets and $\mathcal{V} \subset \mathcal{U}$. Let $T \in D_h(\mathcal{U})$. We set*

$$M_{\mathcal{V}}(T) := \sup\ \{\langle T|\alpha\rangle \mid \alpha\ h-\text{form in } \mathcal{D}(\mathcal{U},)\ \operatorname{supp}\alpha \subset \mathcal{V},\ \|\alpha\| \leq 1\ \forall x \in \mathcal{U}\}\ ,$$

and we say that T is of finite mass if $M_{\mathcal{V}}(T)$ is finite. If $\mathcal{V} = \mathcal{U}$ we shall simply write $M(T)$ instead of $M_{\mathcal{V}}(T)$.

With the previous definitions in mind, we are ready to prove the following regularization-type results for currents.

Theorem 3.1. *Let $\mathcal{U} \subset \mathbb{R}^n$ be an open set, and let T be a h-current in $\mathcal{U}$ of finite mass $M(T)$. Then for any $0 < \epsilon << 1$ there exists $\omega_\epsilon \in C^\infty(\mathcal{U})$ h-form such that, if we set $T_\epsilon := T_{\omega_\epsilon}$ we have:*

i) *$T_\epsilon \to T$ in the sense of currents;*
ii) *$\|\omega_\epsilon\|_{L^1(\mathcal{U})} \leq c_1\, M(T_\epsilon) \leq c_2\, M(T)$;*
iii) *if $\partial T = 0$ then the forms ω_ϵ are closed;*
iv) *if T is compactly supported in $\mathcal{U}$ then the the forms ω_ϵ are compactly supported in an open set $\tilde{U} \subset\subset \mathcal{U}$ depending only on the support of T.*

Proof. The existence of a sequence of currents T_ϵ satisfying i) is proved in Proposition 3 of [9], Chapter 5, Section 2.1. On the other hand, by the subsequent Proposition 6, the currents T_ϵ can be written as $T_\epsilon := T_{\omega_\epsilon}$, where ω_ϵ is a h-form in $C^\infty(\mathcal{U})$. We notice now that, if denote by $\|\xi\|$ the comass norm of a covector $\xi \in \bigwedge^h(\mathbb{R}^n)$ (see Definition 2.1), by formula (4) there exists a geometric constant $c_1 > 0$ such that

$$|\xi| \leq c_1\|\xi\| \qquad \text{for all } \xi \in \textstyle\bigwedge^h(\mathbb{R}^n).$$

Thus, keeping in mind again Proposition 6 of [9], Chapter 5, Section 2.1

$$\|\omega_\epsilon\|_{L^1(\mathcal{U})} \le c_1 \int_{\mathcal{U}} \|\omega_\epsilon\|\, dV = c_1\, M(T_\epsilon) \le c_2\, M(T),$$

by Proposition 2 - iii) of [9], Chapter 5, Section 2.1. This proves ii). Moreover, assertion iv) of the same proposition of [9], yields our iv). Finally, iii) follows from Proposition 4 - ii) of [9], Chapter 5, Section 2.1.

□

We are now able to prove the following Poincaré and Sobolev inequalities for currents.

Proof of Theorem 1.1. Keeping the notations of Theorem 3.1 with $\mathcal{U} = B'$, if we choose a sequence $(\epsilon_k)_{k\in\mathbb{N}}$, $\epsilon_k \to 0$ as $k \to \infty$ and we write for sake of simplicity $\omega_k := \omega_{\epsilon_k}$, by Theorem 3.1 -iii) ω_k are closed, hence by Theorem 2.1-(1), for any $k \in \mathbb{N}$ there exists a $(h-1)$-form

$$\phi_k \in L^q(B) \qquad \text{such that } \; d\phi_k = \omega_k \text{ in } B,$$

and, by Theorem 3.1 - ii),

$$\|\phi_k\|_{L^q(B)} \le C\, \|\omega_k\|_{L^1(B')} \le C\, M(T). \tag{7}$$

Since $q > 1$, it follows from (7) that there exists a $(h-1)$-form $\phi \in L^q(B)$ such that (up to a subsequence)

$$\phi_k \to \phi \qquad \text{weakly in } L^q(B).$$

Therefore

$$\|\phi\|_{L^q(B)} \le \liminf_k \|\phi_k\|_{L^q(B)} \le C\, M(T). \tag{8}$$

In particular, $T_{\phi_k} \to T_\phi$ in the sense of currents. Therefore, keeping in mind Theorem 3.1 - i),

$$\partial T_\phi = \lim_k \partial T_{\phi_k} = \lim_k T_{d\phi_k} = \lim_k T_{\omega_k} = T. \tag{9}$$

This proves Poincaré inequality. As for Sobolev inequality, we carry on the same argument relying on Sobolev inequality for L^1-forms stated in Theorem 2.1-(2) and keeping again into account Theorem 3.1.

□

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